Programming Assignment 2

In this assignment we will automatically create a Lagrange polynomial and a Hermite polynomial that approximate a function to a specified precision. I will walk you through how to do this with for a Lagrange polynomial and then you will be responsible for doing it for a Hermite polynomial.

Consider the following problem. Given a function $f(x)$, an interval $[a, b]$, and a tolerance ε, I want to create a Lagrange polynomial $P(x)$ such that for all $x \in [a, b]$, the absolute error $|P(x) - f(x)| < \varepsilon$. More precisely, our problem is the following

Input: $[a, b]$, an interval,
f, a function $f \in C^{\infty}[a, b]$,
ε, real positive number.

Output: $P(x)$, a Lagrange polynomial such that for all $x \in [a, b]$, $|P(x) - f(x)| < \varepsilon$.

To solve this problem, we must construct a Lagrange polynomial satisfying certain conditions. To construct a Lagrange polynomial we need to know the nodes it will be interpolating. Furthermore, we need to guarantee that the nodes we pick are "good;" that is, that the absolute error is bounded by ε. One way of checking whether we our nodes are "good" is to use the following theorem from our book.

Theorem 3.3. *Suppose $x_0, x_1, \ldots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then for each $x \in [a, b]$, there exists a number $\xi(x) \in (a, b)$ with*

$$f(x) - P(x) = \frac{f^{(n+1)}\big(\xi(x)\big)}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)$$

where $P(x)$ is the Lagrange interpolating polynomial

$$P(x) = \sum_{k=0}^{n} \left[f(x_k) \prod_{\substack{i=0 \\ i \neq k}} \frac{x - x_i}{x_k - x_i} \right].$$

This theorem says that *if I know the nodes* then I can calculate whether the Lagrange polynomial is "good." However, in our case we do not know the nodes. One way to get around this is to simply construct the nodes yourself in a systematic way. I suggest we use the following algorithm to create our nodes.

ALGORITHM: $\{x_0, \ldots, x_n\} \leftarrow$ **CreateNodes**$([a, b], n)$
Input : $[a, b]$, an interval,
n, a positive integer.
Output: $\{x_0, \ldots, x_n\}$, a list of values such that for all i, $x_i = a + ih$ where $h = \frac{b-a}{n}$.

$h \leftarrow \dfrac{b-a}{n}$
for $i = 0$ **to** n **do**
  $x_i \leftarrow a + ih$
return $\{x_0, \ldots, x_n\}$

Example. Suppose $[a, b] = [3, 5]$ and $n = 10$. Then **CreateNodes**$([a, b], n)$ outputs

$$\{x_0, \ldots, x_{10}\} \leftarrow \left\{3, \frac{16}{5}, \frac{17}{5}, \frac{18}{5}, \frac{19}{5}, 4, \frac{21}{5}, \frac{22}{5}, \frac{23}{5}, \frac{24}{5}, 5\right\}. \tag{1}$$

The previous algorithm creates a set of nodes such that the distance between successive nodes x_i, x_{i+1} equals $h = \frac{b-a}{n}$. Now that we have our nodes, we can check whether the Lagrange polynomial using these nodes is "good" by checking whether the condition from Theorem 3.3 holds.

ALGORITHM: *bool* $\leftarrow$ **IsGoodLagrange**($[a,b], f, \varepsilon, \{x_0,\ldots,x_n\}$)

Input : $[a,b]$, an interval,
f, a function $f \in C^\infty[a,b]$,
ε, real positive number,
$\{x_0,\ldots,x_n\}$, list of values.

Output: *bool*, true if and only if $|P(x) - f(x)| < \varepsilon$ for all $x \in [a,b]$ where $P(x)$ is the Lagrange polynomial using nodes $\{x_0,\ldots,x_n\}$, false otherwise.

1 $M \leftarrow \max_{\xi\in(a,b)} \left|f^{(n+1)}(\xi)\right|$
2 $m \leftarrow \max_{x\in[a,b]} |(x-x_0)(x-x_1)\cdots(x-x_n)|$
3 **if** $\dfrac{M}{(n+1)!} \cdot m < \varepsilon$ **then**
4   *bool* $\leftarrow$ true
  else
5   *bool* $\leftarrow$ false
6 **return** *bool*

Example. Suppose $f(x) = e^x$, $[a,b] = [3,5]$, $\varepsilon = 10^{-7}$, and $\{x_0,\ldots,x_{10}\}$ are the nodes (1) generated in the previous example. Then **IsGoodLagrange**($[a,b], f, \varepsilon, \{x_0,\ldots,x_{10}\}$) outputs true.

The if statement in line 3 in the previous algorithm checks whether absolute error is less than the specified ε. This is what we mean by checking whether our Lagrange polynomial is "good," because it checks the condition in Theorem 3.3. In an actual implementation of our algorithm, we can use the `Maximize` or `NMaximize` command in *Mathematica* to calculate M, m in steps 1 and 2, respectively.

Now we can combine the two methods to solve our problem by iterating over n until our "good" condition is reached. We will also assume the existence of a method which creates a Lagrange polynomial through a given set of points, which we call **LagrangePolynomial** (which I gave you an implementation of in *Mathematica*).

ALGORITHM: $P \leftarrow$ **FindGoodLagrangePolynomial**($[a,b], f, \varepsilon$)

Input : $[a,b]$, an interval,
f, a function $f \in C^\infty[a,b]$,
ε, real positive number.

Output: $P(x)$, a Lagrange polynomial such that for all $x \in [a,b]$, $|P(x) - f(x)| < \varepsilon$.

$n \leftarrow 1$
$\{x_0,\ldots,x_n\} \leftarrow$ **CreateNodes**($[a,b], n$)
while IsGoodLagrange($[a,b], f, \{x_0,\ldots,x_n\}$) = false **do**
    $n \leftarrow n+1$
    $\{x_0,\ldots,x_n\} \leftarrow$ **CreateNodes**($[a,b], n$)
$P \leftarrow$ **LagrangePolynomial** $\left(\left\{(x_0, f(x_0)),\ldots,(x_n, f(x_n))\right\}\right)$
return P

Example. Suppose $f(x) = e^x$, $[a,b] = [3,5]$, and $\varepsilon = 10^{-7}$. Calling **FindGoodLagrangePolynomial**($[a,b], f, \varepsilon$) returns

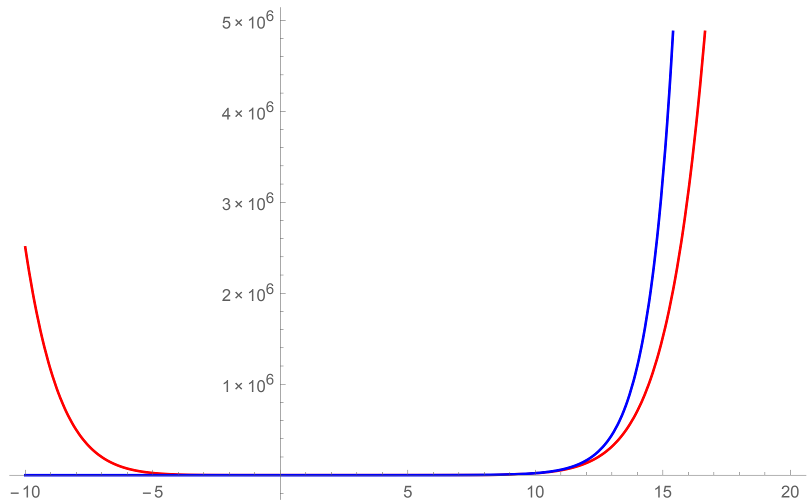
Out[620]=
$$\begin{aligned}
\frac{1}{145152}\, 390625\, e^3 \Big(& (4-x)(5-x)\left(-\tfrac{24}{5}+x\right)\left(-\tfrac{23}{5}+x\right)\left(-\tfrac{22}{5}+x\right)\left(-\tfrac{21}{5}+x\right)\left(-\tfrac{19}{5}+x\right)\left(-\tfrac{18}{5}+x\right)\left(-\tfrac{17}{5}+x\right)\left(-\tfrac{16}{5}+x\right) + \\
& 10\, e^{1/5}\left(\tfrac{21}{5}-x\right)(-5+x)\left(-\tfrac{24}{5}+x\right)\left(-\tfrac{23}{5}+x\right)\left(-\tfrac{22}{5}+x\right)(-4+x)\left(-\tfrac{19}{5}+x\right)\left(-\tfrac{18}{5}+x\right)\left(-\tfrac{17}{5}+x\right)(-3+x) - \\
& 45\, e^{2/5}\left(\tfrac{22}{5}-x\right)(-5+x)\left(-\tfrac{24}{5}+x\right)\left(-\tfrac{23}{5}+x\right)\left(-\tfrac{21}{5}+x\right)(-4+x)\left(-\tfrac{19}{5}+x\right)\left(-\tfrac{18}{5}+x\right)\left(-\tfrac{16}{5}+x\right)(-3+x) + \\
& 120\, e^{3/5}\left(\tfrac{23}{5}-x\right)(-5+x)\left(-\tfrac{24}{5}+x\right)\left(-\tfrac{22}{5}+x\right)\left(-\tfrac{21}{5}+x\right)(-4+x)\left(-\tfrac{19}{5}+x\right)\left(-\tfrac{17}{5}+x\right)\left(-\tfrac{16}{5}+x\right)(-3+x) - \\
& 210\, e^{4/5}\left(\tfrac{24}{5}-x\right)(-5+x)\left(-\tfrac{23}{5}+x\right)\left(-\tfrac{22}{5}+x\right)\left(-\tfrac{21}{5}+x\right)(-4+x)\left(-\tfrac{18}{5}+x\right)\left(-\tfrac{17}{5}+x\right)\left(-\tfrac{16}{5}+x\right)(-3+x) + \\
& 252\, e\,(5-x)\left(-\tfrac{24}{5}+x\right)\left(-\tfrac{23}{5}+x\right)\left(-\tfrac{22}{5}+x\right)\left(-\tfrac{21}{5}+x\right)\left(-\tfrac{19}{5}+x\right)\left(-\tfrac{18}{5}+x\right)\left(-\tfrac{17}{5}+x\right)\left(-\tfrac{16}{5}+x\right)(-3+x) + \\
& 210\, e^{6/5}(-5+x)\left(-\tfrac{24}{5}+x\right)\left(-\tfrac{23}{5}+x\right)\left(-\tfrac{22}{5}+x\right)(-4+x)\left(-\tfrac{19}{5}+x\right)\left(-\tfrac{18}{5}+x\right)\left(-\tfrac{17}{5}+x\right)\left(-\tfrac{16}{5}+x\right)(-3+x) - \\
& 120\, e^{7/5}(-5+x)\left(-\tfrac{24}{5}+x\right)\left(-\tfrac{23}{5}+x\right)\left(-\tfrac{21}{5}+x\right)(-4+x)\left(-\tfrac{19}{5}+x\right)\left(-\tfrac{18}{5}+x\right)\left(-\tfrac{17}{5}+x\right)\left(-\tfrac{16}{5}+x\right)(-3+x) + \\
& 45\, e^{8/5}(-5+x)\left(-\tfrac{24}{5}+x\right)\left(-\tfrac{22}{5}+x\right)\left(-\tfrac{21}{5}+x\right)(-4+x)\left(-\tfrac{19}{5}+x\right)\left(-\tfrac{18}{5}+x\right)\left(-\tfrac{17}{5}+x\right)\left(-\tfrac{16}{5}+x\right)(-3+x) - \\
& 10\, e^{9/5}(-5+x)\left(-\tfrac{23}{5}+x\right)\left(-\tfrac{22}{5}+x\right)\left(-\tfrac{21}{5}+x\right)(-4+x)\left(-\tfrac{19}{5}+x\right)\left(-\tfrac{18}{5}+x\right)\left(-\tfrac{17}{5}+x\right)\left(-\tfrac{16}{5}+x\right)(-3+x) + \\
& e^2\left(-\tfrac{24}{5}+x\right)\left(-\tfrac{23}{5}+x\right)\left(-\tfrac{22}{5}+x\right)\left(-\tfrac{21}{5}+x\right)(-4+x)\left(-\tfrac{19}{5}+x\right)\left(-\tfrac{18}{5}+x\right)\left(-\tfrac{17}{5}+x\right)\left(-\tfrac{16}{5}+x\right)(-3+x)\Big)
\end{aligned}$$

We can convince ourselves that for all $x \in [a,b]$, that the absolute error $|f(x) - P(x)| < \varepsilon$ by plugging in many values:

Out[42]//TableForm=

z	|f(z) - H(z)|
3.	2.13163×10^{-14}
3.2	3.98799×10^{-10}
3.4	1.42109×10^{-14}
3.6	4.56435×10^{-11}
3.8	2.84217×10^{-14}
4.	2.39098×10^{-11}
4.2	1.42109×10^{-14}
4.4	4.8729×10^{-11}
4.6	2.13163×10^{-13}
4.8	4.50854×10^{-10}
5.	2.55795×10^{-13}

If we graph the Lagrange polynomial P and the original function f, we see they closely resemeble each other on the interval $[a,b]$.



As a summary, our method **FindGoodLagrangePolynomial** generates a possible set of $n+1$ nodes (using **CreateNodes**), checks whether they are "good" (using **IsGoodLagrange**), and if they are "good," forms the Lagrange polynomial (using **LagrangePolynomial**).

Your task in this assignment is to solve the following problem.

Input: $[a,b]$, an interval,
f, a function $f \in C^\infty[a,b]$,
ε, real positive number.

Output: $H(x)$, a Hermite polynomial such that for all $x \in [a,b]$, $|H(x) - f(x)| < \varepsilon$.

To solve this problem, you must implement a method like **FindGoodLagrangePolynomial**. Instead, you might want to call it **FindGoodHermitePolynomial** here. To get started, I suggest you do the following:

- Implement **CreateNodes**: There is no modification needed for this method. Think about why you do not have to change the algorithm in the Hermite case. I gave you the code for this.
- Implement **IsGoodHermite**: This is a modified version of **IsGoodLagrange**. In the Hermite case, you have a different check for what is "good." Think about what Theorem in the book you need to use.
- Implement **FindGoodHermitePolynomial**: This is a modification of **FindGoodLagrangePolynomial**. I would suggest that you use the Hermite polynomial code I sent you, `HermitePolynomial` in *Mathematica*.

Check your code using the inputs $f(x) = e^x$, $[a, b] = [3, 5]$, $\varepsilon = 10^{-7}$. Turn in your code and a table showing the absolute error $|H(x) - f(x)|$ for $x = 3, 3.2, 3.4, \ldots, 4.8, 5$ where $H(x)$ is your "good" Hermite polynomial. Plot f and H on the interval $[-10, 20]$.

Programming Assignment 2

Lagrange Polynomial

```
CreateNodes[{a_, b_}, n_] := Module[{h, i, X},
   X = {};
   h = (b - a) / n;
   For[i = 0, i <= n, i++,
    AppendTo[X, a + i * h];
   ];
   Return[X];
  ];
```

```
CreateNodes[{3, 5}, 10]
```

$\left\{3, \frac{16}{5}, \frac{17}{5}, \frac{18}{5}, \frac{19}{5}, 4, \frac{21}{5}, \frac{22}{5}, \frac{23}{5}, \frac{24}{5}, 5\right\}$

```
IsGoodLagrange[{a_, b_}, f_, epsilon_, nodes_] := Module[{n, M, m, df, g, bool},
   n = Length[nodes] - 1;
   df = D[f, {x, n + 1}];
   M = First[NMaximize[{Abs[df], a <= x <= b}, x]];
   g = Product[x - nodes[[i]], {i, 1, n + 1}];
   m = First[NMaximize[{Abs[g], a <= x <= b}, x]];
   If[(M / ((n + 1) !)) * m < epsilon,
    bool = True,
    bool = False
   ];
   Return[bool];
  ];
```

```
nodes = CreateNodes[{3, 5}, 10];
IsGoodLagrange[{3, 5}, Exp[x], 10^(-7), nodes]
```

```
True
```

```
LagrangePolynomial[points_List] := Module[{numPoints, L, P},
  numPoints = Length[points];
  L[k_] := Product[(x - points[[i, 1]])/(points[[k, 1]] - points[[i, 1]]), {i, 1, k - 1}]
    Product[(x - points[[i, 1]])/(points[[k, 1]] - points[[i, 1]]), {i, k + 1, numPoints}];
  P = Sum[points[[k, 2]] L[k], {k, 1, numPoints}];
  Return[Simplify[P]];
 ];

FindGoodLagrangePolynomial[{a_, b_}, f_, epsilon_] := Module[{n, nodes, P, i, fvals, points},
  n = 1;
  nodes = CreateNodes[{a, b}, n];
  While[IsGoodLagrange[{a, b}, f, epsilon, nodes] == False,
   n = n + 1;
   nodes = CreateNodes[{a, b}, n];
  ];

  fvals = {};
  For[i = 1, i <= n + 1, i++,
   AppendTo[fvals, f /. x -> nodes[[i]]];
  ];
  points = Transpose[{nodes, fvals}];
  P = LagrangePolynomial[points];
  Return[P];
 ];
```

```
P = FindGoodLagrangePolynomial[{3, 5}, Exp[x], 10^(-7)]
```

$$
\begin{aligned}
&\frac{1}{145\,152}\, 390\,625\, e^{3} \\
&\Bigg((4-x)\,(5-x)\left(-\frac{24}{5}+x\right)\left(-\frac{23}{5}+x\right)\left(-\frac{22}{5}+x\right)\left(-\frac{21}{5}+x\right)\left(-\frac{19}{5}+x\right)\left(-\frac{18}{5}+x\right)\left(-\frac{17}{5}+x\right)\left(-\frac{16}{5}+x\right) + \\
&\quad 10\, e^{1/5}\left(\frac{21}{5}-x\right)(-5+x)\left(-\frac{24}{5}+x\right)\left(-\frac{23}{5}+x\right)\left(-\frac{22}{5}+x\right)(-4+x)\left(-\frac{19}{5}+x\right)\left(-\frac{18}{5}+x\right) \\
&\quad \left(-\frac{17}{5}+x\right)(-3+x) - 45\, e^{2/5}\left(\frac{22}{5}-x\right)(-5+x)\left(-\frac{24}{5}+x\right)\left(-\frac{23}{5}+x\right)\left(-\frac{21}{5}+x\right)(-4+x) \\
&\quad \left(-\frac{19}{5}+x\right)\left(-\frac{18}{5}+x\right)\left(-\frac{16}{5}+x\right)(-3+x) + 120\, e^{3/5}\left(\frac{23}{5}-x\right)(-5+x)\left(-\frac{24}{5}+x\right)\left(-\frac{22}{5}+x\right) \\
&\quad \left(-\frac{21}{5}+x\right)(-4+x)\left(-\frac{19}{5}+x\right)\left(-\frac{17}{5}+x\right)\left(-\frac{16}{5}+x\right)(-3+x) - 210\, e^{4/5}\left(\frac{24}{5}-x\right)(-5+x) \\
&\quad \left(-\frac{23}{5}+x\right)\left(-\frac{22}{5}+x\right)\left(-\frac{21}{5}+x\right)(-4+x)\left(-\frac{18}{5}+x\right)\left(-\frac{17}{5}+x\right)\left(-\frac{16}{5}+x\right)(-3+x) + 252\, e\,(5-x) \\
&\quad \left(-\frac{24}{5}+x\right)\left(-\frac{23}{5}+x\right)\left(-\frac{22}{5}+x\right)\left(-\frac{21}{5}+x\right)\left(-\frac{19}{5}+x\right)\left(-\frac{18}{5}+x\right)\left(-\frac{17}{5}+x\right)\left(-\frac{16}{5}+x\right)(-3+x) + \\
&\quad 210\, e^{6/5}(-5+x)\left(-\frac{24}{5}+x\right)\left(-\frac{23}{5}+x\right)\left(-\frac{22}{5}+x\right)(-4+x)\left(-\frac{19}{5}+x\right)\left(-\frac{18}{5}+x\right)\left(-\frac{17}{5}+x\right) \\
&\quad \left(-\frac{16}{5}+x\right)(-3+x) - 120\, e^{7/5}(-5+x)\left(-\frac{24}{5}+x\right)\left(-\frac{23}{5}+x\right)\left(-\frac{21}{5}+x\right)(-4+x)\left(-\frac{19}{5}+x\right) \\
&\quad \left(-\frac{18}{5}+x\right)\left(-\frac{17}{5}+x\right)\left(-\frac{16}{5}+x\right)(-3+x) + 45\, e^{8/5}(-5+x)\left(-\frac{24}{5}+x\right)\left(-\frac{22}{5}+x\right)\left(-\frac{21}{5}+x\right) \\
&\quad (-4+x)\left(-\frac{19}{5}+x\right)\left(-\frac{18}{5}+x\right)\left(-\frac{17}{5}+x\right)\left(-\frac{16}{5}+x\right)(-3+x) - 10\, e^{9/5}(-5+x)\left(-\frac{23}{5}+x\right) \\
&\quad \left(-\frac{22}{5}+x\right)\left(-\frac{21}{5}+x\right)(-4+x)\left(-\frac{19}{5}+x\right)\left(-\frac{18}{5}+x\right)\left(-\frac{17}{5}+x\right)\left(-\frac{16}{5}+x\right)(-3+x) + \\
&\quad e^{2}\left(-\frac{24}{5}+x\right)\left(-\frac{23}{5}+x\right)\left(-\frac{22}{5}+x\right)\left(-\frac{21}{5}+x\right)(-4+x)\left(-\frac{19}{5}+x\right)\left(-\frac{18}{5}+x\right)\left(-\frac{17}{5}+x\right)\left(-\frac{16}{5}+x\right)(-3+x)\Bigg)
\end{aligned}
$$

Studying the absolute errors at a lot of points on the interval [3, 5]. Notice the error is always less than 10^{-7}.

```
f = Exp[x];
data = Table[{z, Abs[f - P] /. x -> z}, {z, 3, 5, 0.2}];
TableForm[data, TableHeadings → {None, {"z", "|f(z) - P(z)|"}}]
```

z	|f(z) - P(z)|
3.	1.77636×10^{-14}
3.2	5.68434×10^{-14}
3.4	3.55271×10^{-15}
3.6	2.84217×10^{-14}
3.8	2.84217×10^{-14}
4.	9.9476×10^{-14}
4.2	7.10543×10^{-14}
4.4	1.27898×10^{-13}
4.6	4.83169×10^{-13}
4.8	9.9476×10^{-14}
5.	1.98952×10^{-13}

Comparing the graphs of *P* (red) and *f* (blue).

```
Plot[{P, Exp[x]}, {x, -10, 20}, PlotStyle → {Red, Blue}]
```