Self-Similar Tilings of Nilpotent Lie Groups

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March 27, 2007

Research supported by the National Science Foundation, grant DMS-0354022, Research Experiences for Undergraduates at The University of Akron.

Research Experience for Undergraduates

Mentors at the University of Akron:

- Jeffrey Adler
- Judith Palagallo

Others in my REU group:

- Rebecca Black
- Lisa Lackney
- Phil Hudelson

Self-Similar Tiling Example

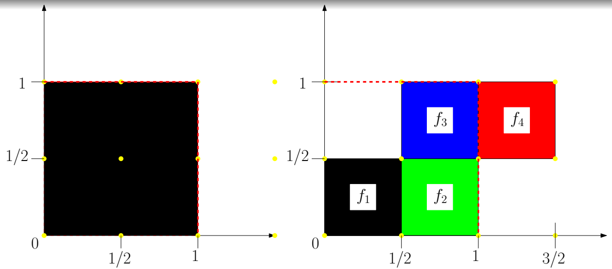
If

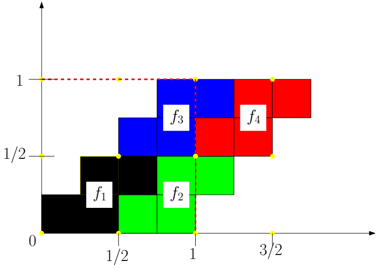
$$M^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

then define the following contraction mappings in $\mathbb{R}^2$

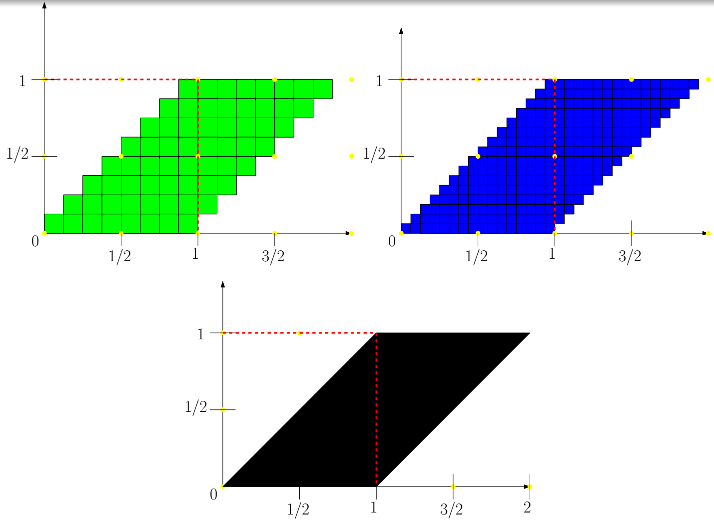
- $f_1(x_1, x_2) := M^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
- $f_2(x_1, x_2) := M^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$
- $f_3(x_1, x_2) := M^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$
- $f_4(x_1, x_2) := M^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$

Self-Similar Tiling Example





Self-Similar Tiling Example



Iterated Function System

Definition

The set of transformations used in the iteration process is called an iterated function system. The limit of this process is called an attractor.

In 1981, Hutchinson [4] developed the mathematical theory behind the covergence of this process.

Metric Spaces

Definition

A metric space is a 2-tuple (X, d) where X is a set and d is a metric on X; that is, a function

$$d \colon X \times X \to \mathbb{R},$$

such that

1. $d(x, y) \geq 0 \quad \text{(non-negativity)},$
2. $d(x, y) = 0 \text{ if and only if } x = y \quad \text{(identity)},$
3. $d(x, y) = d(y, x) \quad \text{(symmetry)},$
4. $d(x, z) \leq d(x, y) + d(y, z) \quad \text{(triangle inequality)}.$

Metric Spaces

Example

The Euclidean space $\mathbb{R}^2$ with the Euclidean metric $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ where

$$d(x, y) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Metric Spaces

Definition

A complete metric space is a metric space in which every Cauchy sequence is convergent.

Tilings

Definition

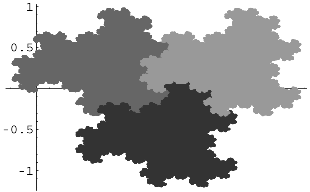
A tiling of a complete metric space X is a locally finite collection $\mathscr{T}$ of non-empty subsets of X such that:

1. For any $A \in \mathscr{T}$, $\mathrm{cl}\,(\mathrm{int}\,A) = A$.
2. For any distinct $A, B \in \mathscr{T}$, $\mathrm{int}\,A \cap \mathrm{int}\,B = \emptyset$.
3. $\bigcup_{A \in \mathscr{T}} A = X$.

Self-Similar Tilings

Definition

A self-similar tiling is a tiling composed of smaller tiles (rep tiles) of the same size, each being the same shape as the whole. We refer to a m-rep tile as an object that can be dissected into m smaller copies of itself.



Group

Definition

A group G is a set of elements together with a binary operation $(*)$ called the group operation that together satisfy the four fundamental properties:

- Closure: If $a, b \in G$, then the product $a * b \in G$.
- Associativity: For all $a, b, c \in G$, $(a * b) * c = a * (b * c)$.
- Identity: There is an identity element $\mathbf{0} \in G$ such that $\mathbf{0} * a = a * \mathbf{0} = a$ for every element $a \in G$.
- Inverse: There must be an inverse or reciprocal of each element. Therefore, G must contain an element $b = a^{-1}$ such that $a * b = b * a = \mathbf{0}$ for each element $a \in G$.

Abelian Group

Definition

An abelian group is a group G for which the elements commute; that is, for all elements $a, b \in G, ab = ba$.

Commutator

Definition

For a group G, the commutator of two subgroups $A, B \subseteq G$ is the subgroup $[A, B]$ where

$$[A, B] = \left\{ aba^{-1}b^{-1} : a \in A, b \in B \right\}.$$

The commutator $[G, G]$ measures the extent to which the group operation on G fails to be commutative.

Nilpotent Group

Definition

Let G be a group, and let $A_0, A_1, A_2, \ldots$ be a sequence of groups with $A_0 = G$ and $A_{i+1} = [G, A_i]$. G is nilpotent if for some n, A_n is trivial.

Example

1. $n \times n$ upper triangular matrices with 1s on the diagonal.
2. Any subgroup of Item 1.
3. Any abelian group.

Lie Group

Definition

A Lie group is a smooth manifold obeying the group properties and that satisfies the additional condition that the group operations are differentiable.

Why Nipotent Lie Groups?

- Every point in a smooth manifold has a neighborhood which resembles Euclidean space.
- Nilpotent Lie groups have a natural automorphic dilation structure.
- Nilpotent Lie groups often have discrete cocompact subgroups (lattices).

Morphisms

Definition

An isomorphism is a bijective map f such that both f and f^{-1} are structure-preserving mappings.

Example

Isomorphisms of two-dimensional Euclidean space $\mathbb{R}^2$ are special cases of linear transformations such as:

- Rotation : $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ Reflection : $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.
- Expansion by k: $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ Contraction by k: $\begin{bmatrix} 1/k & 0 \\ 0 & 1/k \end{bmatrix}$.

Morphisms

Definition

An automorphism Φ is an isomorphism from a mathematical object to itself. For a group G with elements $\alpha, \beta \in G$:

$$\Phi(\alpha * \beta) = \Phi(\alpha) * \Phi(\beta).$$

Expansions

Let (G, d) be a locally compact Hausdorff space.

Definition

A function $\Phi\colon G \to G$ is an expansive map if there exists an $r \in \mathbb{R}$, $r > 1$, such that for $\alpha, \beta \in G$,

$$d\big(\Phi(\alpha), \Phi(\beta)\big) \geq r \cdot d\,(\alpha, \beta)\,.$$

Lattices

Definition

A lattice $\Gamma \subset G$ is a cocompact discrete subgroup of G.

Definition

For $\Gamma \subset G$, Γ is cocompact in G if for a compact set $K \subset G$

$$\bigcup_{\gamma \in \Gamma} (\gamma * K) = G.$$

Example

$\mathbb{Z}^n$ is cocompact in $\mathbb{R}^n$.

Notation

Until otherwise noted[3]:

- Let G be a nilpotent Lie group with group operation $*$. Then G is a locally compact Hausdorff topological group, and has a right-invariant Riemannian metric [6].
- Let $\Gamma \subset G$ be a lattice.
- Let Φ be a continuous expansive automorphism of G such that $\Phi(\Gamma) \subseteq \Gamma$.

Residue System

Definition

A family $\{y_1, \ldots, y_m\} \subset \Gamma$ is a residue system or a complete set of coset representatives of Φ, if $y_1 = \mathbf{0}$ and

$$\Gamma = \bigcup^{*} \{y_i * \Phi(\Gamma) : i = 1, \ldots, m\}.$$

[3] G is a nilpotent Lie group, $\Gamma \subset G$ is a lattice, Φ is a continuous expansive automorphism of G

Self-Similar (Revisited)

Definition ([4])

Define f_i as $f_i(\alpha) = \Phi^{-1}(\alpha) * y_i, i = 1, \dots, m$. A compact set $\mathbf{A} \neq \emptyset$ is self-similar with respect to $f_1, \dots, f_m$ if

$$\mathbf{A} = f_1(\mathbf{A}) \cup \cdots \cup f_m(\mathbf{A}).$$

[3] G is a nilpotent Lie group, $\Gamma \subset G$ is a lattice, Φ is a continuous expansive automorphism of G

The Main Result

Let m equal the cardinality of $\Gamma/\Phi(\Gamma)$. Fix a right Haar measure μ on G.

Theorem

If Φ is a continuous expansive automorphism of G and $\{y_1, \ldots, y_m\}$ is a residue system of Φ, then there is a unique m-rep tile $\mathbf{A}_1$ such that

$$\Phi\left(\mathbf{A}_1\right) = \mathbf{A}_1 \cup \cdots \cup \mathbf{A}_m \text{ with } \mathbf{A}_i = \mathbf{A}_1 * y_i.$$

[3] G is a nilpotent Lie group, $\Gamma \subset G$ is a lattice, Φ is a continuous expansive automorphism of G

Definition

Definition

Let the Heisenberg group be defined by

$$H^{2n+1}(\mathbb{R}) = \left\{(\mathbf{x}, z) : \mathbf{x} \in \mathbb{R}^{2n}, z \in \mathbb{R}\right\}$$

with the group law

$$(\mathbf{x}, z) * \left(\mathbf{x}', z'\right) = \left(\mathbf{x} + \mathbf{x}', z + z' + B\left(\mathbf{x}, \mathbf{x}'\right)\right)$$

where B is a nondegenerate skew-symmetric bilinear form on $\mathbb{R}^{2n}$. For notational purposes let $H = H^{2n+1}(\mathbb{R})$.

Norm

Definition

We define the norm by

$$\left|(\mathbf{x}, z)\right|_H = \left(\|\mathbf{x}\|^4 + |z|^2\right)^{1/4},$$

where $||\cdot||$ is the standard Euclidean norm on $\mathbb{R}^{2n}$.

Automorphisms

Theorem

Any automorphism $\Phi\colon H \to H$ *is of the form* $\Phi\big((\mathbf{x}, z)\big) = (M\mathbf{x}, \omega(\mathbf{x}) + az)$, *where* $M \in \mathrm{GSp}(2n)$ *such that* $B(M\mathbf{v}, M\mathbf{w}) = aB(\mathbf{v}, \mathbf{w})$ *for all* $\mathbf{v}, \mathbf{w} \in \mathbb{R}^{2n}$, *and where* $\omega\colon \mathbb{R}^{2n} \to \mathbb{R}$ *is a linear tranformation.*

Fact

For $M \in \mathrm{GSp}(2n)$ we have the relationship $a^n = \det M$ where M uniquely determines a.

Expansion Maps

Theorem

An automorphism $\Phi\colon H \to H$ *is an expansion map if and only if* $\Phi\big((\mathbf{x}, z)\big) = (M\mathbf{x}, az)$ *where, for some* $0 < c < 1$*,* $\left\|M^{-1}\mathbf{x}\right\| \leq c\left\|\mathbf{x}\right\|$ *and* $\left|a^{-1}\right| \leq c^2$*.*

Group Law for Examples

Consider two examples of fractal tilings on the Heisenberg group, with the group law given by

$$(\mathbf{x}, z) * (\mathbf{x}', z') = \left(\mathbf{x} + \mathbf{x}', z + z' + \frac{1}{2}\left(x_1 x_2' - x_1' x_2\right)\right).$$

Twin Dragon

Example

The first example we will consider is the tiling generated by the automorphism

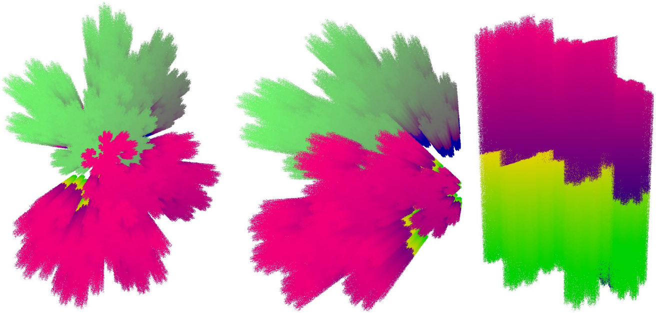
$$\Phi(\mathbf{x}, z) = (M\mathbf{x}, 2z)\,, \text{ where } M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

M is the same matrix that generates a twindragon tiling in $\mathbb{R}^2$. One possible set of residue vectors for this twindragon in the plane is $\{(0,0),(0,1)\}$. Therefore, one possible selection of the four residues is

$$\left\{(0,0,0),(0,1,0),\left(0,0,\frac{1}{2}\right),\left(0,1,\frac{1}{2}\right)\right\}.$$

Twin Dragon

These residue vectors are found by direct application of the previous proposition and allow us to generate the figure below.



Terdragon

Example

The second example of a tiling in H is the one generated by the automorphism

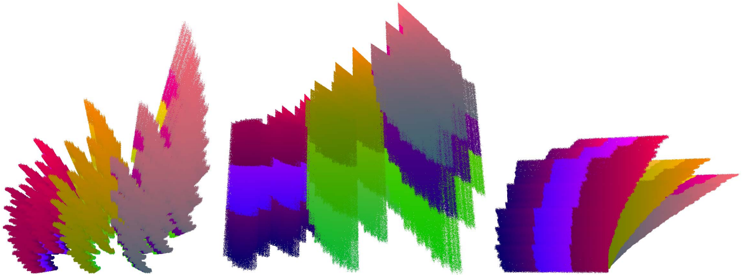
$$\Phi(\mathbf{x}, z) = (M\mathbf{x}, 3z)\,, \text{ where } M = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}.$$

M is the same matrix that generates a terdragon in $\mathbb{R}^2$. One set of residues for this tiling in the plane is $\left\{(0,0),(1,0),(2,0)\right\}$. Then, one possible set of residues for Φ is

$$\left\{(0,0,0),(1,0,0),(2,0,0),\left(0,0,\tfrac{1}{2}\right),\left(1,0,\tfrac{1}{2}\right),\left(2,0,\tfrac{1}{2}\right),(0,0,1),(1,0,1),(2,0,1)\right\}.$$

Terdragon

By following the selection method of residues from the proposition, we generate the tiling shown in the figure below.



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