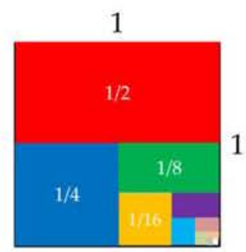
Series

Definition.
Given a sequence, $\{a_n\}_{n=1}^{\infty}$, the associated ***(infinite) series*** is the ordered sum

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

Examples.

$$\{a_n\} = \left\{\frac{1}{2^n}\right\} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots\right\}$$



$$\{a_n\} = \left\{3, \frac{1}{10}, \frac{4}{10^2}, \frac{1}{10^3}, \frac{5}{10^4}, \cdots\right\}$$

Definition.
Given a series, $\sum_{n=1}^{\infty} a_n$, the sequence of ***partial sums***, $\{s_k\}$, is defined for each k as

$$s_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + \cdots + a_k$$

Example.

$$\sum_{n=1}^{\infty} a_n = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \cdots$$

Definition.
Given a series, $\sum_{n=1}^{\infty} a_n$, let s_n denote its nth partial sum.
If $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = S$ exists as a real number,
then $\sum_{n=1}^{\infty} a_n$ is called ***convergent*** and we write $\sum_{n=1}^{\infty} a_n = S$.

Example.

$$\sum_{n=1}^{\infty} a_n = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \cdots$$

$$\{s_n\} = \{3, 3.1, 3.14, 3.141, 3.1415, \dots\}$$

Definition.
Given a series, $\sum_{n=1}^{\infty} a_n$, let s_n denote its nth partial sum.
If $\{s_n\}$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is called ***divergent.***

Example.

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \cdots$$

Definition.
A ***geometric series*** is a series that has the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots + ar^{n-1} + ar^n + \cdots \qquad a \neq 0$$

Observe.

$$\frac{n+1 \text{ term of series}}{n \text{ term of series}} = \frac{ar^n}{ar^{n-1}} = r$$

Example.

Theorem.

Given a geometric series $\sum_{n=1}^{\infty} ar^{n-1}$

- If $\underbrace{|r| < 1}_{-1 < r < 1}$, then $\sum_{n=1}^{\infty} ar^{n-1}$ converges and $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$
- If $\underbrace{|r| \geq 1}_{-1 \leq r \text{ or } r \geq 1}$, then $\sum_{n=1}^{\infty} ar^{n-1}$ diverges

Example.

Example.

Is the series $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ convergent or divergent?

Example.

Write the number $2.\overline{16} = 2.161616\ldots$ as a ratio of integers.

Proof of Theorem.

Given a geometric series $\sum_{n=1}^{\infty} ar^{n-1}$, $a \neq 0$

- If $r = 1$, then $\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} a \implies s_n = na$
$$\implies \lim_{n\to\infty} s_n = \infty$$
$$\implies \text{series diverges}$$

- If $r \neq 1$, then
$$\begin{array}{rl} s_n = & a + ar + ar^2 + \cdots + ar^{n-1} \\ -\quad rs_n = & \quad\; ar + ar^2 + \cdots + ar^{n-1} + ar^n \\ \hline s_n - rs_n = & a - ar^n \end{array}$$
$$s_n = \frac{a\left(1-r^n\right)}{1-r} = \frac{a}{1-r} - \frac{a}{1-r}r^n$$

➢ If $r > 1$ or $r \leq -1$, then $\lim_{n\to\infty} s_n$ diverges $\implies$ series diverges

➢ If $-1 < r < 1$, then $\lim_{n\to\infty} s_n = \frac{a}{1-r} \implies$ series converges

Index Shift.

$$\sum_{n=2}^{\infty} \frac{n+5}{2} = \sum_{i=0}^{\infty} \boxed{\;?\;}$$

Example.

Calculate $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$ for $|x| < 1$

Definition.

A ***telescoping series*** is a series whose partial sums eventually only have a fixed number of terms after cancellation

Example.

Calculate $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) = S$

Conditional Statements.

implication $p \implies q$ — you eat spinach $\implies$ you get strong

$\implies$

↕ logically the same

contrapositive $\sim q \implies \sim p$ — you do not get strong $\implies$ you do not eat spinach

$\implies$

logically not the same (implication ↔ converse)

converse $q \implies p$ — you get strong $\implies$ you eat spinach

Theorem.

$$\sum_{n=1}^{\infty} a_n \text{ is convergent} \implies \lim_{n\to\infty} a_n = 0$$

Example.

Divergence Test. (Contapositive of above Theorem)

$$\begin{bmatrix} \lim_{n\to\infty} a_n \text{ does not exist} \\ \text{or} \\ \lim_{n\to\infty} a_n \neq 0 \end{bmatrix} \implies \sum_{n=1}^{\infty} a_n \text{ is divergent}$$

Example.

If $\{a_n\} = \{1, 1, 1, \dots\}$ is $\sum_{n=1}^{\infty} a_n$ convergent or divergent?

Remark.

The following is **NOT TRUE**. (Converse of previous theorem)

$$\lim_{n\to\infty} a_n = 0 \implies \sum_{n=1}^{\infty} a_n \text{ is convergent}$$

Famous Counterexample.

The ***harmonic series*** is $\sum_{n=1}^{\infty} a_n$ where $a_n = \dfrac{1}{n}$

Theorem.

- $\sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n \qquad c$ constant
- $\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$

Examples.