Self-Similar Tilings of Nilpotent Lie Groups

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Abstract

We construct self-similar fractal tilings on rationally graded nilpotent Lie groups. Specific examples and graphs of fractal tilings in the Heisenberg group are given

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1 Introduction

We are all familiar with tilings; we see them every day on the kitchen floor or on a chess board. Intuitively, we consider a given design a "tiling" if it has a few specific characteristics: it must break the larger set into a collection of smaller sets, which cover the entire set with no gaps and no overlap. We can define this intuitive notion of tiling more rigorously by saying that a "tiling" of a complete metric space X (such as the kitchen floor) is a locally finite collection T of non-empty subsets of X (in other words, if you draw a small enough circle around any point on the kitchen floor, it will intersect only finitely many tiles), that fulfulls the following three conditions:

1. For any $A \in T$, $\mathrm{cl}\,(\mathrm{int}\,A) = A$ (the tiles are nice solid spaces, not just collections of scattered points).

2. For any discrete $A, B \in T$, $\mathrm{int}\,A \cap \mathrm{int}\,B = \emptyset$ (they don't overlap).

3. $\cup A = X$ (they cover the whole space).

Most tilings found on floors or game boards are also "self-similar". A self-similar tiling, intuitively, has the additional characteristic that if you group together several tiles, you can create a larger set with the same shape. For example, four squares on a chess board make up a larger square.

We can extend this mathematical notion of tiling beyond the simple rectangular or triangular tilings found in buildings or board games. For example, we can create tilings of the plane $\mathbb{R}^2$, where each tile has a fractal boundary. Figure 1 is one such tiling of $\mathbb{R}^2$, the fractal cross.

In this paper, we generalize these fractal tilings to spaces other than Euclidean space, $\mathbb{R}^n$. Specifically, we generalize a theorem on $\mathbb{R}^n$ concerning self-similar tilings [2] to apply to any nilpotent Lie group. This theorem guarantees

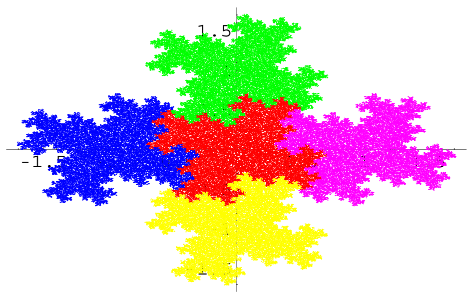


Figure 1: The fractal cross

that given an expansive automorphism that preserves some lattice, we can construct a self-similar tile. Once we have this result for all nilpotent Lie groups, we examine the Heisenberg group, a specific example. In the Heisenberg group, we can describe precisely the form of all automorphisms, and in particular we can describe all automorphisms with the property of being an expansive map. We also classify all lattices in the three-dimensional Heisenberg group, which allows us to construct specific examples of self-similar tilings on this group, which can be represented as plots in $\mathbb{R}^3$. Examples of such tilings were done earlier by Gelbrich [3] and Strichartz [6]. These tilings fulfill the three conditions listed above, and also have fractal boundary. We also present examples of another technique for "lifting" tilings of the plane into the Heisenberg group, which is not restricted to automorphisms of the Heisenberg group (and hence does not guarantee that the "lifted" sets will create a tiling in the Heisenberg group).

It is natural to generalize tilings to nilpotent Lie groups because they are a class of groups on which we can find a metric, an automorphic dilation structure, discrete cocompact subgroups, and a measure, all of which are needed for our notion of tiling. Possible further generalizations and further explorations of the characteristics of these tilings of nilpotent Lie groups will be discussed in a concluding section.

2 Nilpotent Lie Groups

Before presenting the extension of self-similar tiling to the class of nilpotent Lie groups, we will define this class and discuss briefly why it is natural to extend the notion of tiling to these groups.

Definition 2.1. Let G be a group, and let $A_0, A_1, A_2, \ldots$ be a sequence of groups with $A_0 = G$ and $A_{i+1} = [G, A_i]$ equals the group generated by $\left\{gag^{-1}a^{-1} : g \in G, a \in A_i\right\}$. G is *nilpotent* if for some n, A_n is trivial.

For example, the group of $n \times n$ upper triangular matrices with 1s on the diagonal is a nilpotent group. Any abelian group will also clearly be nilpotent, since if G is abelian then $[G, G] = A_1$ is trivial. We look at nilpotent Lie groups in particular because Lie groups are smooth manifolds, giving nice topological properties. In addition, in nilpotent Lie groups we have a group structure that generalizes translations and dilations, and we can find a metric to give us a notion of contraction and expansion maps.

We can characterize a nilpotent Lie group G as $G = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_r}$ with $x \in G$ written as $x = \{x_{ij} : 1 \le i \le r, 1 \le j \le n_i\}$, with the group law

$$(x * x')_{ij} = x_{ij} + x'_{ij} + F_{ij}(x, x') \tag{2.2}$$

where F_{ij} is some polynomial in $x_1, \ldots, x_{i-1}, x'_1, \ldots, x'_{i-1}$. We also require that these polynomials behave well with a notion of dilation. We define a dilation δ_t, $t \in \mathbb{R}$ as

$$(\delta_t x)_{ij} = t^i x_{ij} \tag{2.3}$$

and we require

$$F_{ij}(\delta_t x, \delta_t x') = t^i F_{ij}(x, x').$$

and then these dilations act as automorphisms on G. Not all such collections of polynomials will give a group, but all the groups we will be looking at can be written in this form [6].

As an example, consider the group of 4×4 upper triangular matrices with 1s on the diagonal. This is a group under matrix multiplication. We can think of any element M of this group as

$$M = \begin{bmatrix} 1 & x_{11} & x_{21} & x_{31} \\ 0 & 1 & x_{12} & x_{22} \\ 0 & 0 & 1 & x_{13} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The group law is then

$$\begin{bmatrix} 1 & x_{11} & x_{21} & x_{31} \\ 0 & 1 & x_{12} & x_{22} \\ 0 & 0 & 1 & x_{13} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x'_{11} & x'_{21} & x'_{31} \\ 0 & 1 & x'_{12} & x'_{22} \\ 0 & 0 & 1 & x'_{13} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & x_{11} + x'_{11} & x_{21} + x'_{21} + x_{11}x'_{12} & x_{31} + x'_{31} + x_{11}x'_{22} + x_{21}x'_{13} \\ 0 & 1 & x_{12} + x'_{12} & x_{22} + x'_{22} + x_{12}x'_{13} \\ 0 & 0 & 1 & x_{13} + x'_{13} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We see that this group can be thought of as $\mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}$ with a group law as given in (2.2). In this case we have the polynomials $F_{1j} = 0$ for any appropriate j, $F_{21} = x_{11}x'_{12}$, $F_{22} = x_{12}x'_{13}$, and $F_{31} = x_{11}x'_{22} + x_{21}x'_{13}$. It is easy to verify that these polynomials are compatible with dilations in the sense of (2.3).

A nilpotent Lie group G can be equipped with a right-invariant Riemannian metric d [6].

Definition 2.4. A function $\Phi\colon G \to G$ is a *contraction map* if there exists an $r \in \mathbb{R}, r < 1$, such that for $\alpha, \beta \in G$,

$$d\big(\Phi(\alpha), \Phi(\beta)\big) \le r \cdot d(\alpha, \beta).$$

Similarly, Φ is an *expansive map* if there exists an $r \in \mathbb{R}$, $r > 1$, such that for $\alpha, \beta \in G$,

$$d\big(\Phi(\alpha), \Phi(\beta)\big) \geq r \cdot d(\alpha, \beta).$$

Definition 2.5. We define a *norm* $|\cdot|$ on G by

$$|\alpha| = d(\mathbf{0}, \alpha)$$

where $\mathbf{0}$ is the identity element of G.

3 Self-Similar Tilings

We now present a way of generating self-similar periodic tilings on a nilpotent Lie group. A self-similar tiling is one composed of smaller tiles (rep tiles) of the same size, each being the same shape as the whole. We refer to a m-rep tile as an object that can be dissected into m smaller copies of itself. In [2], Bandt constructs tilings with self-similar tiles in $\mathbb{R}^n$ from matrices of integers. We begin by generalizing Bandt's results to nilpotent Lie groups and later present the Heisenberg group as a specific example.

3.1 Preliminaries

An important property of contraction mappings is how each mapping reduces "area." We summarize some results about measurable sets.

In this section G will denote a locally compact Hausdorff topological group. Let $C_c(G)$ denote the space of real-valued continuous functions on G with compact support. If S is a subset of G then let χ_S denote the characteristic function of S.

Suppose that μ is a measure on G. For a measurable set S,

$$\mu(S) = \int_G \chi_S \, \mathrm{d}\mu.$$

All $f \in C_c(G)$ can be approximated by linear combinations of characteristic functions of measurable sets, so μ gives us a way to define

$$\int_G f \, \mathrm{d}\mu.$$

Definition 3.1. If μ is a Borel measure on G, then μ is a *right Haar measure* on G if for all measurable sets $S \subset G$,

$$\mu(Sg) = \mu(S), \quad \forall g \in G.$$

Theorem 3.2 (Theorem 29C and 29D [5]). *G has a Haar measure that is unique up to a positive real multiple which is finite on compact sets.*

Lemma 3.3. *Let $\Phi : G \to G$ be a continuous automorphism of G and let μ be a Haar measure on G. Then there exists $\ell \in \mathbb{R}$, $\ell > 0$ such that*

$$\mu\big(\Phi(S)\big) = \ell \cdot \mu(S)$$

for all measurable sets $S \subset G$.

Proof. For all measurable subsets $S \subset G$, define a measure μ' on G by $\mu'(S) := \mu\big(\Phi(S)\big)$. From Lemma 3.2 μ' is a positive scalar multiple of μ so

$$\mu'(S) = \mu\big(\Phi(S)\big) = \ell \cdot \mu(S), \quad \ell \in \mathbb{R}.$$

□

Definition 3.4. For groups G, H with $H \subset G$ let G/H denote the set of all *left cosets* of H in G. A *left coset* is of the form

$$gH = \{gh : h \in H\}, \quad g \in G.$$

Suppose $\Gamma \subset G$ is a closed subgroup. Let μ_G and μ_Γ denote Haar measures on G and Γ, respectively.

Theorem 3.5 (Theorem 33C [5])**.** *There is a G-invariant measure $\mu_{G/\Gamma}$ such that for all $f \in C_c(G)$*

$$\int_G f \, d\mu_G = \int_{G/\Gamma} \left(\int_\Gamma f(x\gamma) \, d\mu_\Gamma \right) d\mu_{G/\Gamma}$$

where γ is our variable of integration in the inner integral and x is our variable of integration in the outer integral.

Definition 3.6. A closed subset C of G is called a *fundamental domain* for the right action of Γ on G if

$$G = \bigcup_{\gamma \in \Gamma} C\gamma$$

and for distinct $\gamma, \gamma' \in \Gamma$ we have $\gamma C \cap \gamma' C$ has measure zero.

Corollary 3.7. *If C is a fundamental domain of Γ in G, then*

$$\mu_G(C) = \mu_{G/\Gamma}(G/\Gamma)$$

Proof. Apply Theorem 3.5 with $f = \chi_C$. □

Definition 3.8. A *lattice* $\Gamma \subset G$ is a cocompact discrete subgroup of G.

3.2 Tilings

Let G be a nilpotent Lie group. Then G is a locally compact Hausdorff topological group, and has a right-invariant Riemannian metric. Let $\Gamma \subset G$ be a lattice and let Φ be a continuous expansive automorphism of G such that $\Phi(\Gamma) \subseteq \Gamma$. Let m equal the cardinality of $\Gamma/\Phi(\Gamma)$. Fix a right Haar measure μ on G.

Definition 3.9. A family $\{y_1, \ldots, y_m\} \subset \Gamma$ is a *residue system* of Φ, if $y_1 = \mathbf{0}$ and

$$\Gamma = \coprod \{y_i * \Phi(\Gamma) : i = 1, \ldots, m\}.$$

Definition 3.10 ([4])**.** Define f_i as $f_i(\alpha) = \Phi^{-1}(\alpha) * y_i, i = 1, \ldots, m$. A compact set $\mathbf{A} \neq \emptyset$ is *self-similar* with respect to $f_1, \ldots, f_m$ if

$$\mathbf{A} = f_1(\mathbf{A}) \cup \cdots \cup f_m(\mathbf{A}).$$

Since Φ^{-1} is a contraction mapping, so is each f_i, and with the same contraction constant. Let $\mathscr{C}$ denote the space of nonempty compact sets in G. Then $\mathscr{C}$ is a metric space equipped with the Hausdorff metric. Define $F \colon \mathscr{C} \to \mathscr{C}$ by

$$F(\mathbf{B}) = f_1(\mathbf{B}) \cup \cdots \cup f_m(\mathbf{B}).$$

Hutchinson [4] proved that F is a contraction on $\mathscr{C}$ and therefore has a unique fixed point:

Theorem 3.11 ([4])**.** *Given $f_1, \ldots, f_m$, there is a unique self-similar set $\mathbf{A}$, and for each compact $\mathbf{B}_0 \neq \emptyset$, the sequence $\mathbf{B}_k = F(\mathbf{B}_{k-1}), k = 1, 2, \ldots$ converges to $\mathbf{A}$ in $\mathscr{C}$.*

Proposition 3.12. *There exists a fundamental domain C for the right action of Γ on G.*

Proof. Let $C = \{g \in G : \forall \gamma \in \Gamma, d(g, \gamma) \geq d(g, \mathbf{0})\}$. □

Knowing this, we present the following theorem to show m-rep tiles can be generated from Φ.

Theorem 3.13. *If Φ is a continuous expansive automorphism of G and $\{y_1, \ldots, y_m\}$ is a residue system of Φ, then there is a unique m-rep tile $\mathbf{A}_1$ such that*

$$\Phi(\mathbf{A}_1) = \mathbf{A}_1 \cup \cdots \cup \mathbf{A}_m \textit{ with } \mathbf{A}_i = \mathbf{A}_1 * y_i.$$

Proof. For $i = 1, \ldots, m$ define f_i as in Definition 3.10. Define $F \colon \mathscr{C} \to \mathscr{C}$ as in Theorem 3.11. Given some compact set $\mathbf{B}_0 \subset G$ then the sequence $\mathbf{B}_k = F(\mathbf{B}_{k-1}), k = 1, 2, \ldots$ converges to a set $\mathbf{A}$ in $\mathscr{C}$ that does not depend on the choice of $\mathbf{B}_0$. For $i = 1, \ldots, m$, let $\mathbf{A}_i = f_i(\mathbf{A})$. Then

$$\mathbf{A} = \Phi(\mathbf{A}_1) \text{ and } \mathbf{A}_i = \mathbf{A}_1 * y_i.$$

We construct $\mathbf{A}$ starting with $\mathbf{B}_0 = \{\mathbf{0}\}$. Thus,

$$\begin{aligned}
\mathbf{B}_1 &= F(\mathbf{B}_0) = f_1(\mathbf{B}_0) \cup \cdots \cup f_m(\mathbf{B}_0) = \{y_1, \ldots, y_m\} \\
\mathbf{B}_k &= F(\mathbf{B}_{k-1}) = f_1(\mathbf{B}_{k-1}) \cup \cdots \cup f_m(\mathbf{B}_{k-1}) \\
&= \left\{\Phi^{-k+1}\left(y_{i_1}\right) * \Phi^{-k+2}\left(y_{i_2}\right) * \cdots * \Phi^{-1}\left(y_{i_{k-1}}\right) * y_{i_k} : i_1, \ldots, i_k \in \{1, \ldots, m\}\right\}.
\end{aligned}$$

Since $\mathbf{B}_0 \subseteq \mathbf{B}_1 \subseteq \cdots$, this sequence is increasing and therefore $\mathbf{A} = \mathrm{cl}(\cup \mathbf{B}_k)$. Since $\mathbf{A}$ is compact, $\mathbf{A}$ is bounded; in other words $\mathbf{A} \subseteq \mathbf{U}_c := \left\{y : |y| \leq c\right\}$ for some $c \in \mathbb{R}$. We must show that $\mathbf{A}$ has positive measure.

Claim 1. There are finitely many points, $x_1, \ldots, x_q \in \Gamma$ such that

$$\mathbf{U}_c \subseteq \mathbf{A} * \{\Phi(x_1), \ldots, \Phi(x_q)\}.$$

Proof. Pick $\delta > 0$ such that each $u \in G$ has distance $\leq \delta$ from Γ. For any $\varepsilon > 0$, we can find k such that $\delta \cdot r^{k-1} < \varepsilon$, since $r < 1$. This means, by the properties of the contraction map, that any u has a distance $\leq \varepsilon$ from the lattice $\Phi^{-k+1}(\Gamma)$. Since the y_i form a residue system, we know that each $z \in \Gamma$ can be written as $z = y_{i_1} * \Phi(x)$ for some $x \in \Gamma$ and some $i_1 \in \{1, \ldots, m\}$. Similarly, $x = y_{i_2} * \Phi(x')$ for some $x' \in \Gamma$ and $i_2 \in \{1, \ldots, m\}$. Therefore, we have $z = y_{i_1} * \Phi(y_{i_2}) * \Phi^2(x')$. Iterating this process we have that for any k,

$$z = y_{i_1} * \Phi\left(y_{i_2}\right) * \Phi^2\left(y_{i_3}\right) * \cdots * \Phi^{k-1}\left(y_{i_k}\right) * \Phi^k\left(x\right).$$

Since Φ is an automorphism, we may apply Φ^{-k+1} to both sides of the equality to yield

$$\Phi^{-k+1}(z) = \Phi^{-k+1}(y_{i_1}) * \Phi^{-k+2}\left(y_{i_2}\right) * \Phi^{-k+3}\left(y_{i_3}\right) * \cdots * y_{i_k} * \Phi\left(x\right)$$

where $\Phi^{-k+1}(y_{i_1}) * \Phi^{-k+2}\left(y_{i_2}\right) * \Phi^{-k+3}\left(y_{i_3}\right) * \cdots * y_{i_k} \in \mathbf{B}_k$ for some k. This implies that

$$\Phi^{-k+1}(\Gamma) = \mathbf{B}_k * \bigcup \{\Phi(x) : x \in \Gamma\}.$$

Intersecting both sides of this equation with the compact set $\mathbf{U}_c$, we obtain a finite set, so

$$\Phi^{-k+1}\left(\Gamma\right) \cap \mathbf{U}_c \subseteq \mathbf{B}_k * \left\{\Phi\left(x_1\right), \ldots, \Phi\left(x_q\right)\right\},$$

where $x_1, \ldots, x_q \in \Gamma$ are the points for which $|\Phi(x)| \leq c$. Now taking the closure of the union over $k = 1, 2, \ldots$ gives

$$\mathbf{U}_c \subseteq \mathbf{A} * \{\Phi(x_1), \ldots, \Phi(x_q)\}.$$

This follows from the fact that

$$\mathbf{A} = \mathrm{cl}\left(\bigcup_i \mathbf{B}_i\right) \quad i = 1, 2, \ldots$$

and that for every $u \in \mathbf{U}_c$, there is some $x \in \Gamma$ such that $d\big(\Phi^{-k+1}(x), u\big) \leq \varepsilon$. Therefore, the closure of

$$\bigcup_i \big(\Phi^{-k+1}(\Gamma) \cap \mathbf{U}_c\big)$$

is the set $\mathbf{U}_c$. This proves Claim 1.

We now verify that $\mathrm{int}(\mathbf{A}) \neq \emptyset$. Assume for a contradiction that $\mathrm{int}(\mathbf{A}) = \emptyset$. We know that $\mathrm{int}(\mathbf{U}_c) \neq \emptyset$ since G is a Lie group. From Claim 1 we have that $\mathbf{U}_c \subseteq \mathbf{A} * \{\Phi(x_1), \ldots, \Phi(x_q)\}$ which implies that $\mathbf{U}_c$ is a subset of a finite union of sets with empty interior. The Baire Category Theorem implies $\mathrm{int}(\mathbf{U}_c) = \emptyset$ which is a contradiction. Therefore, $\mathrm{int}(\mathbf{A}) \neq \emptyset$.

We want to show our contraction map Φ^{-1} reduces measure by a factor of m, the number of coset representatives of $\Gamma/\Phi(\Gamma)$.

Claim 2. Let $\Gamma' = \Phi^{-1}\Gamma$. The expansion factor ℓ of Φ is equal to the index $[\Gamma' : \Gamma]$.

Proof. From Proposition 3.12, we know there exists a fundamental domain C for the right action of Γ on G. Then $\Phi^{-1}(C)$ is a fundamental domain for Γ'. Let S be a set of left coset representatives for Γ in Γ'. Then

$$C_1 := \bigcup_{s \in S} s\Phi^{-1}(C)$$

is another fundamental domain for Γ'. The union is disjoint up to measure 0. Now,

$$\mu(C_1) = \sum_{s \in S} \mu\big(\Phi^{-1}(C) * s\big)$$

and since μ is right invariant, we have

$$\begin{aligned} \mu(C_1) &= \sum_{s \in S} \mu\big(\Phi^{-1}(C)\big) \\ &= [\Gamma' : \Gamma]\mu\big(\Phi^{-1}(C)\big) \\ &= [\Gamma' : \Gamma]\ell^{-1}\mu(C). \end{aligned}$$

By Corollary 3.7 we know

$$\mu(C) = \mu(C_1).$$

Therefore, $[\Gamma' : \Gamma]\ell^{-1} = 1$ which implies $[\Gamma' : \Gamma] = \ell$. This proves Claim 2.

Thus, Φ^{-1} reduces measure by a factor of m. Therefore, $\mu(\mathbf{A}_i) = \mu(\mathbf{A})/m$ for $\mathbf{A}_i = f_i(\mathbf{A})$. We know

$$\mathbf{A} = F(\mathbf{A}) = f_1(\mathbf{A}) \cup \cdots \cup f_m(\mathbf{A}) = \mathbf{A}_1 \cup \cdots \cup \mathbf{A}_m.$$

Pick distinct i and j. Then

$$\mu(\mathbf{A}) \leq \sum_{k=1}^{m} \mu(\mathbf{A}_k) - \mu(\mathbf{A}_i \cap \mathbf{A}_j)$$

so,

$$\mu\left(\mathbf{A}\right) + \mu\left(\mathbf{A}_i \cap \mathbf{A}_j\right) \leq \sum_{k=1}^{m} \mu\left(\mathbf{A}_k\right) = m \cdot \frac{\mu(\mathbf{A})}{m} = \mu(\mathbf{A}).$$

This means $\mu(\mathbf{A}_i \cap \mathbf{A}_j) = 0$ which implies $\mathrm{int}(\mathbf{A}_i) \cap \mathrm{int}(\mathbf{A}_j) = \emptyset$ for a particular $i \neq j$. However, i and j are arbitrary, so this is true for all distinct combinations of $i, j, i \neq j$. Since $\Phi(\mathbf{A}_1) = \mathbf{A}$ and the sets $\mathbf{A}_2, \ldots, \mathbf{A}_m$ are congruent to $\mathbf{A}_1$ with

$$\Phi(\mathbf{A}_1) = \mathbf{A}_1 \cup \cdots \cup \mathbf{A}_m,$$

then $\mathbf{A}_1$ is a m-rep tile. □

Definition 3.14. A finite group of $\mathbf{S}$ of isometries of G is a *symmetry group* of Φ if $\Phi\mathbf{S} = \mathbf{S}\Phi$.

With Definition 3.14. we may now produce a more general extension of Theorem 3.13 to include symmetry groups.

Theorem 3.15. *Let Φ be a continuous expansive automorphism on G, let $\{y_1, \ldots, y_m\}$ be a residue system of Φ, let $s_1, \ldots, s_m$ be contained in a symmetry group $\mathbf{S}$ of Φ, and suppose*

$$\Gamma = \coprod \left\{ s_i^{-1}\big(y_i * \Phi(\Gamma)\big) : i = 1, \ldots, m \right\}.$$

For $i = 1, \ldots, m$, define $f_i \colon G \to G$ by $f_i(\alpha) = s_i\big(\Phi^{-1}(\alpha) * y_i\big)$. Then there exists a set $\mathbf{A}$, self-similar with respect to f_i that has a nonempty interior.

Proof. With a few changes to Theorem 3.13 we will be able to verify Theorem 3.15. Since Φ^{-1} is a contraction mapping, so is each f_i, and with the same contraction constant. Define $F \colon \mathscr{C} \to \mathscr{C}$ as in Theorem 3.11.

We know that since $\mathbf{S}$ is a group, and $\Phi^{-1}s\Phi \in \mathbf{S}$ for $s \in \mathbf{S}$, it follows from $s_{1_i}, \ldots, s_{i_k} \in \mathbf{S}$ that $s_{i_k}\Phi^{-1}s_{i_{k-1}}\Phi^{-1} \cdot \cdots \cdot s_{i_1}\Phi^{-1} \cdot \Phi^k = t \in \mathbf{S}$. As before, we construct the self-similar set $\mathbf{A}$ starting with $\mathbf{B}_0 = \{\mathbf{0}\}$. Thus,

$$\begin{aligned}
\mathbf{B}_1 &= F(\mathbf{B}_0) = f_1(\mathbf{B}_0) \cup \cdots \cup f_m(\mathbf{B}_0) = \{y_1, \ldots, y_m\} \\
\mathbf{B}_k &= F(\mathbf{B}_{k-1}) = f_1(\mathbf{B}_{k-1}) \cup \cdots \cup f_m(\mathbf{B}_{k-1}) \\
&= \big\{ s_{i_k}\Phi^{-1} \cdot \cdots \cdot s_{i_3}\Phi^{-1}s_{i_2}\Phi^{-1}(y_{i_1}) * \cdots * s_{i_k}\Phi^{-1}s_{i_{k-1}}\Phi^{-1}(y_{i_{k-2}}) \\
&\qquad * s_{i_k}\Phi^{-1}(y_{i_{k-1}}) * y_{i_k} : i_1, \ldots, i_k \in \{1, \ldots, m\} \big\}.
\end{aligned}$$

Since $\mathbf{B}_0 \subseteq \mathbf{B}_1 \subseteq \cdots$, this sequence is increasing and therefore $\mathbf{A} := \mathrm{cl}(\cup \mathbf{B}_k)$ and $\mathbf{A}$ is compact and $\mathbf{A}$ is bounded; in other words $\mathbf{A} \subseteq \mathbf{U}_c := \big\{ y : |y| \leq c \big\}$ for some $c \in \mathbb{R}$. It remains to prove the following claim:

Claim 3. There are finitely many points, $x_1, \ldots, x_q \in \Gamma$ such that

$$\mathbf{U}_c \subseteq s\left(\mathbf{A} * \big\{\Phi(x_1), \ldots, \Phi(x_q)\big\}\right), s \in S.$$

Proof. Since the y_i form a residue system, we know that each $z \in \Gamma$ can be written as $z = s_{i_1}^{-1}\big(y_{i_1} * \Phi(x)\big) = s_{i_1}^{-1}(y_{i_1}) * s_{i_1}^{-1}\Phi(x)$ for some $x \in \Gamma$. Similarly, $x = s_{i_2}^{-1}\big(y_{i_2} * \Phi(x')\big) = s_{i_2}^{-1}(y_{i_2}) * s_{i_2}^{-1}\Phi(x')$ for some $x' \in \Gamma$. Therefore, we have

$z = s_{i_1}^{-1}(y_{i_1}) * s_{i_1}^{-1}\Phi s_{i_2}^{-1}(y_{i_2}) * s_{i_1}^{-1}\Phi s_{i_2}^{-1}\Phi(x')$. Iterating this process we have that for any k,

$$\begin{aligned} z = s_{i_1}^{-1}(y_{i_1}) * \cdots * s_{i_1}^{-1}\Phi \bullet \cdots \bullet s_{i_{k-1}}^{-1}(y_{i_{k-1}}) \\ * s_{i_1}^{-1}\Phi \bullet \cdots \bullet s_{i_k}^{-1}(y_{i_k}) * s_{i_1}^{-1}\Phi s_{i_2}^{-1}\Phi \bullet \cdots \bullet s_{i_k}^{-1}\Phi(x) \end{aligned}$$

for arbitrary $x \in \Gamma$. Since Φ is an automorphism we may apply Φ^{-k+1} to both sides to get

$$\begin{aligned} \Phi^{-k+1}(z) = \Phi^{-k+1}s_{i_1}^{-1}(y_{i_1}) * \cdots * \Phi^{-k+1}s_{i_1}^{-1}\Phi \bullet \cdots \bullet s_{i_{k-1}}^{-1}(y_{i_{k-1}}) \\ * \Phi^{-k+1}s_{i_1}^{-1}\Phi \bullet \cdots \bullet s_{i_k}^{-1}(y_{i_k}) * \Phi^{-k+1}s_{i_1}^{-1}\Phi s_{i_2}^{-1}\Phi \bullet \cdots \bullet s_{i_k}^{-1}\Phi(x). \end{aligned}$$

For t from above we have

$$t^{-1} = \Phi^{-k+1}s_{i_1}^{-1}\Phi s_{i_2}^{-1}\Phi \bullet \cdots \bullet \Phi s_{i_k}^{-1}$$

which implies that

$$\begin{aligned} \Phi^{-k+1}(z) &= t^{-1}s_{i_k}\Phi^{-1} \bullet \cdots \bullet s_{i_3}\Phi^{-1}s_{i_2}\Phi^{-1}(y_{i_1}) * \cdots * t^{-1}s_{i_k}\Phi^{-1}(y_{i_{k-1}}) * t^{-1}y_{i_k} * t^{-1}\Phi(x) \\ &= t^{-1}(s_{i_k}\Phi^{-1} \bullet \cdots \bullet s_{i_3}\Phi^{-1}s_{i_2}\Phi^{-1}(y_{i_1}) * \cdots * s_{i_k}\Phi^{-1}(y_{i_{k-1}}) * y_{i_k}) * t^{-1}\Phi(x) \\ &= t^{-1}(b) * t^{-1}\Phi(x), \text{ with } b \in \mathbf{B}_k \\ &= t^{-1}\left(b * \Phi(x)\right). \end{aligned}$$

Thus

$$\Phi^{-k+1}(\Gamma) \subseteq \bigcup \left\{ t^{-1}\big(\mathbf{B}_k * \Phi(x)\big) : t \in \mathbf{S}, x \in \Gamma \right\}.$$

Now, restricting the lefthand side of the above expression to points contained in $\mathbf{U}_c$, $\mathbf{U}_c$ can only contain a finite number of points in Γ, so

$$\Phi^{-k+1}(\Gamma) \cap \mathbf{U}_c \subseteq \bigcup \left\{ t^{-1}\left(\mathbf{B}_k * \left\{\Phi(x_1), \ldots, \Phi(x_q)\right\}\right) : t \in \mathbf{S} \right\},$$

where $x_1, \ldots, x_q \in \Gamma$ are the points for which $|\Phi(x)| \leq c$. Now taking the closure of the union over $k = 1, 2, \ldots$ gives

$$\mathbf{U}_c \subseteq s\left(\mathbf{A} * \{\Phi(x_1), \ldots, \Phi(x_q)\}\right).$$

From Claim 3 we know that there are finitely many copies of $s\big(\mathbf{A} * \Phi(x)\big)$ of $\mathbf{A}$ which cover $\mathbf{U}_c$. The proof that $\mathbf{A}$ has nonempty interior follows as in Theorem 3.13 $\square$

4 Heisenberg Group

4.1 Preliminaries

A nontrivial example of a stratified nilpotent Lie group is the Heisenberg group. In order to create specific examples of tilings on the Heisenberg group, we first identify a general form for all automorphisms. Then we classify automorphisms that have the properties of being contraction maps, expansion maps, or isometries. In what follows, we consider the group $H = H^{2n+1}(\mathbb{R}) = \big\{(\mathbf{x}, z) : \mathbf{x} \in \mathbb{R}^{2n}, z \in \mathbb{R}\big\}$ with the group law

$$(\mathbf{x}, z) * (\mathbf{x}', z') = (\mathbf{x} + \mathbf{x}', z + z' + B(\mathbf{x}, \mathbf{x}'))$$

where B is a nondegenerate skew-symmetric bilinear form. We use the norm $\big|(\mathbf{x}, z)\big|_H = \Big(\|\mathbf{x}\|^4 + |z|^2\Big)^{1/4}$, where $||\cdot||$ is the standard Euclidean norm on $\mathbb{R}^{2n}$ which gives us the right-invariant metric d_H defined by $d_H(\alpha, \beta) = \big|\alpha * \beta^{-1}\big|_H$ for any $\alpha, \beta \in H^{2n+1}(\mathbb{R})$.

4.2 Automorphisms

4.2.1 Properties of Automorphisms on $H^{2n+1}(\mathbb{R})$

Theorem 4.1. *Any automorphism $\Phi\colon H \to H$ is of the form $\Phi\big((\mathbf{x}, z)\big) = (M\mathbf{x}, \omega(\mathbf{x}) + az)$, where $M \in \mathrm{GSp}(2n)$ such that $B(M\mathbf{v}, M\mathbf{w}) = aB(\mathbf{v}, \mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^{2n}$, and where $\omega\colon \mathbb{R}^{2n} \to \mathbb{R}$ is a linear tranformation.*

Proof. We know, by properties of automorphism, that Φ maps the center, Z, into itself. In the case of the Heisenberg group, the center can be defined as $Z = \{(\mathbf{0}, z) : z \in \mathbb{R}\}$. Then we can think of an automorphism on Z as represented by simply an automorphism on $\mathbb{R}$, say $f(\mathbf{x}) = a\mathbf{x}$ where $a \in \mathbb{R}, a \neq 0$. Hence, we know that $\Phi\big((\mathbf{0}, z)\big) = (\mathbf{0}, az)$ for some a. Similarly, Φ induces an automorphism g on H/Z, where $g\big(Z * (\mathbf{x}, z)\big) = Z * \big(\Phi(\mathbf{x}, z)\big)$ for any $(\mathbf{x}, z) \in H$. Since any coset $Z * (\mathbf{x}, z)$ can be thought of as $\big\{(\mathbf{x}, r) : r \in \mathbb{R}\big\} \subset H$, we can relate any automorphism on H/Z to an automorphism on $\mathbb{R}^{2n}$, defined by an invertible $2n \times 2n$ matrix M. Then if $\big[(\mathbf{x}, z)\big]$ is the equivalence class of $(\mathbf{x}, z)$ mod Z, $g\Big(\big[(\mathbf{x}, z)\big]\Big) = \big[(M\mathbf{x}, z)\big] = \Big[\Phi\big((\mathbf{x}, z)\big)\Big]$. Then we know two things about our automorphism:

$$\begin{aligned}
\Phi\big((\mathbf{0}, z)\big) &= (\mathbf{0}, az)\\
\Phi\big((\mathbf{x}, z)\big) &= \big(M\mathbf{x}, h(\mathbf{x}, z)\big)
\end{aligned}$$

where $h(\mathbf{x}, z)$ is some real number. Combining these two facts, and the fact that Φ is an automorphism, we can focus on a function depending only on $\mathbf{x}$, since

$$\begin{aligned}
\Phi\big((\mathbf{x}, z)\big) &= \Phi\big((\mathbf{x}, 0) * (\mathbf{0}, z)\big)\\
&= \Phi\big((\mathbf{x}, 0)\big) * \Phi(\mathbf{0}, z)\\
&= \big(M\mathbf{x}, h(\mathbf{x}, 0)\big) * (\mathbf{0}, az)\\
&= (M\mathbf{x}, h(\mathbf{x}, 0) + az)\\
&= \big(M\mathbf{x}, h(\mathbf{x}, z)\big).
\end{aligned}$$

Hence we know that $h(\mathbf{x}, z) = h(\mathbf{x}, 0) + az$, so we restrict our attention to the function $\omega(\mathbf{x})$, where $h(\mathbf{x}, z) = \omega(\mathbf{x}) + az$. We now have a formula for

$\Phi\big((\mathbf{x},z)\big)$, but we need to know more about the function ω and the relationship between a and M. We first look at what conditions must hold for Φ to be an automorphism.

Using the given group law and our formula for the automorphism, we have on one hand

$$\begin{aligned}\Phi\big((\mathbf{x},z)*(\mathbf{x}',z')\big) &= \Phi\big((\mathbf{x}+\mathbf{x}', z+z'+B(\mathbf{x},\mathbf{x}')\big)\\ &= \Big(M(\mathbf{x}+\mathbf{x}'), \omega(\mathbf{x}+\mathbf{x}') + a\big(z+z'+B(\mathbf{x},\mathbf{x}')\big)\Big)\\ &= \big(M\mathbf{x}{+}M\mathbf{x}', \omega(\mathbf{x}+\mathbf{x}') + az + az' + aB(\mathbf{x},\mathbf{x}')\big)\end{aligned}$$

and on the other hand,

$$\begin{aligned}\Phi\big((\mathbf{x},z)\big)*\Phi\big((\mathbf{x}',z')\big) &= \big(M\mathbf{x}, \omega(\mathbf{x})+az\big)*\big(M\mathbf{x}', \omega(\mathbf{x}')+az'\big)\\ &= \big(M\mathbf{x}{+}M\mathbf{x}', \omega(\mathbf{x})+az+\omega(\mathbf{x}')+az'+B(M\mathbf{x},M\mathbf{x}')\big).\end{aligned}$$

Since these must be equal in order for us to have an automorphism, it's clear that we must have the relation

$$\omega(\mathbf{x}+\mathbf{x}') + aB(\mathbf{x},\mathbf{x}') = \omega(\mathbf{x}) + \omega(\mathbf{x}') + B(M\mathbf{x},M\mathbf{x}');$$

or rearranging,

$$\omega(\mathbf{x}+\mathbf{x}') = \omega(\mathbf{x}) + \omega(\mathbf{x}') + B(M\mathbf{x},M\mathbf{x}') - aB(\mathbf{x},\mathbf{x}'). \tag{4.2}$$

We can now begin to build an understanding of the function ω. First, we show that ω preserves multiplication by a real scalar: $\omega(r\mathbf{x}) = r\omega(\mathbf{x})$ for $r \in \mathbb{R}$. Notice that since B is a skew-symmetric bilinear form, we have the property

$$B(\mathbf{v}, c\mathbf{v}) = cB(\mathbf{v},\mathbf{v}) = 0$$

for any scalar, $c \in \mathbb{R}$ since by definition of skew-symmetric, $B(\mathbf{v},\mathbf{v}) = -B(\mathbf{v},\mathbf{v})$ for any vector. Hence $B(\mathbf{v},\mathbf{v}) = 0$ for all $\mathbf{v} \in \mathbb{R}^{2n}$. A simple proof by induction shows that $\omega(n\mathbf{x}) = n\omega(\mathbf{x})$ for any integer n. For the base case, clearly $\omega(1\mathbf{x}) = 1\omega(\mathbf{x})$. Now, assume $\omega(n\mathbf{x}) = n\omega(\mathbf{x})$ is true for n. Then we have, by (4.2),

$$\begin{aligned}\omega\big((n+1)\mathbf{x}\big) &= \omega(n\mathbf{x}+\mathbf{x})\\ &= \omega(n\mathbf{x}) + \omega(\mathbf{x}) + B\big(M(n\mathbf{x}), M\mathbf{x}\big) - aB(n\mathbf{x},\mathbf{x})\\ &= (n+1)\omega(\mathbf{x}).\end{aligned}$$

Hence by induction, we have for all positive integers, $\omega(n\mathbf{x}) = n\omega(\mathbf{x})$.

Now consider $\omega(\mathbf{x}) = \omega\big(n(\mathbf{x}/n)\big) = n\omega(\mathbf{x}/n)$. Dividing the left and right by n gives us that $(1/n)\omega(\mathbf{x}) = \omega(\mathbf{x}/n)$, for any positive integer n. Now we see that $\omega\big((m/n)\mathbf{x}\big) = \omega\big(m(\mathbf{x}/n)\big) = m\omega(\mathbf{x}/n) = (m/n)\omega(\mathbf{x})$. Finally, note that we already know that $\omega(\mathbf{0}) = 0$, since we know $\Phi\big((\mathbf{0},z)\big) = (\mathbf{0}, az)$. This gives us that $\omega(\mathbf{0}) = \omega(\mathbf{x}-\mathbf{x}) = \omega(\mathbf{x}) + \omega(-\mathbf{x}) = 0$, hence $\omega(-\mathbf{x}) = -\omega(\mathbf{x})$. So, for any rational number q, $\omega(q\mathbf{x}) = q\omega(\mathbf{x})$. Since we are only interested in continuous automorphisms, this suffices to show that

$$\omega(r\mathbf{x}) = r\omega(\mathbf{x})$$

for any $r \in \mathbb{R}$.

We can rearrange (4.2) as follows

$$\omega(\mathbf{x} + \mathbf{x}') - \big(\omega(\mathbf{x}) + \omega(\mathbf{x}')\big) = B(M\mathbf{x}, M\mathbf{x}') - aB(\mathbf{x}, \mathbf{x}'). \tag{4.3}$$

Fix $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^{2n}$. (4.3) must hold for $r\mathbf{x}, r\mathbf{x}'$, where r is an arbitrary real number. That means that

$$\begin{aligned} \omega(r\mathbf{x} + r\mathbf{x}') - \big(\omega(r\mathbf{x}) + \omega(r\mathbf{x}')\big) &= B\big(M(r\mathbf{x}), M(r\mathbf{x}')\big) - aB(r\mathbf{x}, r\mathbf{x}') \\ r\Big(\omega(\mathbf{x} + \mathbf{x}') - \big(\omega(\mathbf{x}) + \omega(\mathbf{x}')\big)\Big) &= r^2\big(B(M\mathbf{x}, M\mathbf{x}') - aB(\mathbf{x}, \mathbf{x}')\big). \end{aligned}$$

Since this must be true for any r, we have

$$\omega(\mathbf{x} + \mathbf{x}') - \big(\omega(\mathbf{x}) + \omega(\mathbf{x}')\big) = B(M\mathbf{x}, M\mathbf{x}') - aB(\mathbf{x}, \mathbf{x}') = 0,$$

for any arbitrary $\mathbf{x}, \mathbf{x}'$. Hence we now know two important things:

$$\begin{aligned} \omega(\mathbf{x} + \mathbf{x}') &= \omega(\mathbf{x}) + \omega(\mathbf{x}') \\ B(M\mathbf{x}, M\mathbf{x}') &= aB(\mathbf{x}, \mathbf{x}'). \end{aligned}$$

This shows us that ω is a linear transformation, and shows the necessary relationship between M and a. $\square$

Now that we have a general form for automorphisms on the Heisenberg group, we can begin to classify characteristics of M and ω that give certain desired properties to the automorphism.

Corollary 4.4. *For $M \in \mathrm{GSp}(2n)$ we have the relationship $a^n = \det M$ where M uniquely determines a.*

Proof. To see this, let the matrix $A = M\left(b^{-1}I_{2n}\right)$ for some $b \in \mathbb{C}$ such that $b^2 = a$. Note that for any $\mathbf{v} \in \mathbb{R}^{2n}$, $A\left(b\mathbf{v}\right) = M\left(b^{-1}I_{2n}\right)\left(b\mathbf{v}\right) = M\mathbf{v}$. Hence we have

$$\begin{aligned} B\left(M\mathbf{v}, M\mathbf{w}\right) &= B\big(A\left(b\mathbf{v}\right), A\left(b\mathbf{w}\right)\big) \\ &= aB\left(A\mathbf{v}, A\mathbf{w}\right). \end{aligned}$$

We already know from the properties of the automorphism that $B\left(M\mathbf{v}, M\mathbf{w}\right) = aB\left(\mathbf{v}, \mathbf{w}\right)$, hence $B\left(A\mathbf{v}, A\mathbf{w}\right) = B\left(\mathbf{v}, \mathbf{w}\right)$. This means that $A \in \mathrm{Sp}\left(2n\right)$, and it's known that all symplectic matrices have determinant 1. Hence $\det A = 1 = \left(\det M\right)\left(\det(b^{-1}I_{2n})\right) = \left(\det M\right)\left(b^{-2n}\right) = a^{-n}\det M$. Therefore, $\det M = a^n$. $\square$

4.2.2 Expansions and Contractions

Given an automorphism $\Phi\big((\mathbf{x}, z)\big) = (M\mathbf{x}, \omega(\mathbf{x}) + az)$, the inverse will be

$$\Phi^{-1}\big((\mathbf{x}, z)\big) = \Big(M^{-1}\mathbf{x}, a^{-1}\big(z - \omega(M^{-1}\mathbf{x})\big)\Big).$$

In constructing tilings, it is important to understand how an automorphism acts on the distance between two points. Given the metric we are using on the Heisenberg group, which defines the norm, we can focus on how a transformation acts on the norm. We can now find restrictions on M and ω that make Φ a contraction or expansion.

Proposition 4.5. *An automorphism $\Phi\colon H \to H$ is a contraction map if and only if $\Phi\big((\mathbf{x}, z)\big) = (M\mathbf{x}, az)$ where for some $0 < c < 1$, $\|M\mathbf{x}\| \leq c\|\mathbf{x}\|$ and $|a| \leq c^2$.*

Proof. Given $\Phi\big((\mathbf{x}, z)\big) = (M\mathbf{x}, \omega(\mathbf{x}) + az)$, we want

$$\Big(\|M\mathbf{x}\|^4 + |\omega(\mathbf{x}) + az|^2\Big)^{1/4} \leq c\big(\|\mathbf{x}\|^4 + |z|^2\big)^{1/4} \tag{4.6}$$

for some $0 < c < 1$, for all $(\mathbf{x}, z) \in H$.

First we show $\omega(\mathbf{x}) = 0$ for all $\mathbf{x}$. Assume there exists some non-zero $\mathbf{x}$ for which $\omega(\mathbf{x}) \neq 0$. Then fixing this $\mathbf{x}$, (4.6) must hold for $(\mathbf{x}, z)$ replaced by $(r\mathbf{x}, 0)$ where r is any non-zero real number. Hence we have

$$\begin{gathered}\Big(\big\|M(r\mathbf{x})\big\|^4 + \big|\omega(r\mathbf{x})\big|^2\Big)^{1/4} \leq c\big(\|r\mathbf{x}\|^4\big)^{1/4} \\ r^4\|M\mathbf{x}\|^4 + r^2\big|\omega(\mathbf{x})\big|^2 \leq r^4 c^4\|\mathbf{x}\|^4 \\ \|M\mathbf{x}\|^4 + r^{-2}\big|\omega(\mathbf{x})\big|^2 \leq c^4\|\mathbf{x}\|^4.\end{gathered}$$

Since $\omega(\mathbf{x}) \neq 0$, it's possible to choose r such that $r^{-2}\big|\omega(\mathbf{x})\big|^2 > \|\mathbf{x}\|^4$, which clearly means this inequality does not hold. Hence, we have a contradiction, so we must have $\omega(\mathbf{x}) = 0$ for any $\mathbf{x}$. Then

$$\|M\mathbf{x}\|^4 \leq c^4\|\mathbf{x}\|^4,$$

so

$$\|M\mathbf{x}\| \leq c\|\mathbf{x}\|$$

for all $\mathbf{x} \in \mathbb{R}^{2n}$. Finally, (4.6) must also hold for $(\mathbf{0}, z)$ for any z. Then substituting in again gives

$$\begin{gathered}|az|^{1/2} \leq c|z|^{1/2} \\ |a| \leq c^2.\end{gathered}$$

□

Proposition 4.7. *Let $\Phi(\mathbf{x}, z) = (M\mathbf{x}, az)$ be a contraction automorphism on H. Let M be a $2n \times 2n$ matrix with eigenvalues λ_i corresponding to a basis of eigenvectors $\mathbf{v}_i$, $i = 1, \ldots, 2n$. Then the contraction constant of Φ is the maximum of the $|\lambda_i|$.*

Proof. M has eigenvalues $\lambda_1, \ldots, \lambda_{2n}$. From Corollary 4.4 we know $a^n = \det M$, so if λ_k is the max of the eigenvalues, then $a = \left(\lambda_1 \cdot \cdots \cdot \lambda_{2n}\right)^{1/n} \leq \lambda_k^2$. Now let $\mathbf{v}_1, \ldots, \mathbf{v}_{2n}$ be a basis of eigenvectors. Let $(\mathbf{x}, z) \in H$. Then

$$\begin{aligned}
|\Phi(\mathbf{x}, z)| &= |(M\mathbf{x}, az)| \\
&= \left(\|M\mathbf{x}\|^4 + |az|^2\right)^{1/4} \\
&= \left(\|M(x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n)\|^4 + |az|^2\right)^{1/4} \\
&= \left(\|\lambda_1 x_1\mathbf{v}_1 + \cdots + \lambda_n x_n\mathbf{v}_n\|^4 + |az|^2\right)^{1/4} \\
&\leq \left(\|\lambda_k x_1\mathbf{v}_1 + \cdots + \lambda_k x_n\mathbf{v}_n\|^4 + |\lambda_k^2 z|^2\right)^{1/4} \\
&\leq |\lambda_k| \left(\|x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n\|^4 + |z|^2\right)^{1/4} \\
&= |\lambda_k| \left(\|\mathbf{x}\|^4 + |z|^2\right)^{1/4} \\
&= |\lambda_k| \left|(\mathbf{x}, z)\right| .
\end{aligned}$$

□

Remark. In a contraction map $\Phi\big((\mathbf{x}, z)\big) = (M\mathbf{x}, az)$, the condition that $|a| \leq c^2$ follows directly from the condition that $\|M\mathbf{x}\| \leq c\|\mathbf{x}\|$ and the relationship $a = (\det M)^{1/n}$. From Proposition 4.7 we know that c is the maximum eigenvalue of M, and we know that $\det M = a^n \leq c^{2n}$. So $|a| \leq c^2$.

Now that we have the conditions necessary for a contraction map, it follows easily to find the conditions necessary for an automorphism to have a contraction map as its inverse.

Proposition 4.8. *An automorphism $\Phi\colon H \to H$ is an expansion map if and only if $\Phi\big((\mathbf{x}, z)\big) = (M\mathbf{x}, az)$ where, for some $0 < c < 1$, $\left\|M^{-1}\mathbf{x}\right\| \leq c\left\|\mathbf{x}\right\|$ and $\left|a^{-1}\right| \leq c^2$.*

Proof. Given an automorphism $\Phi\big((\mathbf{x}, z)\big) = (M\mathbf{x}, \omega\left(\mathbf{x}\right) + az)$, we know that $\Phi^{-1}\big((\mathbf{x}, z)\big) = \Big(M^{-1}\mathbf{x}, a^{-1}\big(z - \omega(M^{-1}\mathbf{x})\big)\Big)$. Since we know the conditions necessary for a contraction map, we know that for Φ^{-1} to be contracting, for some $0 < c < 1$, $\left\|M^{-1}\mathbf{x}\right\| \leq c\left\|\mathbf{x}\right\|$, $\left|a^{-1}\right| \leq c^2$, and $-a^{-1}\omega\left(M^{-1}\mathbf{x}\right) = 0$ for all $\mathbf{x} \in \mathbb{R}^{2n}$. This last condition means that $\omega\left(\mathbf{x}\right) = 0$ for all $\mathbf{x} \in \mathbb{R}^{2n}$. □

Finally, by a parallel proof to the proof for contraction maps, except with the inequalities replaced by equalities, we can find a form for automorphisms that expand or contract the norm by a constant factor.

Proposition 4.9. *An automorphism* $\Phi\colon H \to H$ *has the property that* $\left|\Phi(\mathbf{x}, z)\right|_H = c\left|(\mathbf{x},z)\right|_H$ *for some non-zero* $c \in \mathbb{R}$ *if and only if* $\Phi\big((\mathbf{x}, z)\big) = (M\mathbf{x}, az)$ *where* $\|M\mathbf{x}\| = c\,\|\mathbf{x}\|$ *for all* $\mathbf{x} \in \mathbb{R}^{2n}$*, and* $|a| = c^2$*. Specifically, if* $\|M\mathbf{x}\| = \|\mathbf{x}\|$ *(in other words, M is orthogonal), and* $|a| = 1$*, then* $\left|\Phi(\mathbf{x}, z)\right|_H = \left|(\mathbf{x}, z)\right|_H$.

Proof. The proof is parallel to the proof for the form of contraction maps with the inequalities replaced by equality. □

Remark. Note again that the condition on a follows directly; given $\|M\mathbf{x}\| = c\,\|\mathbf{x}\|$, we know that c is the only eigenvalue of M, and hence $\det M = a^n = c^{2n}$, hence $|a| = c^2$.

Given this classification of automorphisms on the Heisenberg group, we look specifically at automorphisms that preserve certain lattices on $H^3(\mathbb{R})$, and we construct examples of tilings on this group.

4.3 Lattices

In order to create specific examples of self-similar tilings in the Heisenberg group, we need information about lattices. This allows us to find automoprhisms that preserve a lattice, and to find a residue system. For the remainder of the paper we restrict H to $H^3(\mathbb{R})$ unless otherwise noted.

For an element $h \in H$, let $\overline{h}$ denote the image under the natural map $H \longrightarrow \mathbb{R}^2$. Similarly, if $S \subset H$ then $\bar{S}$ will denote the image of S under the natural map. For a set $S = \{h_1, h_2\}$ in H, and $k \in \mathbb{N}$, let $\Gamma_{S,k}$ denote the group generated by S, together with $[h_1, h_2]/k$, the commutator divided by k with $[h_1, h_2] = h_1 h_2 h_1^{-1} h_2^{-1}$.

Lemma 4.10. *If* Γ *is a lattice in* H *then* $\bar{\Gamma}$ *is a lattice in* $\mathbb{R}^2$*.*

Proof. Because Γ is cocompact, $\Gamma \cdot K = H$ for some compact set K. This implies that $\bar{\Gamma} \cdot \bar{K} = \mathbb{R}^2$, and since the image under a continuous map of a compact set is also compact, $\bar{K}$ is compact, showing that $\bar{\Gamma}$ is cocompact in $\mathbb{R}^2$.

Next we suppose that $\bar{\Gamma}$ is not discrete. Then the identity of $\mathbb{R}^2$ must be a limit point. Choose a series $v_1, v_2, \ldots$ of nonzero elemnts of $\bar{\Gamma}$ such that $\lim v_i = \mathbf{0}$. Therefore we have $h_i = (v_i, z_i) \in H$ such that $v_i \to 0$. Since we have $\bar{\Gamma}$ is cocompact in $\mathbb{R}^2$, then $\bar{\Gamma}$ spans $\mathbb{R}^2$; so we can pick some $h = (v, z) \in \Gamma$ such that for infinitely many i, $v \notin \operatorname{span}(v_i)$. For such i, $[h, h_i] \neq 0$. Now replace the sequence h_i with a subsequence so that $[h, h_i] \neq 0$ for all h_i. But $[h, h_i] = 2B(v, v_i) \to 0$ as $i \to \infty$, which is a contradiction since Γ is discrete. □

Theorem 4.11. *All lattices have the form* $\Gamma_{S,k}$ *where* $\bar{S}$ *is a linearly independent set.*

Proof. Let Γ be a lattice and let Z be the center of H. Consider $\Gamma \cap Z$. Since Γ is a lattice $\Gamma \cap Z$ is a discrete subgroup of Z, and we know that $\Gamma \cap Z \simeq \mathbb{Z}$. Let γ be a generator. Choose α and β such that $\bar{\alpha}$ and $\bar{\beta}$ are generators for $\bar{\Gamma}$. Now look at $[\alpha, \beta]$. Since $[\alpha, \beta] \in \mathbb{Z}$ we know $[\alpha, \beta] = \gamma^k$ for some $k \in \mathbb{Z}$. Hence $\Gamma \simeq \Gamma_{\{\alpha,\beta\},k}$. □

Remark. The converse is false. For example, suppose the bilinear form fixed by the group law is

$$B = \begin{bmatrix} 0 & 1/2 \\ -1/2 & 0 \end{bmatrix}.$$

Let $S = \{(1, 0, \pi), (0, 1, 0)\}$. Here $\bar{S}$ is linearly independent and $[(1, 0, \pi), (0, 1, 0)] = (0, 0, 1)$. In this case $\Gamma_{S,1} \cap Z$ is isomorphic to the subgroup of $\mathbb{R}$ generated by 1 and π and is not discrete. Thus $\Gamma_{S,1}$ is not a lattice.

Theorem 4.12. $\Gamma_{S,k} \cong \Gamma_{S',k}$.

Proof. We will show that there exists an automorphism Φ of H such that $\Phi(\Gamma_{S,k}) = \Gamma_{S',k}$. Write $S = \{h_1, h_2\}$ and $S' = \{h'_1, h'_2\}$ where $h_i = (\mathbf{x}_i, z_i)$ and $h'_i = (\mathbf{x}'_i, z'_i)$ for $i = 1, 2$. Since $\overline{S}$ and $\overline{S}'$ are both bases for $\mathbb{R}^2$ there exists a symplectic similitude matrix M such that $MS = S'$ and there exists a linear transformation $\omega \colon \mathbb{R}^2 \to \mathbb{R}$ such that $\omega(\mathbf{x}_i) + az_i = z'_i$ for $i = 1, 2$, where $a = \det M$. Define $\Phi \colon H \to H$ by $\Phi(\mathbf{x}, z) = (M\mathbf{x}, \omega(\mathbf{x}) + az)$ and recall that this is an automorphism, so $\Phi([h_1, h_2]) = [\Phi(h_1), \Phi(h_2)]$. It follows that

$$\Phi\left(\frac{[h_1, h_2]}{k}\right) = \frac{[h'_1, h'_2]}{k},$$

so $\Phi(\Gamma_{S,k}) = \Gamma_{S',k}$. □

Theorem 4.13. *If $\Gamma_{S,k}$ and $\Gamma_{S',k'}$ are lattices and $k \neq k'$, then $\Gamma_{S,k} \ncong \Gamma_{S',k'}$.*

Proof. Let $(\Gamma_{S,k})'$ and $(\Gamma_{S',k'})'$ denote the respective commutator subgroups of $\Gamma_{S,k}$ and $\Gamma_{S',k'}$. We will show that $(\Gamma_{S,k})'$ has index k in the center $Z(\Gamma_{S,k})$ of $\Gamma_{S,k}$. Note that

$$Z(\Gamma_{S,k}) = \left\langle \frac{[h'_1, h'_2]}{k} \right\rangle$$

and $(\Gamma_{S,k})' = \langle [h_1, h_2] \rangle$. Suppose we have an isomoprphism $\Gamma_{S,k} \to \Gamma_{S',k'}$, then we obtain an isomorphism $Z(\Gamma_{S,k})/\Gamma'_{S,k} \to Z(\Gamma_{S',k'})/\Gamma'_{S',k'}$ where $Z(\Gamma_{S,k})/\Gamma'_{S,k}$ has order k and $Z(\Gamma_{S',k'})/\Gamma'_{S',k'}$ has order k'. But then $k = k'$, a contradiction. Therefore, $\Gamma_{S,k} \ncong \Gamma_{S',k'}$. □

Now that we have the form for expansion maps and lattices, we construct specific examples of tilings in the Heisenberg group. Before showing these examples

4.4 Self-Similar Tilings on $H^3(\mathbb{R})$

We may now apply Theorem 3.13 to show examples of tilings in the Heisenberg group, along with all necessary information to plot them. The following proposition gives us a method of choosing residue vectors to generate images of tilings in in $H^{2n+1}(\mathbb{R})$.

Proposition 4.14. *Let $\Phi(\mathbf{x}, y) = (M\mathbf{x}, ay)$ be an expansion on H^{2n+1} and M an invertible $2n \times 2n$ matrix with integer entries. For this proposition, fix a bilinear form such that*

$$2B(\mathbf{x}, \mathbf{x}') \in \mathbb{Z}, \quad \mathbf{x}, \mathbf{x}' \in \mathbb{Z}^{2n}$$

Let

$$C_{2n} = \left\{ \mathbf{v} \in \mathbb{Z}^{2n} : \mathbf{v} = M \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_{2n} \end{pmatrix},\ v_1, \ldots, v_{2n} \in [0, 1), \right\}$$

and let

$$C_{\mathbb{Z}} = \{ z \in [0, a/2) : 2z \in \mathbb{Z} \}.$$

Then we define our set of residue vectors of Φ to be

$$R = \big\{ (\mathbf{x}, y) : \mathbf{x} \in C_{2n}, y \in C_{\mathbb{Z}} \big\}.$$

Now let Z_H be the lattice on the Heisenberg group defined by

$$\mathbb{Z}_H = \big\{ (\mathbf{x}, y/2) \in H_{2n+1} : \mathbf{x} \in \mathbb{Z}^{2n}, z \in \mathbb{Z} \big\}.$$

R is a complete set of coset representors of distinct cosets of Z_H mod ΦZ_H.

Proof. If $(\mathbf{x}, y) \in \mathbb{Z}_H$ then $\Phi(\mathbf{x}, y) = (M\mathbf{x}, ay) \in \mathbb{Z}_H$ because M has integer entries and a is an integer determined by M. Also for any two elements $\mathbf{v}, \mathbf{v}' \in \mathbb{Z}_H$ we have $\mathbf{v} * \mathbf{v}' \in \mathbb{Z}_H$. If $\mathbf{v} = \mathbf{w} * \mathbf{v}'$ then $\mathbf{w} \in \mathbb{Z}_H$ and if $\mathbf{v} = \mathbf{v}' * \mathbf{w}$ then $\mathbf{w} \in \mathbb{Z}_H$. Now we wish to show that any $(\mathbf{x}, z) \in \mathbb{Z}_H$ can be written as equivalent to a residue mod $\Phi\mathbb{Z}_H$. Any

$$(\mathbf{x}, z) = (M\mathbf{v}, z)$$

for some $\mathbf{v} \in \mathbb{R}^n$. Then

$$(\mathbf{x}, z) = (\mathbf{x}_a, z_a) * (\mathbf{x}_b, z_b) = (M\mathbf{v}_a, z_a) * (M\mathbf{v}_b, z_b) = (M\mathbf{v}, z)$$

where $M\mathbf{v}_a = \mathbf{x}_a$ and $M\mathbf{v}_b = \mathbf{x}_b$ and $\mathbf{x} = \mathbf{x}_a + \mathbf{x}_b$. Then

$$z = z_a + z_b + B(\mathbf{x}_a, \mathbf{x}_b).$$

Since $\mathbf{x}_a, \mathbf{x}_b \in \mathbb{Z}^{2n}$ we have $2B(\mathbf{x}_a, \mathbf{x}_b) \in \mathbb{Z}$. Then $2z_a, 2z_b \in \mathbb{Z}$. Now it remains to describe how to pick $\mathbf{v}_a, \mathbf{v}_b, z_a$ and z_b. If $\mathbf{v} = (v_1, \ldots, v_{2n})$, then let

$$\mathbf{v}_b = \big(\lfloor \mathbf{v}_1 \rfloor, \ldots, \lfloor \mathbf{v}_{2n} \rfloor \big)$$

and

$$\mathbf{v}_a = \mathbf{v} - \mathbf{v}_b.$$

Let $q = z - B(\mathbf{x}_a, \mathbf{x}_b)$ and then define $z_a = q - \lfloor q/(a/2) \rfloor$ and $z_b = q - z_a$. Then $(\mathbf{x}_a, z_a) \in R$ and therefore $(\mathbf{x}, z)$ is equivalent to a residue mod ΦZ_H. This means that every element of $\mathbb{Z}_H$ is equivalent to a residue mod $\mathbb{Z}_H$. Now it remains to show that each residue represents a distinct coset of $\mathbb{Z}_H$. Consider any two residues,

$$(\mathbf{x}, z) = (M\mathbf{v}, z), (\mathbf{x}', z') = (M\mathbf{v}', z').$$

If they are equivalent, then

$$(M\mathbf{v}, z) = (M\mathbf{v}', z') * (M\mathbf{p}, ar)$$

where $(\mathbf{p}, r) \in \mathbb{Z}_H$. Then $\mathbf{v} = \mathbf{v}' + \mathbf{p}$ but since $\mathbf{v} = (v_1, \ldots, v_{2n})$ and $\mathbf{v}' = (v_1', \ldots, v_{2n}')$ where each v_i and $v_i' \in [0, 1)$ we must have $\mathbf{p} = \mathbf{0}$. Then $B(\mathbf{x}', M\mathbf{p}) = 0$. Then if we let $q = z - B(\mathbf{x}', M\mathbf{p}) = z$ then $ar = \lfloor v/(a/2) \rfloor$. But since $v \in [0, a/2)$ we have $ar = 0$. So

$$(M\mathbf{p}, r) = \mathbf{0},$$

and the two residues are the same. Then two residues are equivalent if and only if they are equal. □

Remark. We know for tilings in $\mathbb{R}^{2n}$ the number of tiles for a transformation M is simply $\det M$, so C_{2n} has $\det M$ elements. $C_{\mathbb{Z}}$ has a elements, so the number of tiles of Φ is the number of residue vectors, $a \cdot \det M$. The important of this result is seen when we apply Theorem 3.13 and Theorem 3.15 to the Heisenberg group.

We will now consider two examples of fractal tilings on the Heisenberg group, with the group law given by

$$(\mathbf{x}, z) * (\mathbf{x}', z') = \left(\mathbf{x} + \mathbf{x}', z + z' + \frac{1}{2}\left(x_1 x_2' - x_1' x_2\right)\right).$$

Example 4.15. The first example we will consider is the tiling generated by the automorphism

$$\Phi(\mathbf{x}, z) = (M\mathbf{x}, 2z), \text{ where } M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$

with $\det M = 2$ and $a = 2$. M is the same matrix that generates a twin dragon tiling in $\mathbb{R}^2$. One possible set of residue vectors for this twindragon in the plane is $\{(0, 0), (0, 1)\}$. Therefore, one possible selection of the four residues in H is

$$\left\{(0, 0, 0), (0, 1, 0), \left(0, 0, \frac{1}{2}\right), \left(0, 1, \frac{1}{2}\right)\right\}.$$

These residue vectors are found by direct application of the previous proposition and allow us to generate Figure 2.

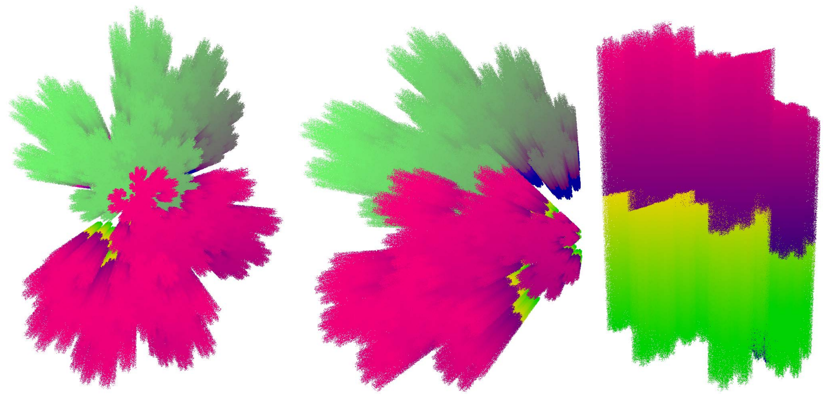


Figure 2: Self-similar tiling of $H^3(\mathbb{R})$ with the twin dragon

Example 4.16. The second example of a tiling in H is the one generated by the automorphism

$$\Phi(\mathbf{x}, z) = (M\mathbf{x}, 3z)\,, \text{ where } M = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}.$$

M is the same matrix that generates a terdragon in $\mathbb{R}^2$. One set of residues for this tiling in the plane is $\{(0,0),(1,0),(2,0)\}$. Then, one possible set of residues for Φ in H is

$$\left\{(0,0,0),(1,0,0),(2,0,0),\left(0,0,\frac{1}{2}\right),\left(1,0,\frac{1}{2}\right),\left(2,0,\frac{1}{2}\right),(0,0,1),(1,0,1),(2,0,1)\right\}.$$

By following the selection method of residues from the proposition, we generate the tiling shown in Figure 3.

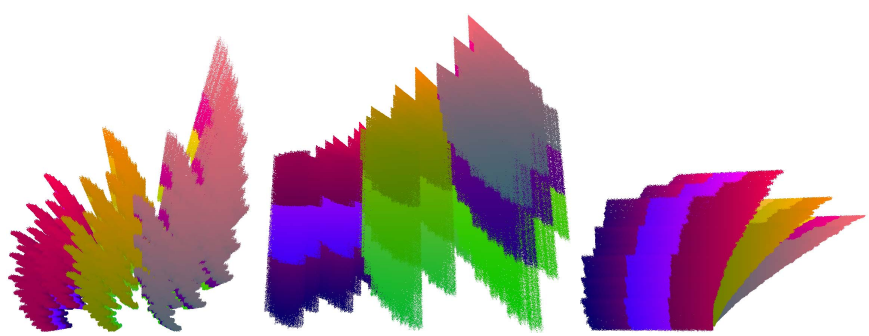


Figure 3: Self-similar tiling of $H^3(\mathbb{R})$ with the terdragon

4.5 Horizontal Lifts

Another way to look at fractals on the Heisenberg group using contraction maps is with horizontal lifts. We will consider collections of Lipschitz contractions of the Heisenberg group with respect to the Heisenberg metric. In our case we will consider the Heisenberg group $H \equiv \mathbb{R}^3$ with the group law

$$(\mathbf{x}, z) * (\mathbf{x}', z') = \big(\mathbf{x} + \mathbf{x}', z + z' + B(\mathbf{x}, \mathbf{x}')\big)$$

where $B(\mathbf{x}, \mathbf{x}') = 2(x_1' \cdot x_2 - x_1 \cdot x_2')$ with $\mathbf{x} = (x_1, x_2)$ and $\mathbf{x}' = (x_1', x_2')$. Throughout this section we will restrict our attention to affine contractions of H that arise as lifts of affine mappings of $\mathbb{R}^2$ as described below.

4.5.1 Preliminaries

Definition 4.17. A function $f : X \to Y$ between metric spaces is called *r-Lipschitz*, $r > 0$, if

$$d\big(f(x), f(y)\big) \leq r \cdot d(x, y)$$

for all $x, y \in X$. Moreover, f is Lipschitz if it is r-Lipschitz for some $r < \infty$. The infimum of those values r for which the above inequality holds for all $x, y \in X$ is called the Lipschitz constant of f.

Definition 4.18. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$. A map $F : H \to H$ is called a *lift* of f if $\pi \circ F = f \circ \pi$.

Here, $\pi : H \to \mathbb{R}^2$ denotes the projection map

$$\pi(\mathbf{x}, z) = \mathbf{x}$$

where π is a mapping of our horizontal lift onto the xy-plane.

The following results using $c = (2 + \sqrt{3})^{1/4}$ are given in [1].

Theorem 4.19 (Existence and uniqueness of horizontal Lipschitz lifts)**.** *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be r-Lipschitz with* $\det Df \equiv \lambda$. *Then there exists a cr-Lipschitz lift $F : (H, d_H) \to (H, d_H)$. If $\widetilde{F}$ is another Lipschitz lift of f, then $\widetilde{F}(\mathbf{x}, z) = F(\mathbf{x}, z + \tau)$ for some $\tau \in \mathbb{R}$. Conversely, if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is Lipschitz with Lipschitz lift, then there exists $\lambda \in \mathbb{R}$ so that* $\det Df \equiv \lambda$.

The proof of the theorem gives an explicit formula for a Lipschitz lift F on H as follows:

$$F(\mathbf{x}, z) = (f(\mathbf{x}), \lambda z + h_0(\mathbf{x}))$$

where h_0 is any function such that

$$\nabla h_0 = 2(\lambda \cdot J - Df^* \cdot Jf),$$

$J(x_1, x_2) = (-x_2, x_1)$ and Df^* is the conjugate transpose of Df.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a general Lipschitz map. Without further assumptions, many different functions $F : H \to H$ serve as lifts of f. However, if we require in addition that F be Lipschitz with respect to d_H, then F is uniquely determined by the formulas above. (This is the second result in [1]).

Proposition 4.20. *The Lipschitz lifts of a Lipschitz affine map $f(\mathbf{x}) = M\mathbf{x}+\mathbf{d}$, $\mathbf{d}, \mathbf{x} \in \mathbb{R}^2$, are given explicitly by*

$$F(\mathbf{x}, z) = \big(f(\mathbf{x}), \det Mz - 2M^t \cdot J(\mathbf{d}) \cdot \mathbf{x} + \tau\big), \tau \in \mathbb{R}.$$

Proof. Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\mathbf{d} = (e, f)$ and $\mathbf{x} = (x_1, x_2)$ with $\lambda = \det M$ and $a, b, c, d, e, f \in \mathbb{R}$. Then

$$\begin{aligned}
\nabla h_0 &= 2(\lambda \cdot J - Df^* \cdot Jf) \\
&= 2\Big((-\lambda x_2, \lambda x_1) - \big(-\lambda x_2 + (ce - af), \lambda x_1 + (bf - de)\big)\Big) \\
&= 2(af - ce, bf - de) \\
&= 2\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{pmatrix} f \\ -e \end{pmatrix} = -2M^t J(\mathbf{d}).
\end{aligned}$$

So $h = -2M^t J(\mathbf{d}) \cdot \mathbf{x} + \tau$. □

Note that if M is invertible, $d = 0$ and $\tau = 0$, then F is a contracting automorphism on H as described in the earlier section.

Proposition 4.21. *The Lipschitz constants of a Lipschitz affine map $f(\mathbf{x}) = M\mathbf{x} + \mathbf{d}$, $\mathbf{d}, \mathbf{x} \in \mathbb{R}^2$ and its corresponding Lipschitz lift F agree.*

Proof. Let λ be the largest eigenvalue of matrix M and $\mathbf{x} = (x_1, x_2)$ and $\mathbf{x}' = (x_1', x_2')$, then $\det M \leq \lambda^2$. Since $\|M\|_E$ is the square root of the largest eigenvalue of MM^t, we know that $\det M \leq \|M\|_E^2$. Then

$$\begin{aligned}
d_H\big(F(\mathbf{x}, t), F(\mathbf{x}', t)\big) &= \Big(\big\|M(x_1 - x_1', x_2 - x_2')\big\|_E^4 + \big|\det M\big(t - t' + 2(x_1'x_2 - x_1x_2')\big)\big|^2\Big)^{1/4} \\
&\leq \Big(\|M\|_E^4 \big\|(x_1 - x_1', y_1 - y_1')\big\|_E^2 + |\det M|^2 \big|t - t' + 2(x_1'x_2 - x_1x_2')\big|^2\Big)^{1/4} \\
&\leq \|M\| \Big(\big\|(x_1 - x_1', y_1 - y_1')\big\|_E^2 + \big|t - t' + 2(x_1'x_2 - x_1x_2')\big|^2\Big)^{1/4} \\
&= \|M\| \, d_H\big((\mathbf{x}, t), (\mathbf{x}'t)\big).
\end{aligned}$$

□

Example 4.22. In the case of $f(\mathbf{x}) = M\mathbf{x}$, where

$$M = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, a \neq 0$$

we have $F(\mathbf{x}, z) = \big(a\mathbf{x}, a^2 z + \tau\big)$ where $\tau \in \mathbb{R}$ is an arbitrary constant. So the Lipschitz constant for f and for F is a, and the contraction constant for either is $1/a$. In particular, if $\tau = 0$ then $F(\mathbf{x}, z) = \big(a\mathbf{x}, a^2 z\big)$, the dilation map.

Example 4.23. Suppose $f(\mathbf{x}) = I\mathbf{x} + \mathbf{d}$, where $\mathbf{d} \in \mathbb{R}^2, \mathbf{d} \neq 0$. Then

$$F(\mathbf{x}, z) = (\mathbf{d}, \tau) * (\mathbf{x}, z)$$

where again τ is an arbitrary constant. Note that F is a left translation in the Heisenberg metric.

Example 4.24. Suppose that $f(\mathbf{x}) = M\mathbf{x} + \mathbf{d}$ where

$$M = \begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

$\mathbf{d} \in \mathbb{R}^2, \mathbf{d} \neq 0$ with $\det M = a^2 + b^2 = \lambda$. Then

$$F(\mathbf{x}, z) = \big(M\mathbf{x} + \mathbf{d}, \lambda z - 2M^t \cdot (-d_2, d_1) \cdot \mathbf{x} + \tau\big)$$

where τ is an arbitrary constant and $\mathbf{d} = (d_1, d_2)$. The contraction constant for f and for F is $\sqrt{\lambda}$.

In the examples above, both f and F are similarities with their respective metrics. Thus the Lipschitz constant agrees with the operator norm of the linear part of f.

Proposition 4.25 ([1])**.** *Let $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be Lipschitz maps with* $\det Df_i \equiv \lambda_i$, $i = 1, 2$. *For each i let F_i be a Lipschitz lift of f_i. Then $F_1 \circ F_2$ is a Lipschitz lift of $f_1 \circ f_2$.*

Theorem 4.26 ([1])**.** *] Let $\mathcal{F} = \{f_1, ...f_M\}$ be an iterated function system on $\mathbb{R}^2$, where each map f_i is r_i-Lipschitz for some r_i and satisfies* $\det Df_i \equiv \lambda_i$. *For each i, let F_i be a lift of f_i to H. Then $\mathcal{F}_H = \{F_1, \ldots, F_M\}$ is an iterated function system on H. Denoting by K, respectively K_H, the invariant set for $\mathcal{F}$, respectively $\mathcal{F}_H$, we have*

$$\pi(K_H) = K.$$

These invariant sets on H are called horizontal fractals.

4.5.2 Examples

We present several examples of lifts of common iterated function systems in $\mathbb{R}^2$. Notice that none of these horizontal lifts are tilings in H. We present one example in full detail and give only the F functions for the remaining ones.

Example 4.27 (Sierpinksi gasket)**.** First we define $\mathbf{e}_1 = (0, 0), \mathbf{e}_2 = (1/4, \sqrt{3}/4), \mathbf{e}_3 = (1/2, 0)$ to be non-rotational translation vectors in $\mathbb{R}^2$. Let M be an invertible 2×2 matrix such that $M \cdot (x_1, x_2) = (x_1/2, x_2/2)$. In other words, M is a dilation that scales (x_1, x_2) by $1/2$. We define each f_i in the following way:

$$
\begin{aligned}
f_1(x_1, x_2) &= \left(\frac{x_1}{2}, \frac{x_2}{2}\right) + \mathbf{e}_1 = \left(\frac{x_1}{2}, \frac{x_2}{2}\right) \\
f_2(x_1, x_2) &= \left(\frac{x_1}{2}, \frac{x_2}{2}\right) + \mathbf{e}_2 = \left(\frac{x_1}{2} + \frac{1}{4}, \frac{x_2}{2} + \frac{\sqrt{3}}{4}\right) \\
f_3(x_1, x_2) &= \left(\frac{x_1}{2}, \frac{x_2}{2}\right) + \mathbf{e}_3 = \left(\frac{x_1}{2} + \frac{1}{2}, \frac{x_2}{2}\right).
\end{aligned}
$$

The iterated function system defined as $\mathcal{F} = \{f_1, f_2, f_3\}$ has the Sierpinski gasket as its invariant set. For each f_i, we must find λ_i, $\lambda_i J$, Jf_i, Df_i^*, and ∇h_0^i. For our example we have,

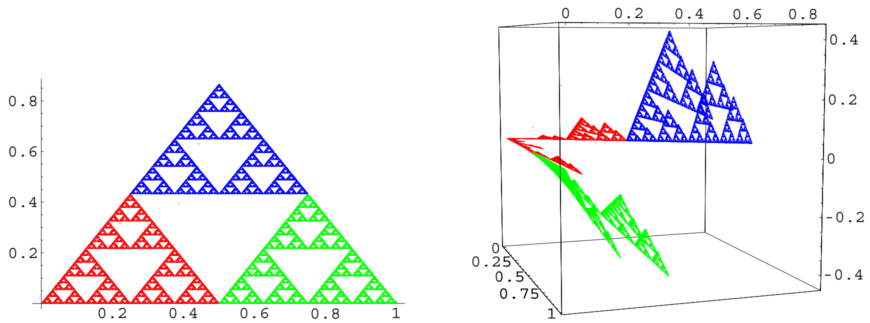
$$
\begin{aligned}
Df_i^* &= \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \\
\lambda_i &= \frac{1}{4} \quad \forall i \in \{1, 2, 3\}.
\end{aligned}
$$

i	$\lambda_i J$	Jf_i
1	$(-x_2/4, x_1/4)$	$(-x_2/2, x_1/2)$
2	$(-x_2/4, x_1/4)$	$(-x_2/2, x_1/2 + 1/2)$
3	$(-x_2/4, x_1/4)$	$(-x_2/2 - \sqrt{3}/4, x_1/2 + 1/4)$

$$
\begin{aligned}
\nabla h_0^1 &= 2\left(\lambda_1 J - Df_1^* \cdot Jf_1\right) = 2 \cdot (0, 0) = (0, 0) \\
h_0^1 &= \tau_1 \\
\nabla h_0^2 &= 2\left(\lambda_2 J - Df_2^* \cdot Jf_2\right) = 2 \cdot \left(0, -\frac{1}{4}\right) = \left(0, -\frac{1}{2}\right) \\
h_0^2 &= -\frac{y}{2} + \tau_2 \\
\nabla h_0^3 &= 2\left(\lambda_3 J - Df_3^* \cdot Jf_3\right) = 2 \cdot \left(\frac{\sqrt{3}}{4}, -\frac{1}{4}\right) = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) \\
h_0^3 &= \frac{\sqrt{3}x}{4} - \frac{y}{4} + \tau_3.
\end{aligned}
$$

The arbitrary constants, $(\tau_1, \tau_2, \tau_3) \in \mathbb{R}^3$ are formed from the partial integration of the gradient ∇h_0^i. Now, we may derive our lift functions F_i,

$$
\begin{aligned}
F_1(x_1, x_2, z) &= \left(\frac{x_1}{2}, \frac{x_2}{2}, \frac{z}{4} + \tau_1\right) \\
F_2(x_1, x_2, z) &= \left(\frac{x_1}{2} + \frac{1}{4}, \frac{x_2}{2} + \frac{\sqrt{3}}{4}, \frac{z}{4} - \frac{x_2}{2} + \tau_2\right) \\
F_3(x_1, x_2, z) &= \left(\frac{x_1}{2} + \frac{1}{2}, \frac{x_2}{2}, \frac{z}{4} + \frac{\sqrt{3}x_1}{4} - \frac{x_2}{4} + \tau_3\right).
\end{aligned}
$$

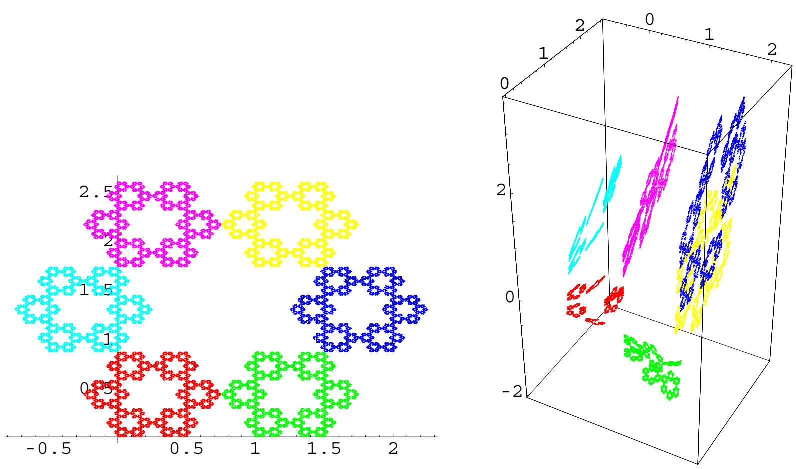


(a) Projection of the horizontal lift onto $\mathbb{R}^2$

(b) Lift of a Sierpinski gasket with $(\tau_1, \tau_2, \tau_3) = (0, 0, 0)$

Figure 4: Horizontal lift of Example 4.27

Example 4.28 (Hexagasket)**.** We present the horizontal lift formed from functions listed in Appendix A.1. An example of a lift of the hexagasket is shown in Figure 5.

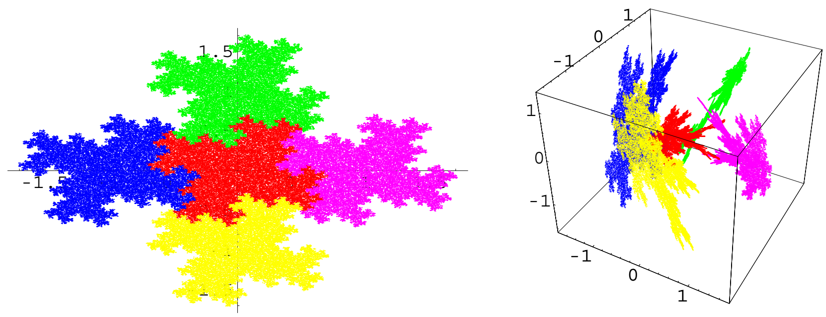


(a) Projection of the horizontal lift onto $\mathbb{R}^2$

(b) Lift of a Hexagasket with $(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6) = (0, 0, 0, 0, 0, 0)$

Figure 5: Horizontal lift of Example 4.28

Example 4.29 (Fractal cross)**.** We present the horizontal lift formed from functions listed in Appendix A.2. An example of a lift of the fractal cross is shown in Figure 6.

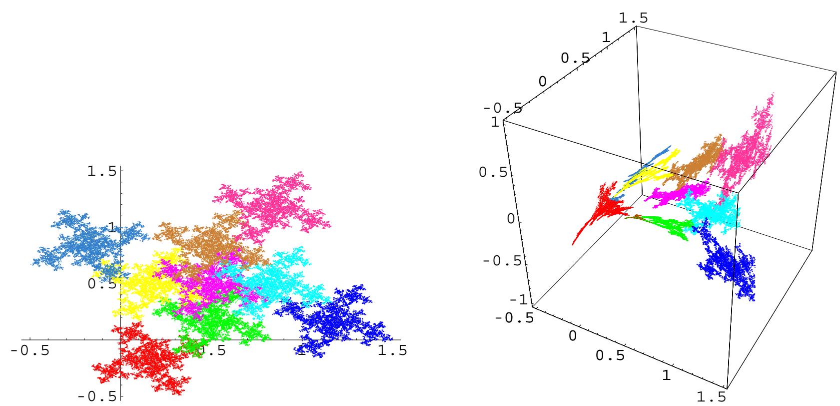


(a) Projection of the horizontal lift onto $\mathbb{R}^2$

(b) Lift of a Fractal cross with $(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5) = (0, 0, 0, 0, 0)$

Figure 6: Horizontal lift of Example 4.29

Example 4.30 (Spiral)**.** We present the horizontal lift formed from functions listed in Appendix A.3. An example of a lift of the fractal cross is shown in Figure 7.



(a) Projection of the horizontal lift onto $\mathbb{R}^2$

(b) Lift of a spiral with $(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7, \tau_8, \tau_9) = (0, 0, 0, 0, 0, 0, 0, 0, 0)$

Figure 7: Horizontal lift of Example 4.30

5 Future Work

The work we have done suggests several directions for further investigation. While we did characterize all automorphisms on the general Heisenberg group, we have only investigated lattices (discrete cocompact subgroups) in the three-dimensional case. A classification of Heisenberg lattices in general would allow investigation of automorphisms that preserve various lattices, which would be helpful in developing a deeper understanding of self-similar tiling in higher dimensional Heisenberg groups, and in creating varied examples of such tilings.

Additionally, now that we have a result proving we're able to construct a self-similar tiling in a rational graded nilpotent Lie group given an expansive map that preserves a lattice, a natural next step would be to examine the form of such maps in the general nilpotent Lie groups. To investigate the form of these automorphisms we would first need to know the specific restrictions on the polynomials F_{ij} that will guarantee we get a group. If we could classify automorphisms and lattices on general nilpotent Lie groups, that would allow the creation of specific examples of self-similar tilings on these groups.

Finally, we focused on nilpotent Lie groups because we knew they would allow us to have a metric, a measure, and lattices; they are not, however, necessarily the only class of groups that have these characteristics. A natural direction for further research would be to generalize our result to include even more general groups.

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A Lift Functions

For each example we present the list of affine functions $f_1, f_2, \ldots, f_k$ that generate a fractal in $\mathbb{R}^2$ and the corresponding lift functions $F_1, F_2, \ldots, F_k$ that form a fractal in H.

A.1 Example 4.28

$$
\begin{aligned}
f_1(x_1, x_2) &= \left(\frac{x_1}{3}, \frac{x_2}{3}\right) \\
f_2(x_1, x_2) &= \left(\frac{x_1}{3} + 1, \frac{x_2}{3}\right) \\
f_3(x_1, x_2) &= \left(\frac{x_1}{3} + \frac{3}{2}, \frac{x_2}{3} + \frac{\sqrt{3}}{2}\right) \\
f_4(x_1, x_2) &= \left(\frac{x_1}{3} + 1, \frac{x_2}{3} + \sqrt{3}\right) \\
f_5(x_1, x_2) &= \left(\frac{x_1}{3}, \frac{x_2}{3} + \sqrt{3}\right) \\
f_6(x_1, x_2) &= \left(\frac{x_1}{3} - \frac{1}{2}, \frac{x_2}{3} + \frac{\sqrt{3}}{2}\right) \\
F_1(x_1, x_2, z) &= \left(f_1(x_1, x_2), \frac{z}{9} + \tau_1\right) \\
F_2(x_1, x_2, z) &= \left(f_2(x_1, x_2), \frac{z}{9} - \frac{2}{3}x_2 + \tau_2\right) \\
F_3(x_1, x_2, z) &= \left(f_3(x_1, x_2), \frac{z}{9} + \frac{\sqrt{3}x_1}{3} - x_2 + \tau_3\right) \\
F_4(x_1, x_2, z) &= \left(f_4(x_1, x_2), \frac{z}{9} + \frac{2\sqrt{3}x_1}{3} - \frac{2x_2}{3} + \tau_4\right) \\
F_5(x_1, x_2, z) &= \left(f_5(x_1, x_2), \frac{z}{9} + \frac{2\sqrt{3}x_1}{3} + \tau_5\right) \\
F_6(x_1, x_2, z) &= \left(f_6(x_1, x_2), \frac{z}{9} + \frac{\sqrt{3}x_1}{3} + \frac{x_2}{3} + \tau_6\right)
\end{aligned}
$$

A.2 Example 4.29

$$
\begin{aligned}
f_1(x_1, x_2) &= \left(\frac{2x_1}{5} + \frac{x_2}{5}, \frac{-x_1}{5} + \frac{2x_2}{5}\right) \\
f_2(x_1, x_2) &= \left(\frac{2x_1}{5} + \frac{x_2}{5}, \frac{-x_1}{5} + \frac{2x_2}{5} + 1\right) \\
f_3(x_1, x_2) &= \left(\frac{2x_1}{5} + \frac{x_2}{5} - 1, \frac{-x_1}{5} + \frac{2x_2}{5}\right) \\
f_4(x_1, x_2) &= \left(\frac{2x_1}{5} + \frac{x_2}{5} + 1, \frac{-x_1}{5} + \frac{2x_2}{5}\right) \\
f_5(x_1, x_2) &= \left(\frac{2x_1}{5} + \frac{x_2}{5}, \frac{-x_1}{5} + \frac{2x_2}{5} - 1\right) \\
F_1(x_1, x_2, z) &= \left(f_1(x_1, x_2), \frac{z}{5} + \tau_1\right) \\
F_2(x_1, x_2, z) &= \left(f_2(x_1, x_2), \frac{z}{5} + \frac{4x_1}{5} + \frac{2x_2}{5} + \tau_2\right) \\
F_3(x_1, x_2, z) &= \left(f_3(x_1, x_2), \frac{z}{5} - \frac{2x_1}{5} + \frac{4x_2}{5} + \tau_3\right) \\
F_4(x_1, x_2, z) &= \left(f_4(x_1, x_2), \frac{z}{5} + \frac{2x_1}{5} - \frac{4x_2}{5} + \tau_4\right) \\
F_5(x_1, x_2, z) &= \left(f_5(x_1, x_2), \frac{z}{5} - \frac{4x_1}{5} - \frac{2x_2}{5} + \tau_5\right)
\end{aligned}
$$

A.3 Example 4.30

$$
\begin{aligned}
f_1(x_1,x_2) &= \left(\frac{x_1}{3}, -\frac{x_2}{3}\right)\\
f_2(x_1,x_2) &= \left(\frac{x_1}{3}+\frac{1}{3}, \frac{x_2}{3}\right)\\
f_3(x_1,x_2) &= \left(\frac{x_1}{3}+1, \frac{x_2}{3}\right)\\
f_4(x_1,x_2) &= \left(\frac{x_1}{3}, \frac{x_2}{3}+\frac{1}{3}\right)\\
f_5(x_1,x_2) &= \left(\frac{x_1}{3}+\frac{1}{3}, \frac{x_2}{3}+\frac{1}{3}\right)\\
f_6(x_1,x_2) &= \left(\frac{x_1}{3}+\frac{2}{3}, \frac{x_2}{3}+\frac{1}{3}\right)\\
f_7(x_1,x_2) &= \left(-\frac{x_1}{3}, \frac{x_2}{3}+\frac{2}{3}\right)\\
f_8(x_1,x_2) &= \left(\frac{x_1}{3}+\frac{1}{3}, \frac{x_2}{3}+\frac{2}{3}\right)\\
f_9(x_1,x_2) &= \left(\frac{x_1}{3}+\frac{2}{3}, \frac{x_2}{3}+1\right)\\
F_1(x_1,x_2,z) &= \left(f_1(x_1,x_2), \frac{z}{9}+\frac{4x_1\cdot x_2}{9}+\tau_1\right)\\
F_2(x_1,x_2,z) &= \left(f_2(x_1,x_2), \frac{z}{9}-\frac{2x_2}{9}+\tau_2\right)\\
F_3(x_1,x_2,z) &= \left(f_3(x_1,x_2), \frac{z}{9}-\frac{2x_2}{3}+\tau_3\right)\\
F_4(x_1,x_2,z) &= \left(f_4(x_1,x_2), \frac{z}{9}+\frac{2x_1}{9}+\tau_4\right)\\
F_5(x_1,x_2,z) &= \left(f_5(x_1,x_2), \frac{z}{9}+\frac{2x_1}{9}-\frac{2x_2}{9}+\tau_5\right)\\
F_6(x_1,x_2,z) &= \left(f_6(x_1,x_2), \frac{z}{9}+\frac{2x_1}{9}-\frac{4x_2}{9}+\tau_6\right)\\
F_7(x_1,x_2,z) &= \left(f_7(x_1,x_2), \frac{z}{9}-\frac{4x_1}{9}+\tau_7\right)\\
F_8(x_1,x_2,z) &= \left(f_8(x_1,x_2), \frac{z}{9}+\frac{4x_1}{9}-\frac{2x_2}{9}+\tau_8\right)\\
F_9(x_1,x_2,z) &= \left(f_9(x_1,x_2), \frac{z}{9}+\frac{2x_1}{3}+\frac{4x_2}{9}+\tau_9\right)
\end{aligned}
$$