Series

Definition.

Given a sequence, $\{a_n\}_{n=1}^{\infty}$, the associated ***(infinite) series*** is the ordered sum

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

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Examples.

$$\{a_n\} = \left\{\frac{1}{2^n}\right\} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots\right\}$$

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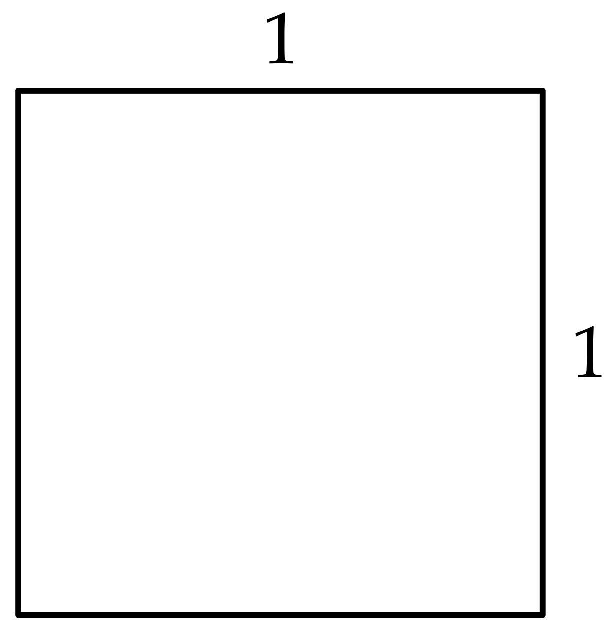

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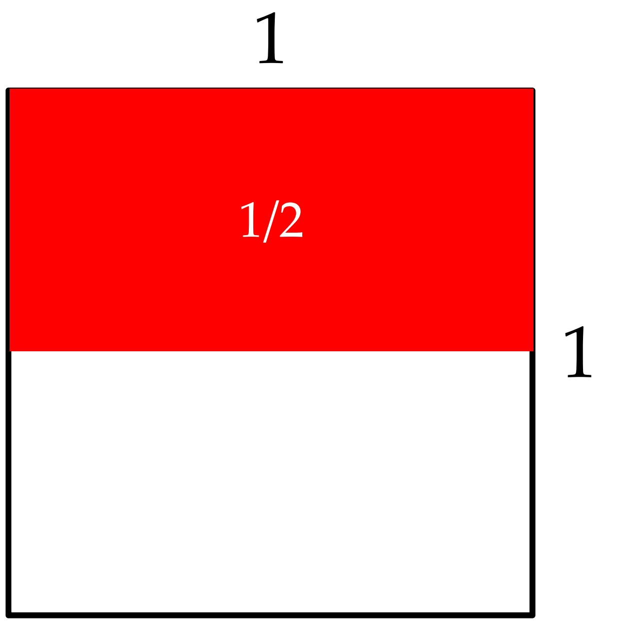

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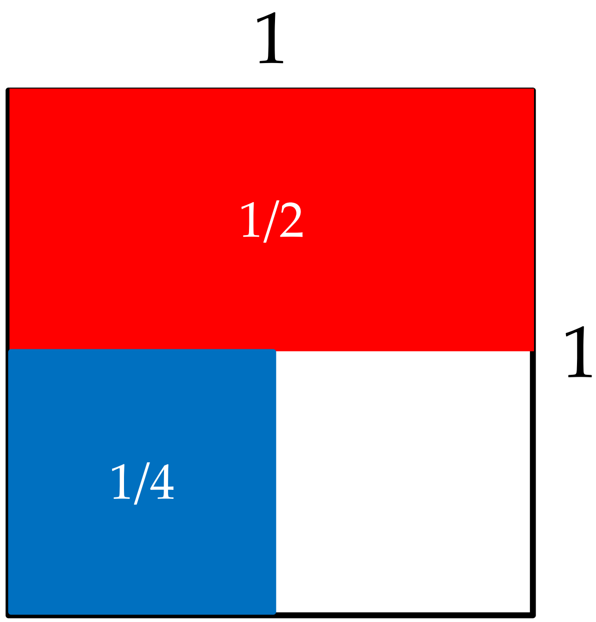

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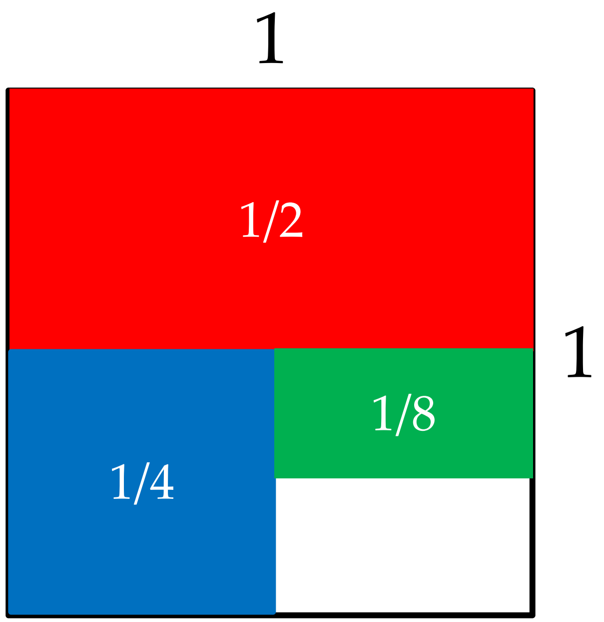

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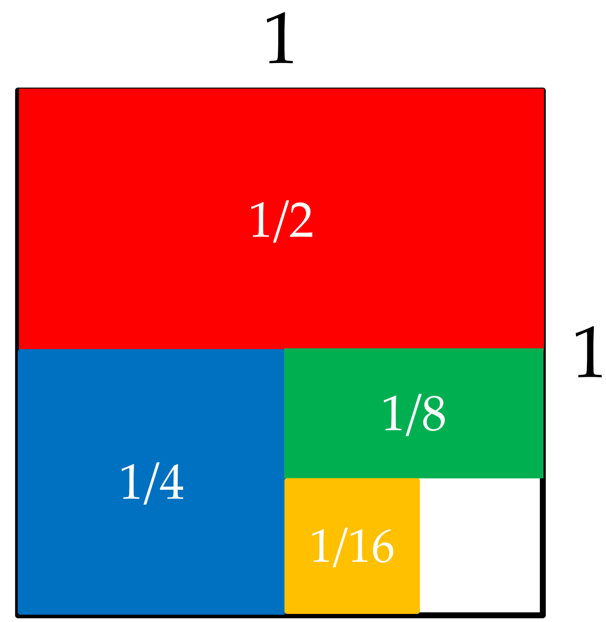

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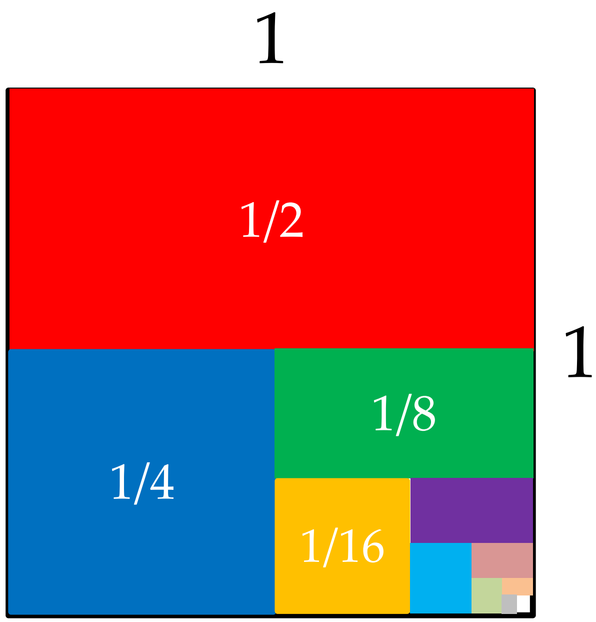

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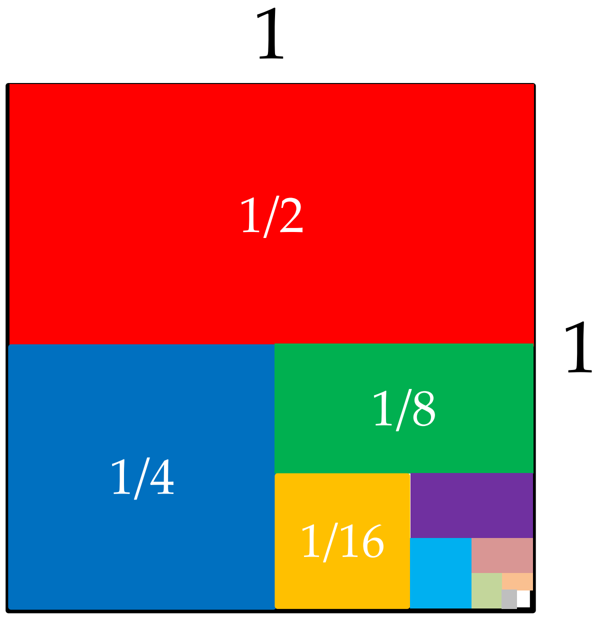

$$\{a_n\} = \left\{3, \frac{1}{10}, \frac{4}{10^2}, \frac{1}{10^3}, \frac{5}{10^4}, \cdots\right\}$$

Definition.

Given a series, $\sum_{n=1}^{\infty} a_n$, the sequence of ***partial sums,*** $\{s_k\}$, is defined for each k as

$$s_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + \cdots + a_k$$

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Given a series, $\sum_{n=1}^{\infty} a_n$, let s_n denote its nth partial sum.

If $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = S$ exists as a real number,

then $\sum_{n=1}^{\infty} a_n$ is called ***convergent*** and we write $\sum_{n=1}^{\infty} a_n = S$.

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$$\{s_n\} = \{3, 3.1, 3.14, 3.141, 3.1415, \ldots\}$$

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If $\{s_n\}$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is called ***divergent.***

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Example.

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \cdots$$

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A ***geometric series*** is a series that has the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots + ar^{n-1} + ar^n + \cdots \qquad a \neq 0$$

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Theorem.

Given a geometric series $\sum_{n=1}^{\infty} ar^{n-1}$

- If $\underbrace{|r| < 1}_{-1 < r < 1}$, then $\sum_{n=1}^{\infty} ar^{n-1}$ converges and $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$
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Example.

Is the series $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$ convergent or divergent?

Example.

Write the number $2.\overline{16} = 2.161616\ldots$ as a ratio of integers.

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Index Shift.

$$\sum_{n=2}^{\infty} \frac{n+5}{2} = \sum_{i=0}^{\infty} \boxed{?}$$

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Example.

Calculate $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$ for $|x| < 1$

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A ***telescoping series*** is a series whose partial sums eventually only have a fixed number of terms after cancellation

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Example.
Calculate $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)=S$

Conditional Statements.

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implication

$p \implies q$

you eat spinach $\implies$ you get strong

$\implies$

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implication | you eat spinach $\implies$ you get strong

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contrapositive | you do not get strong $\implies$ you do not eat spinach

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logically the same

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Conditional Statements.

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$\implies$

logically not the same

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Theorem.

$$\sum_{n=1}^{\infty} a_n \text{ is convergent} \Longrightarrow \lim_{n\to\infty} a_n = 0$$

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Divergence Test.

$$\left(\begin{array}{c} \lim\limits_{n\to\infty} a_n \text{ does not exist} \\ \text{or} \\ \lim\limits_{n\to\infty} a_n \neq 0 \end{array} \right) \implies \sum_{n=1}^{\infty} a_n \text{ is divergent}$$

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Example.

If $\{a_n\} = \{1, 1, 1, \ldots\}$ is $\sum_{n=1}^{\infty} a_n$ convergent or divergent?

Remark.

The following is **NOT TRUE**.

$$\lim_{n\to\infty} a_n = 0 \implies \sum_{n=1}^{\infty} a_n \text{ is convergent}$$

Remark.

The following is **NOT TRUE**. (Converse of previous theorem)

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Famous Counterexample.

The ***harmonic series*** is $\sum_{n=1}^{\infty} a_n$ where $a_n = \dfrac{1}{n}$

Theorem.

- $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n \qquad c \text{ constant}$
- $\displaystyle\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$

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Examples.