Connectivity in Semi-Algebraic Sets

by
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Ph.D. defense for partial fulfillment of the requirements for the Degree of Doctor of Philosophy at North Carolina State University

Applied Mathematics

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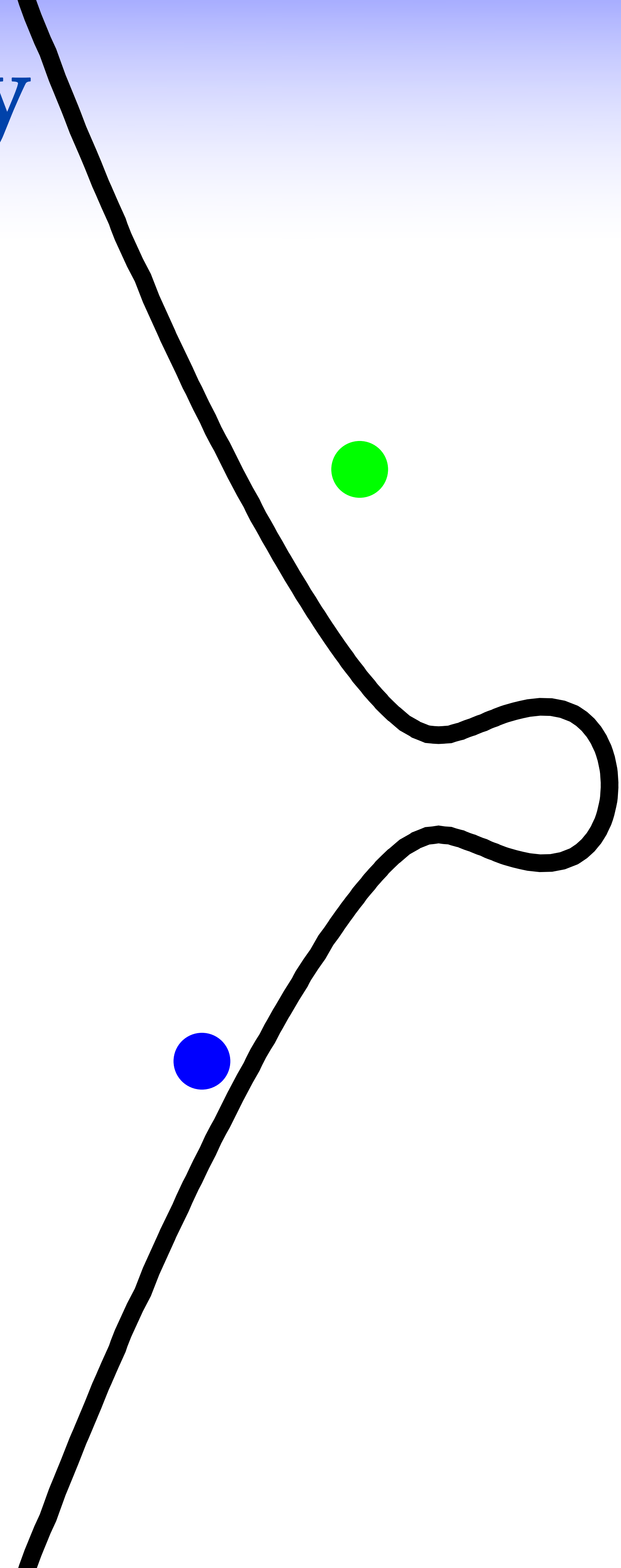
Problem: Connectivity

Problem: Connectivity

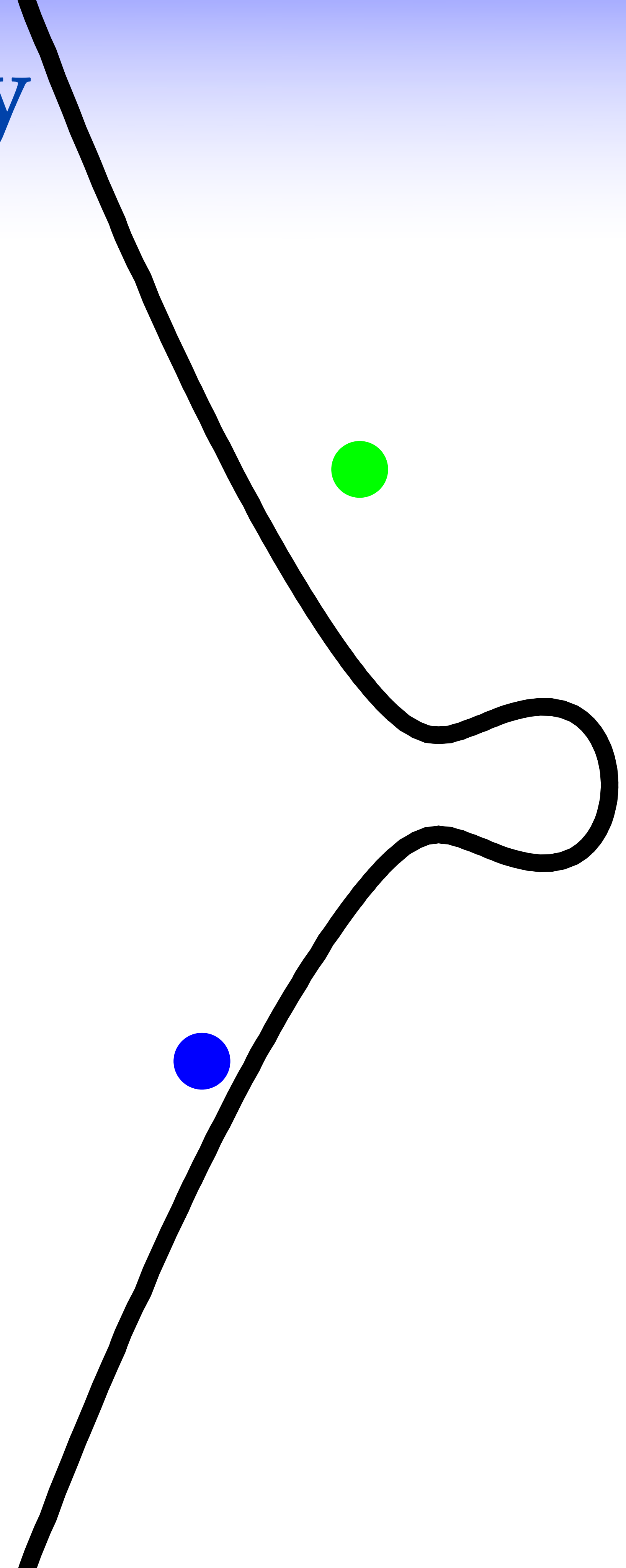
Can I connect and using a continuous path that does not cross the black curve?



Problem: Connectivity

Can I connect ● and ● using a continuous path that does not cross the black curve?

NO



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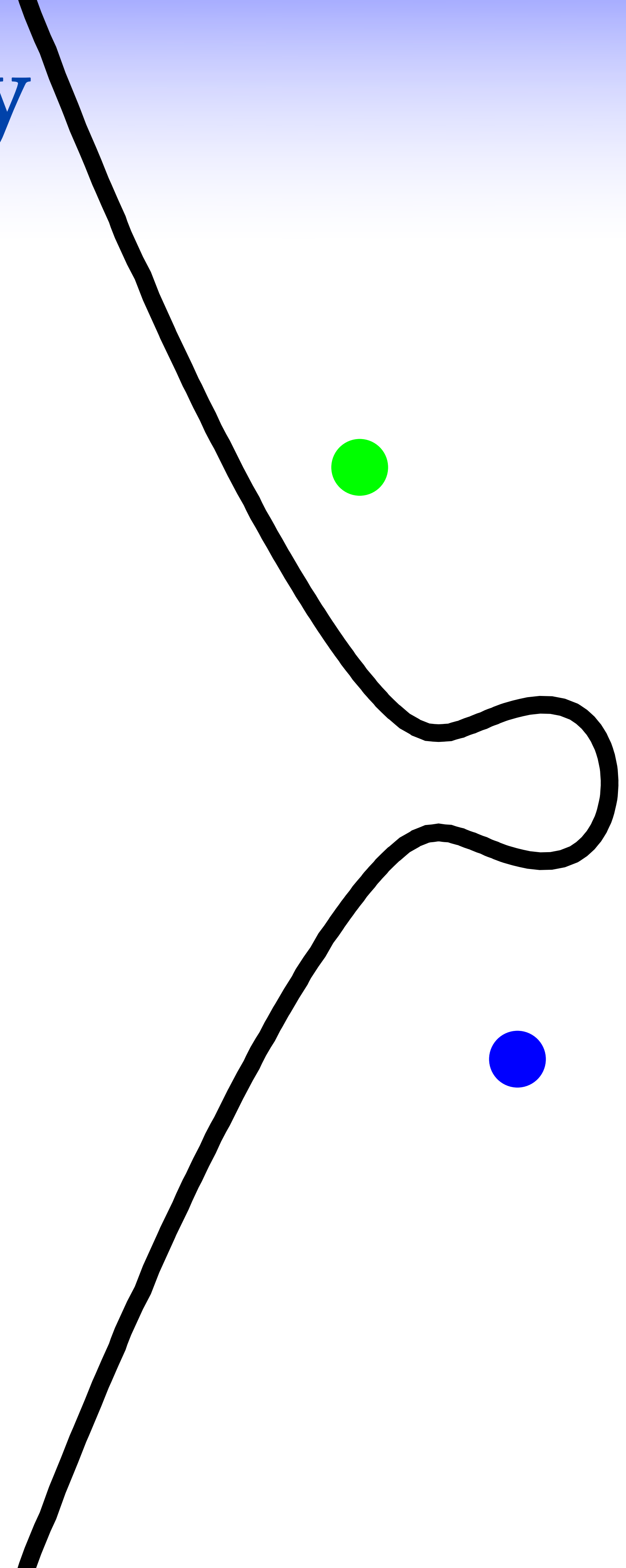

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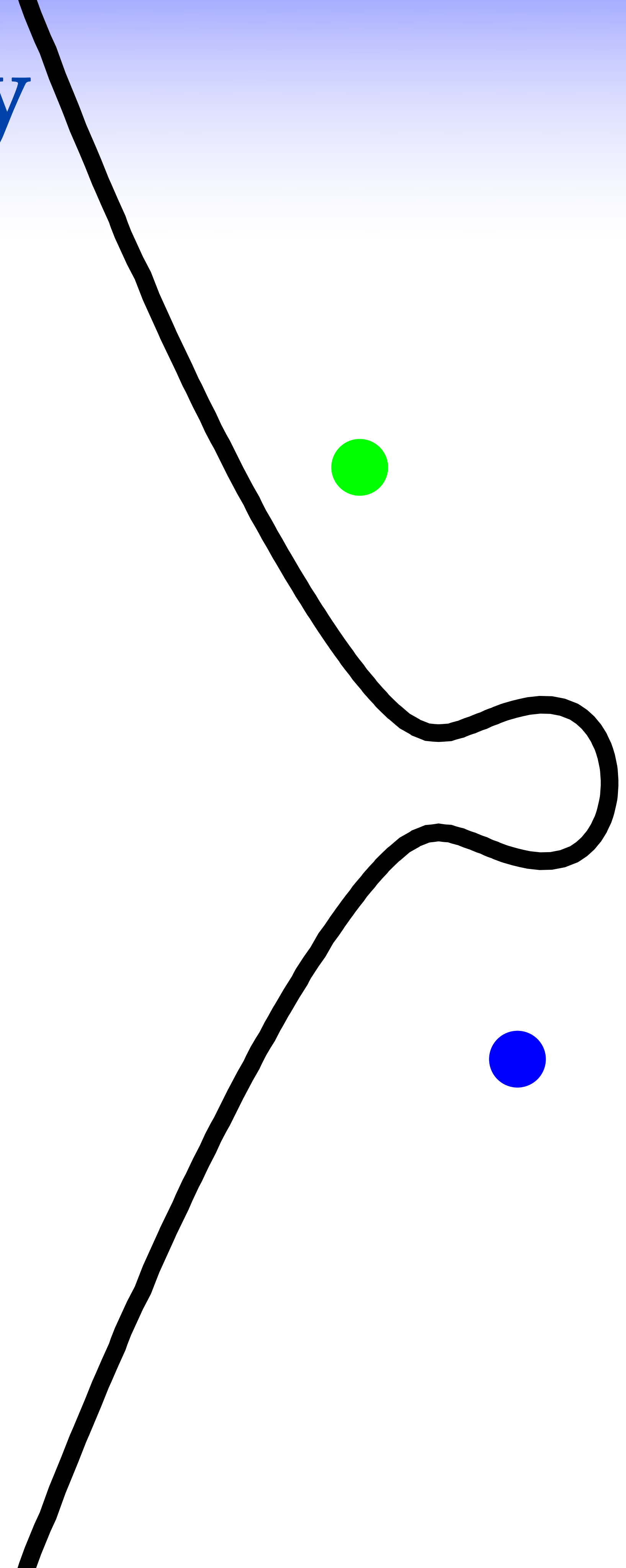

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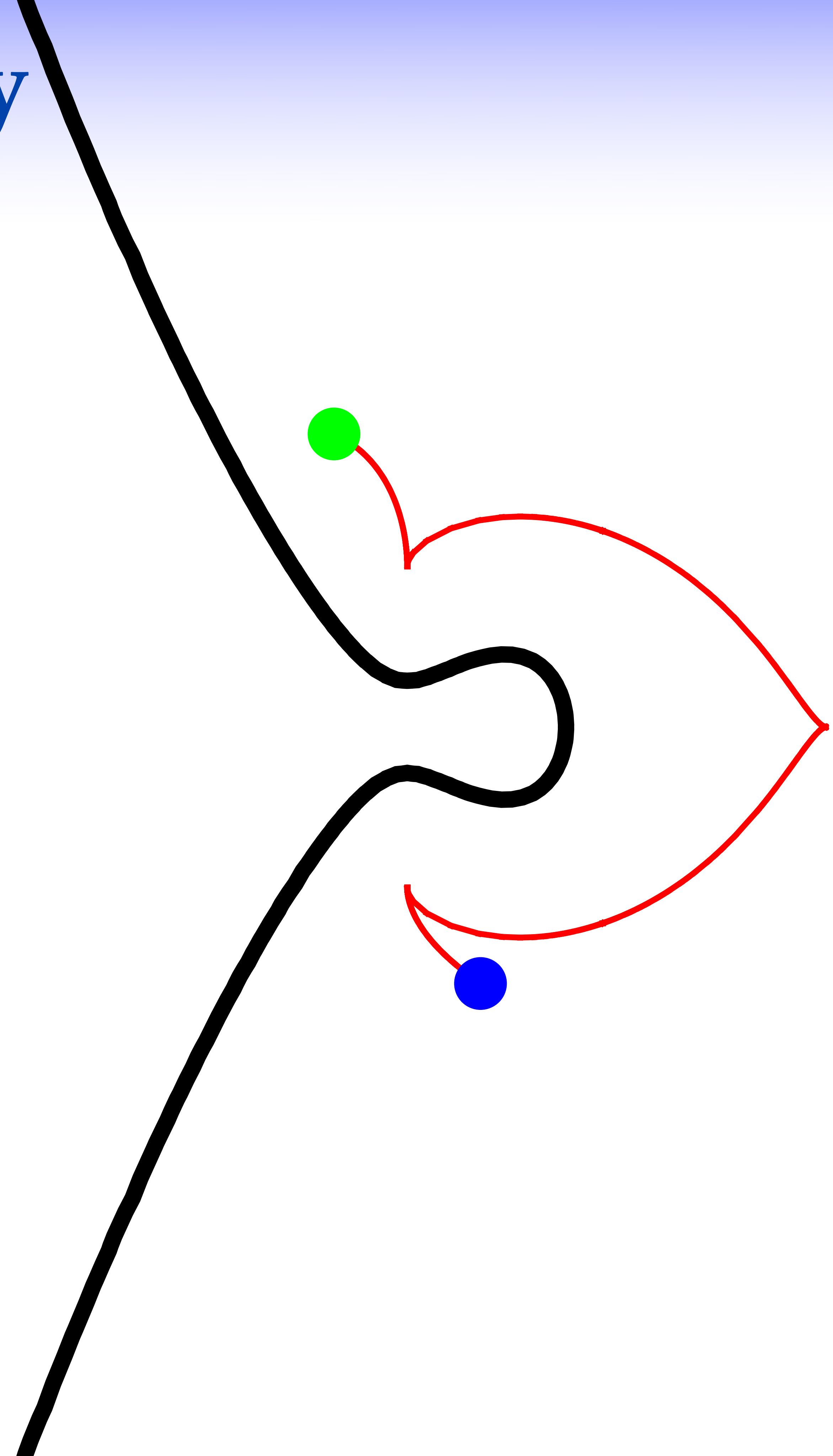

Problem: Connectivity

Can I connect ● and ● using a continuous path that does not cross the black curve?

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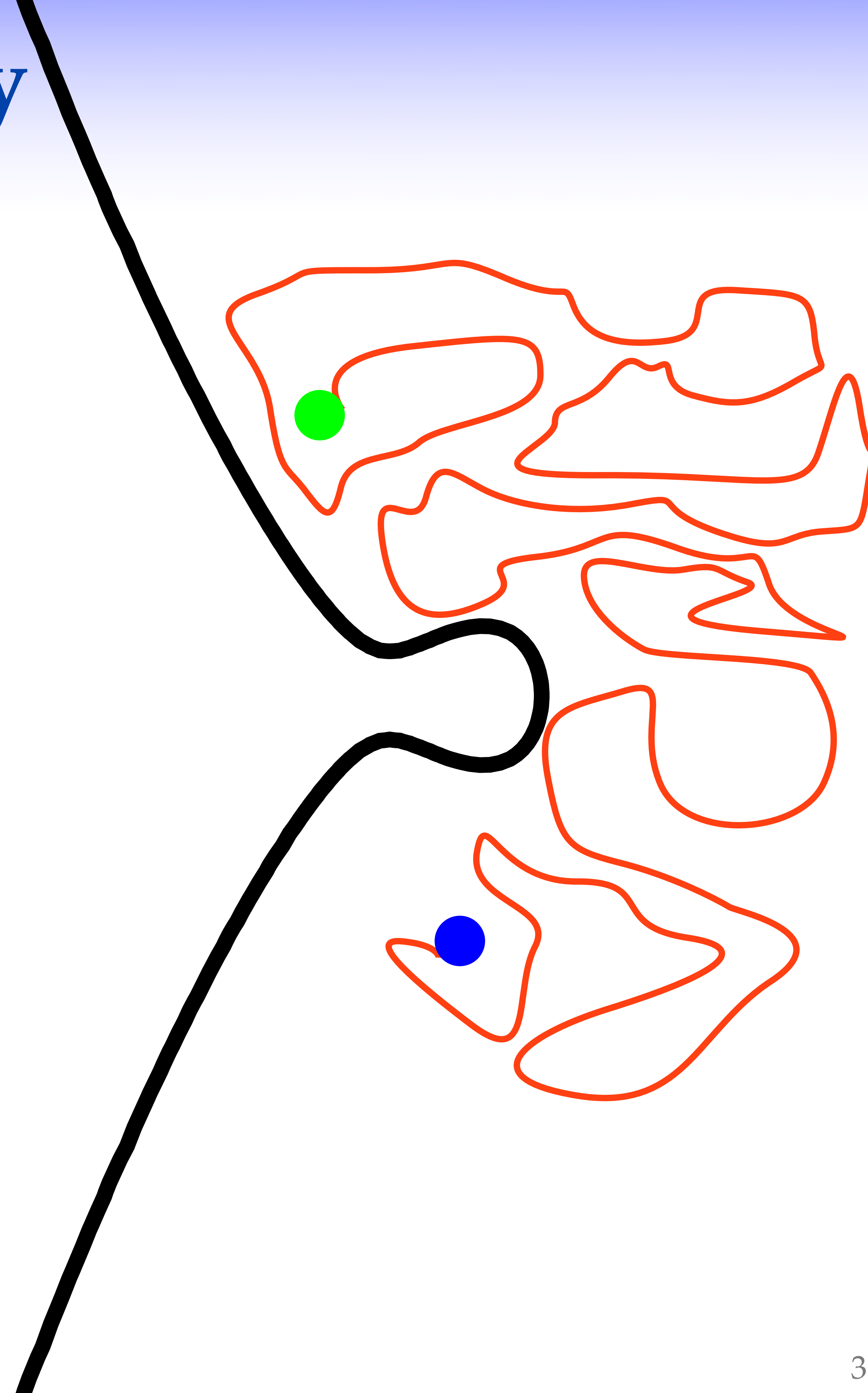
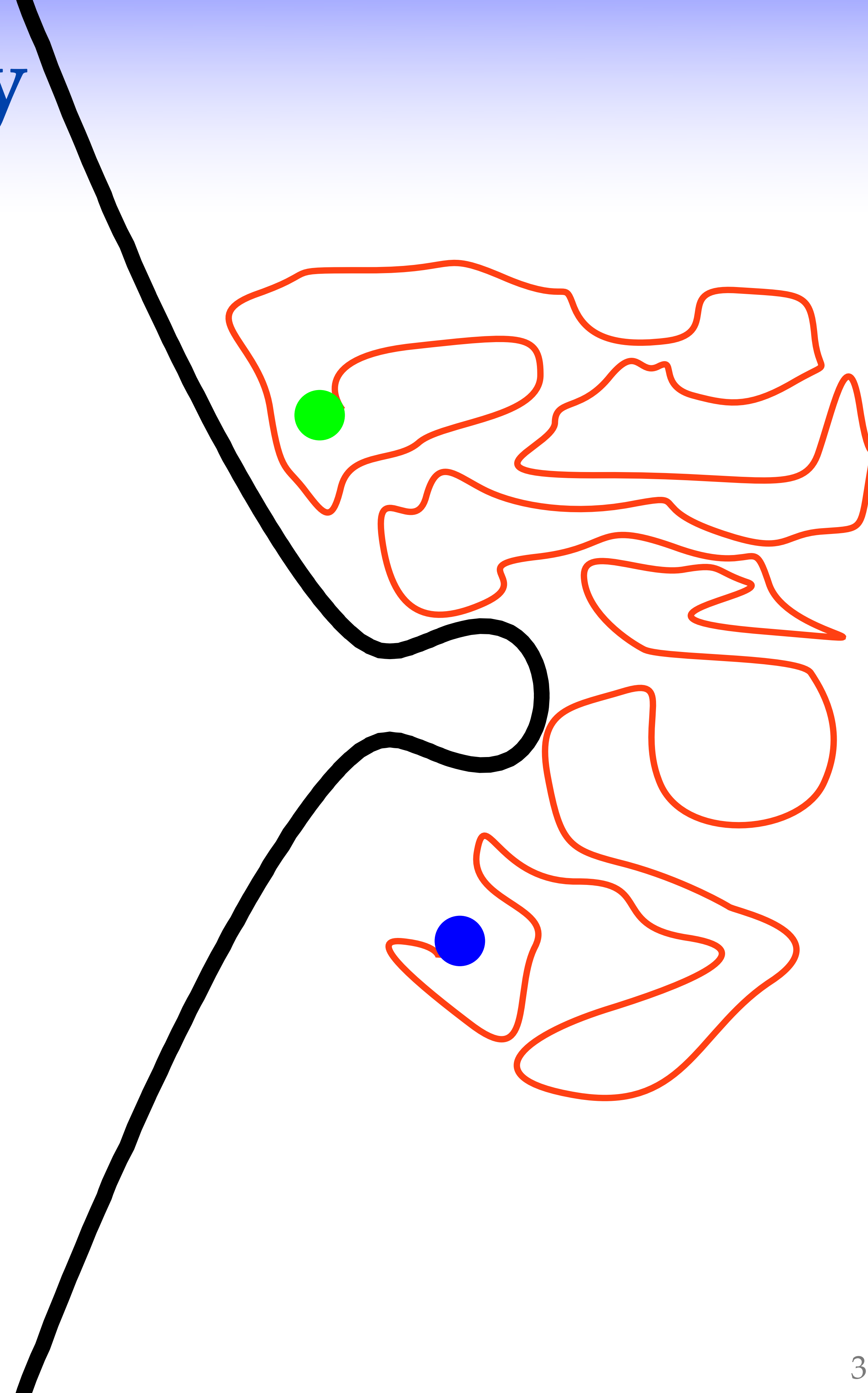


Problem: Connectivity

Can I connect ● and ● using a continuous path that does not cross the black curve?

YES

Do ● and ● lie in a same connected region?



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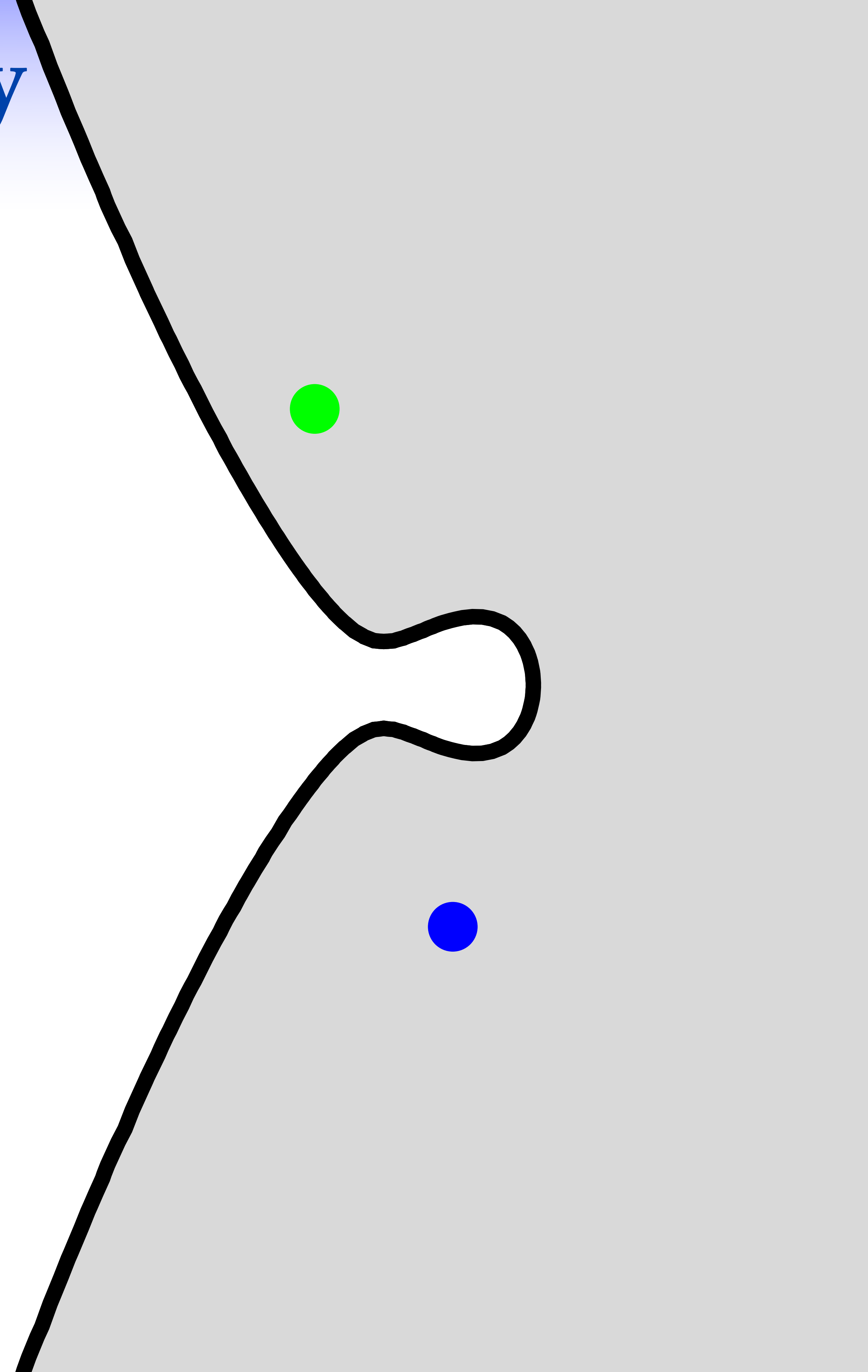

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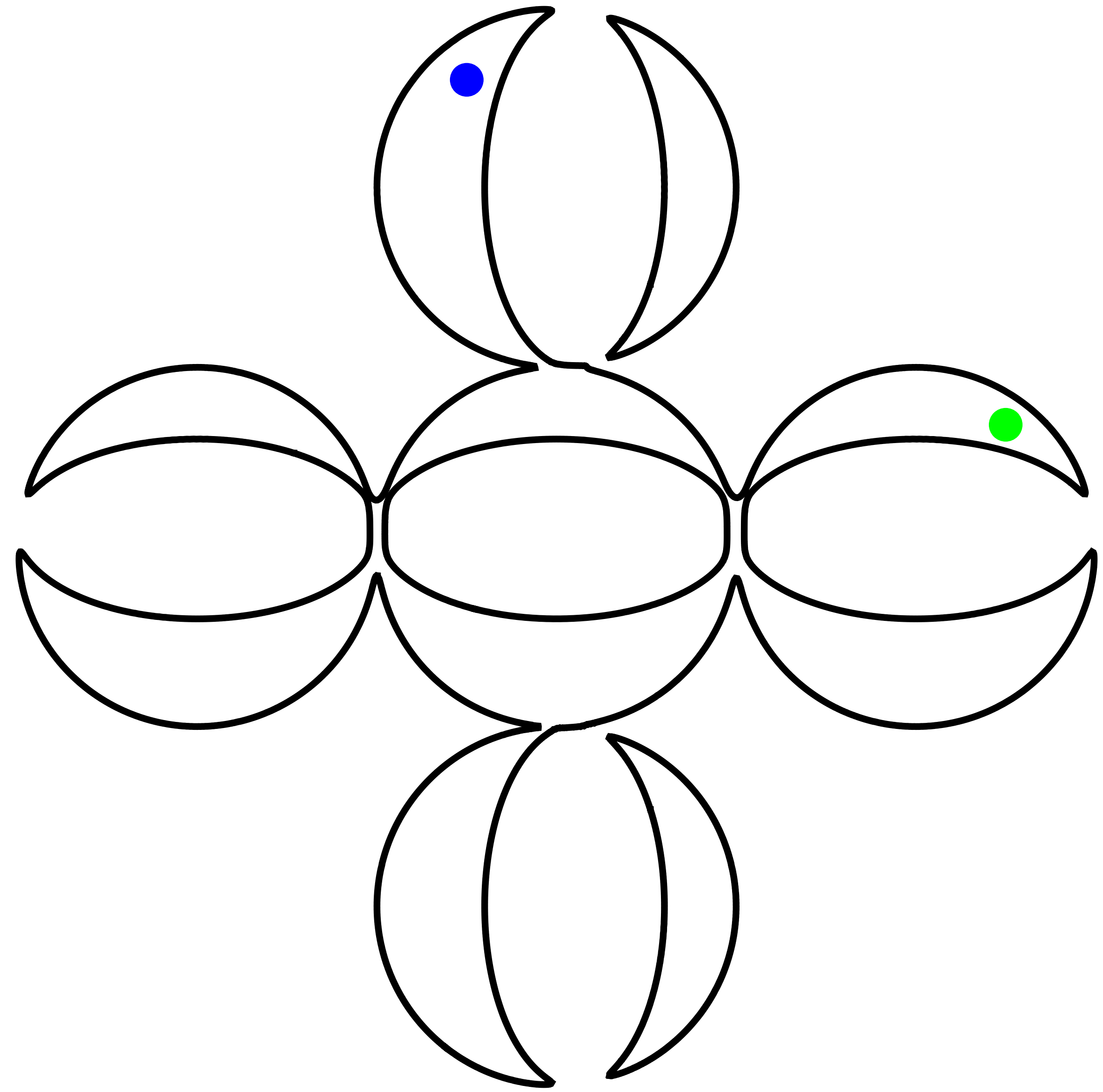
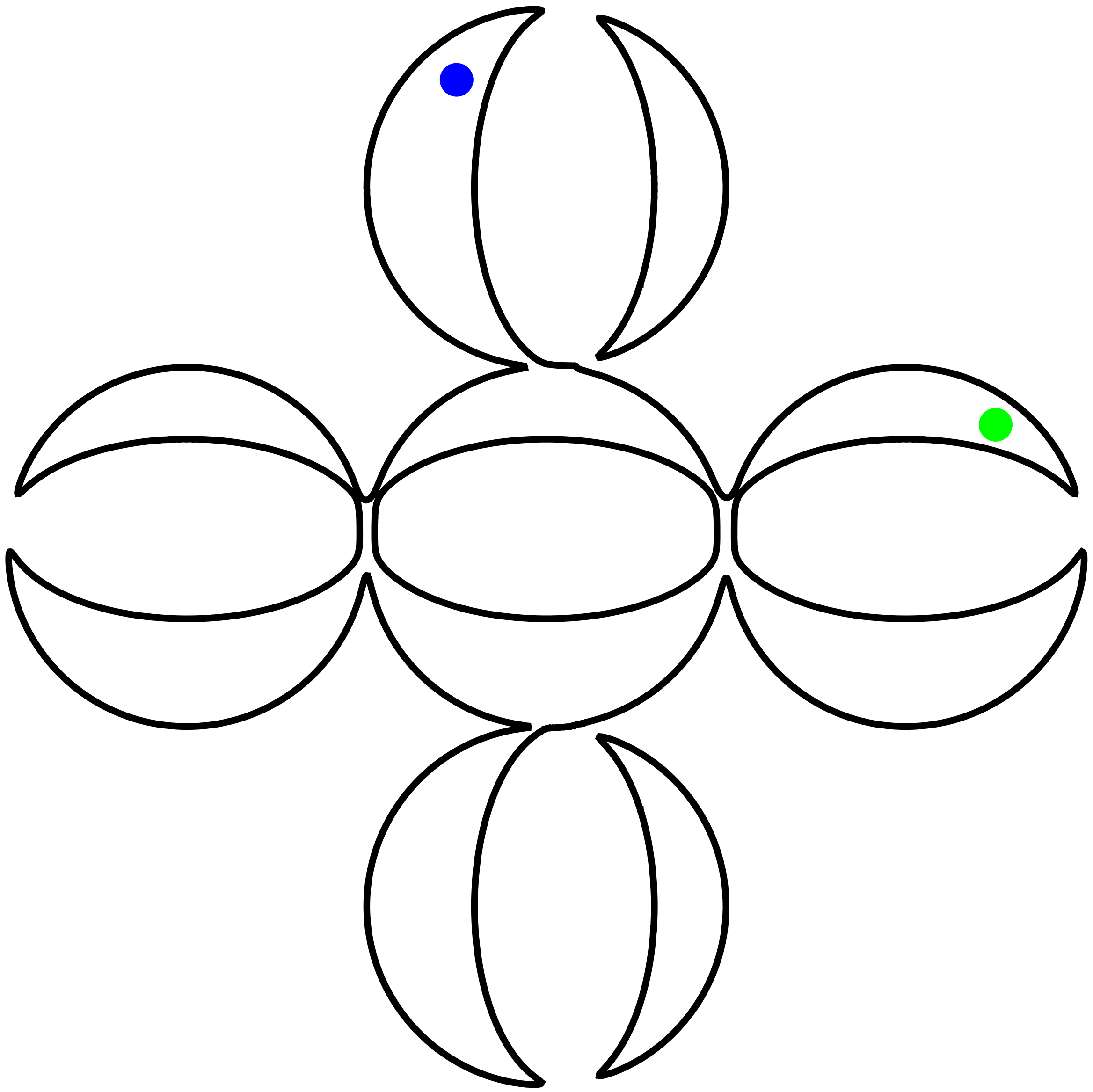


Problem: Connectivity

Can I connect ● *and* ● *using a continuous path that does not cross the black curve?*

YES

Do ● *and* ● *lie in a same connected region?*



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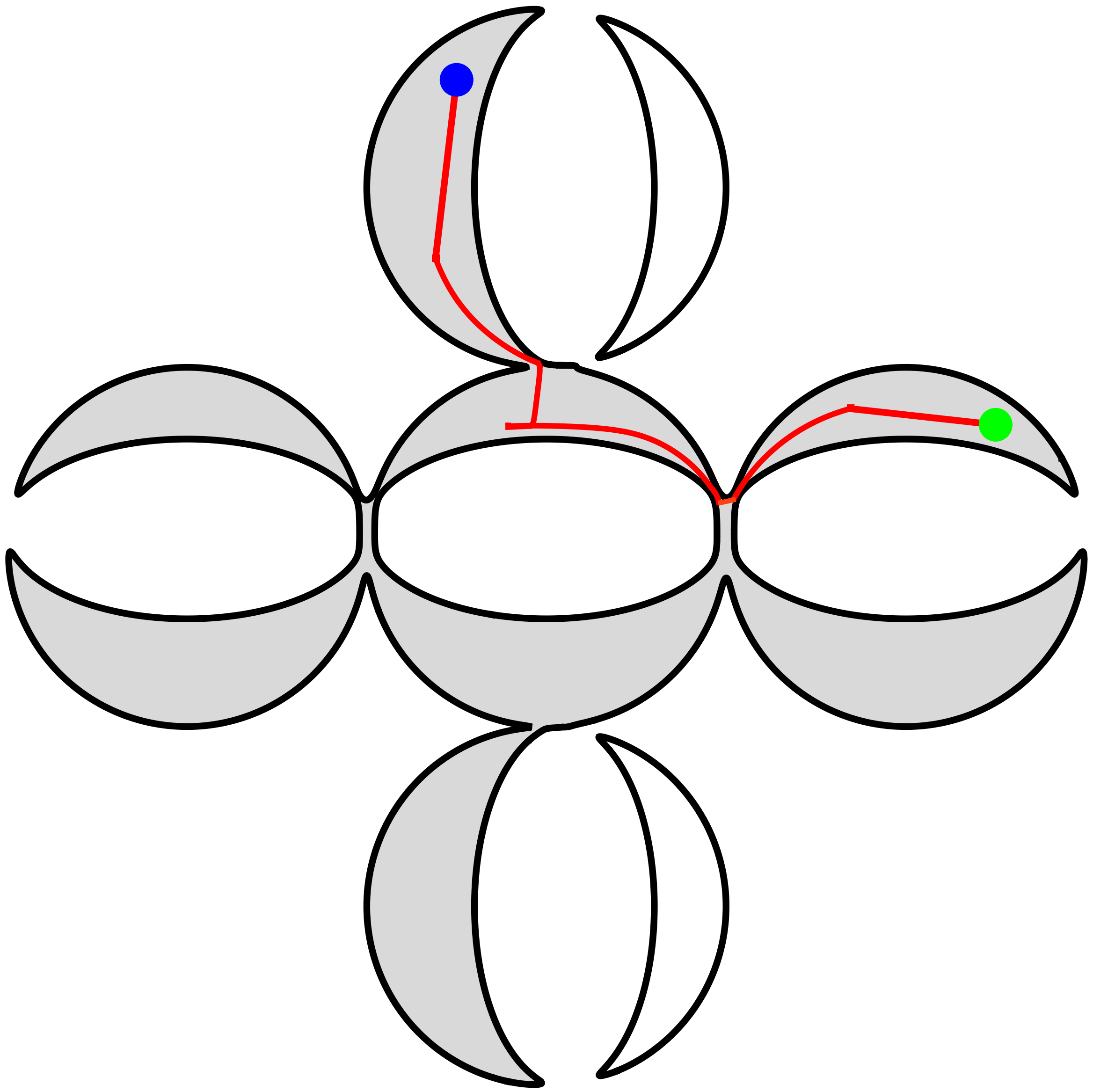

Problem: Connectivity

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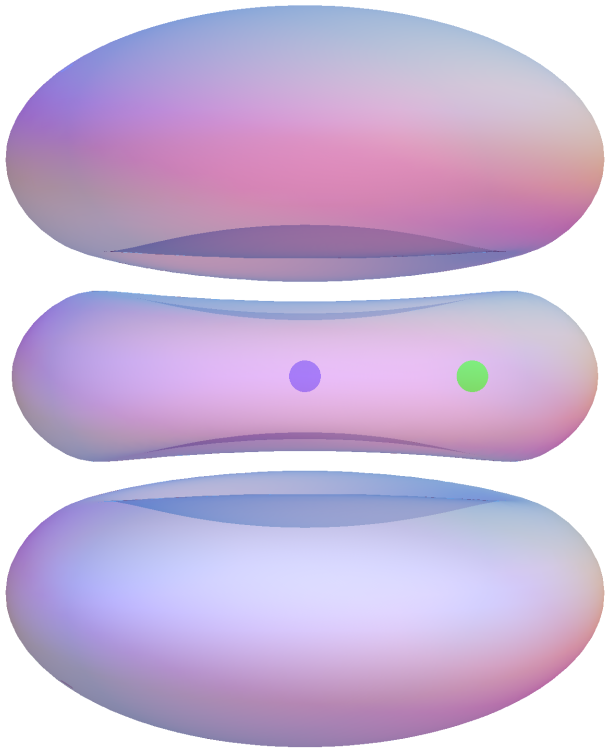
Can I connect ● and ● using a continuous path that does not cross the surface?



Do ● and ● lie in a same connected region?

Problem: Connectivity

Can I connect ● and ● using a continuous path that does not cross the surface?

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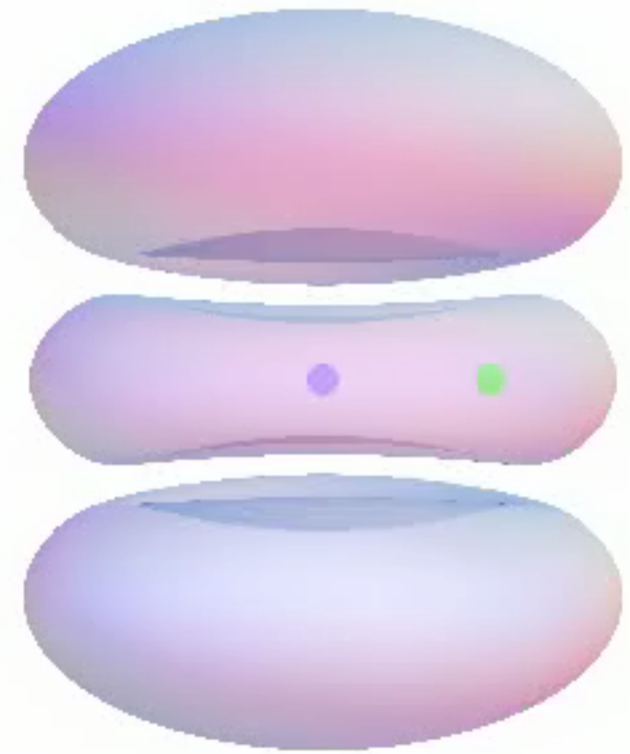

Do and lie in a same connected region?

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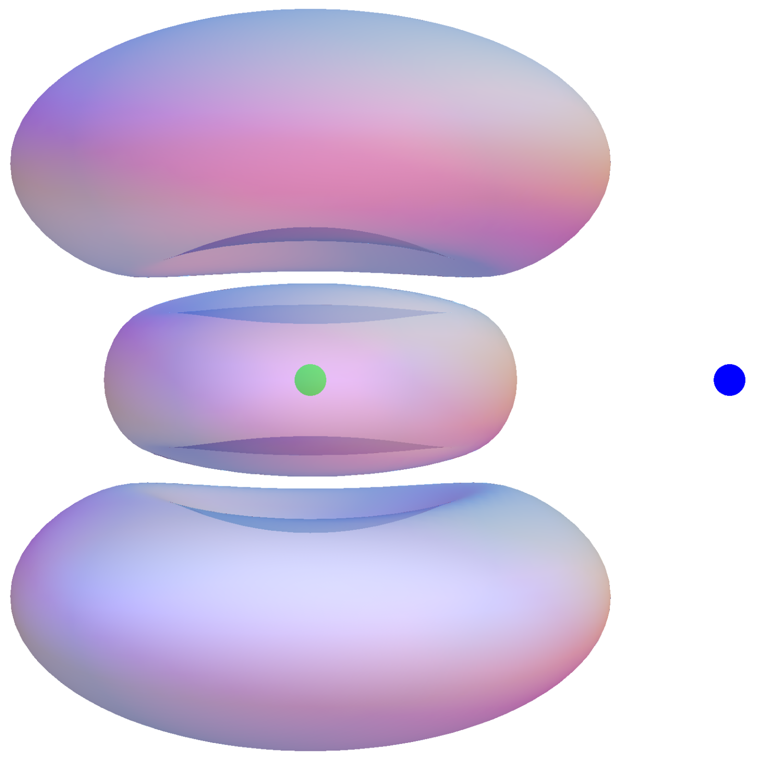

Problem: Connectivity

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$$f = 10x_1^3 - 10x_1^2 + 10x_2^2 - 1$$

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Do $\left(-\frac{3}{2}, -\frac{7}{4}\right)$ *and* $\left(-\frac{1}{2}, 2\right)$ *lie in a same connected region of* $\{f \neq 0\}$?

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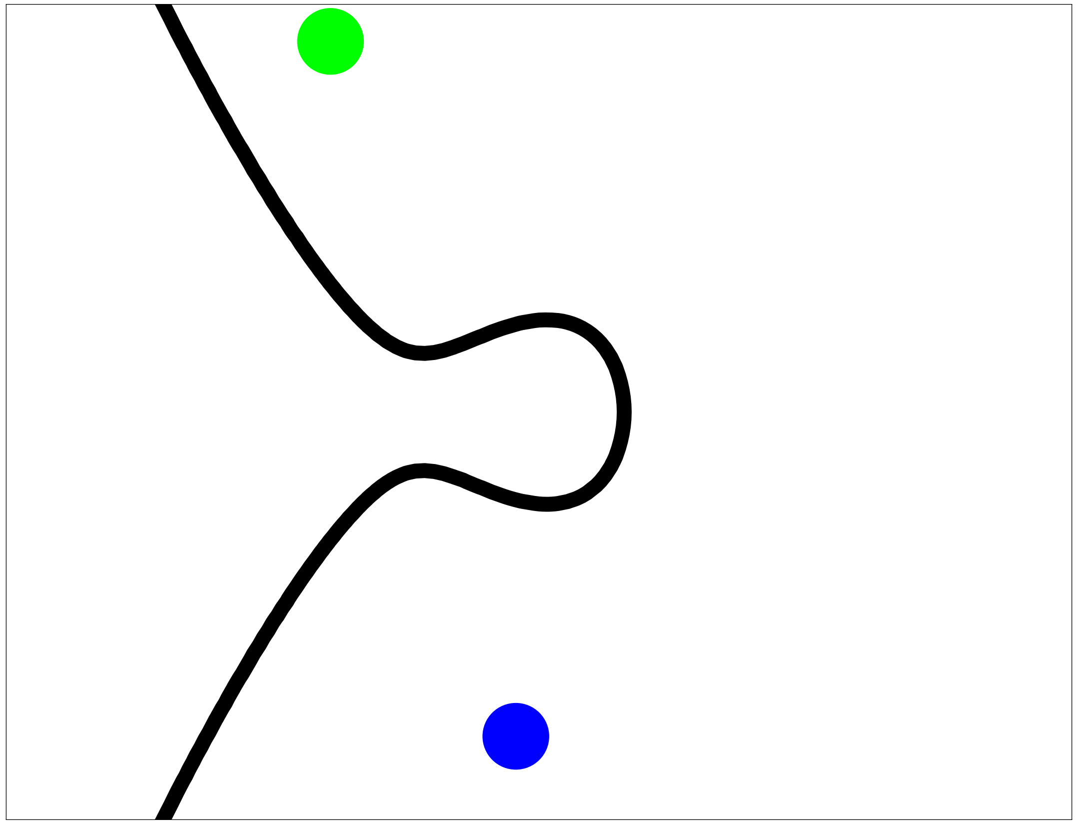

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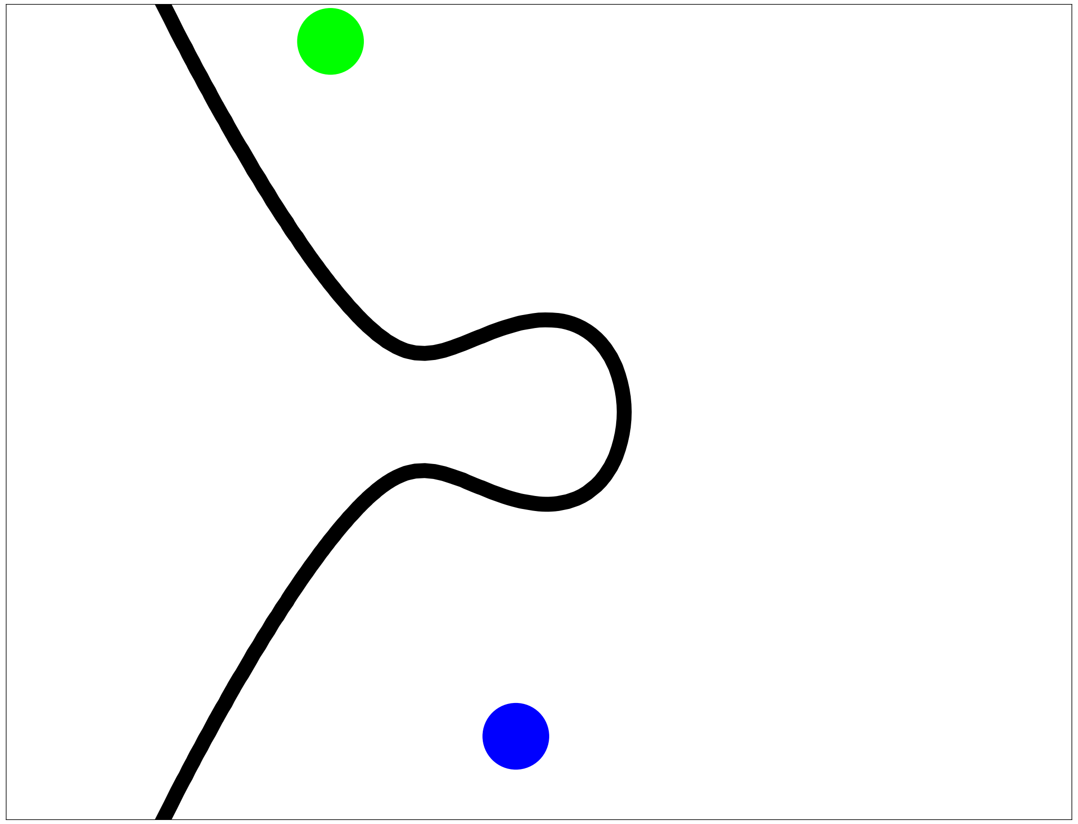


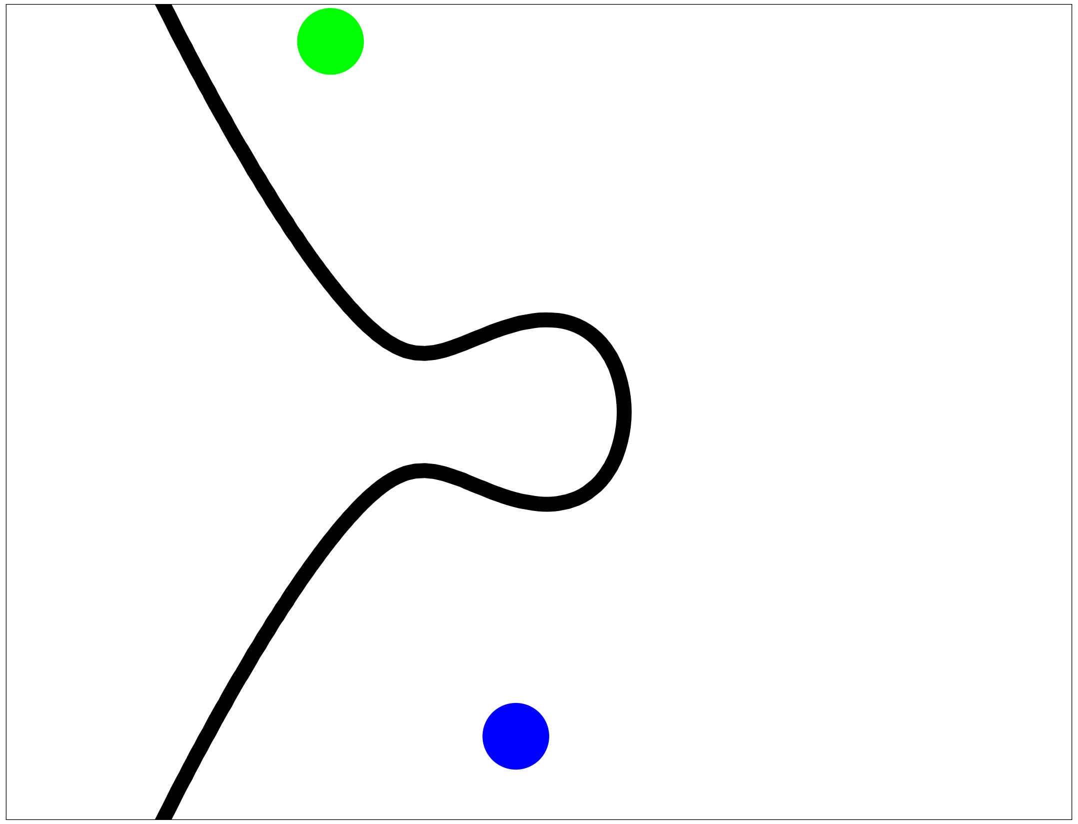
Problem: Connectivity

Input

$f \in \mathbb{Z}[\mathsf{x}_1, \ldots, \mathsf{x}_n]$, squarefree,
finitely many singular points,
$n \geq 2$, $\deg(f) \geq 1$

$p, q \in \mathbb{Q}^n \cap \{f \neq 0\}$

$$f = 10x_1^3 - 10x_1^2 + 10x_2^2 - 1$$



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Output

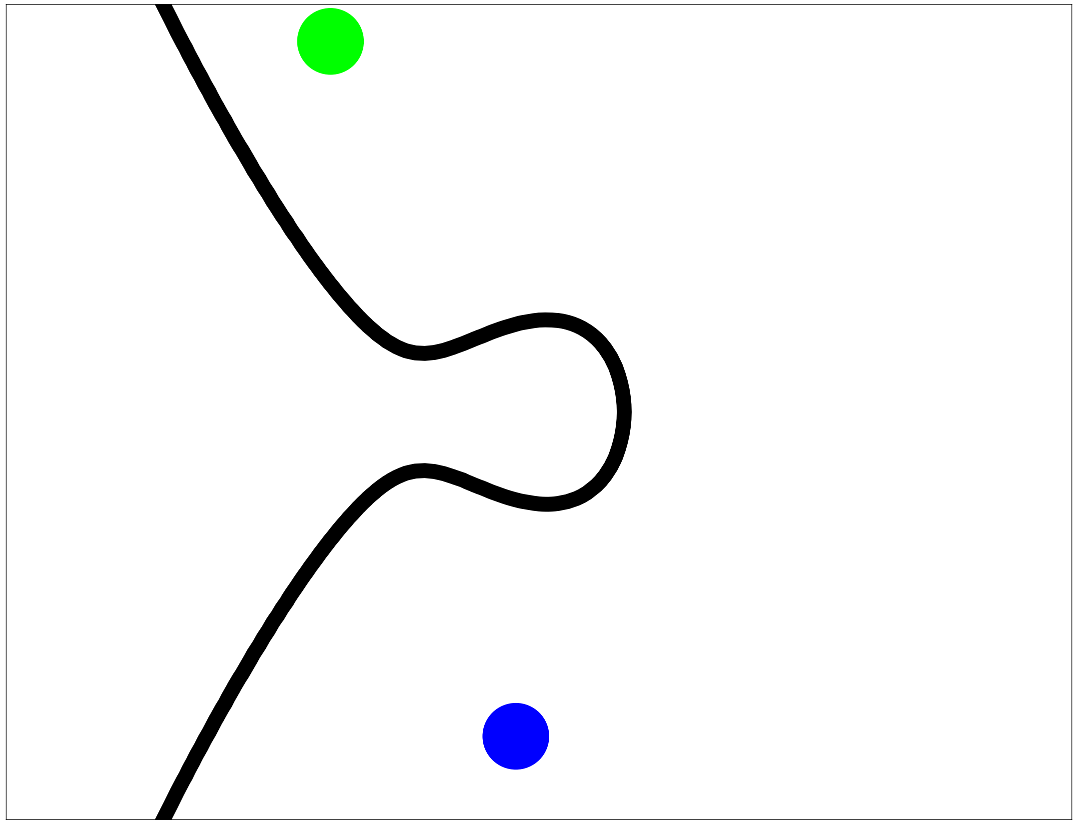
True

if p, q are in a same semi-algebraically connected component of $\{f \neq 0\}$

False

otherwise

$f = 10x_1^3 - 10x_1^2 + 10x_2^2 - 1$



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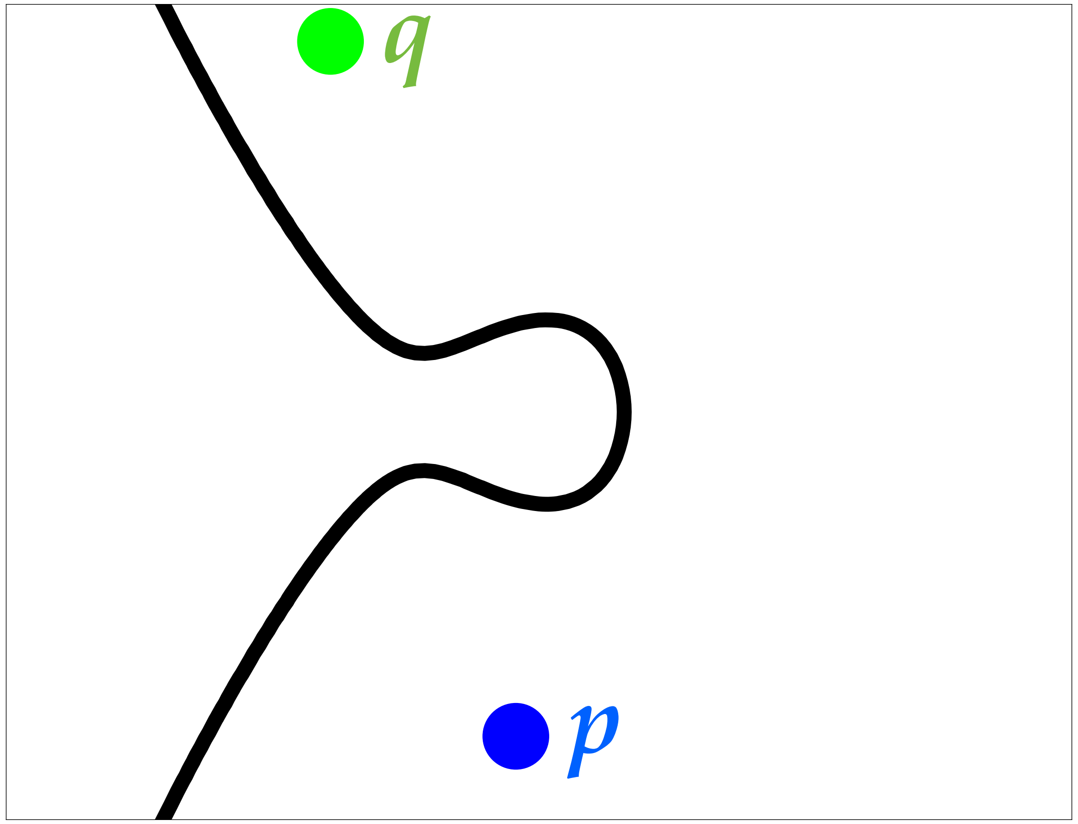

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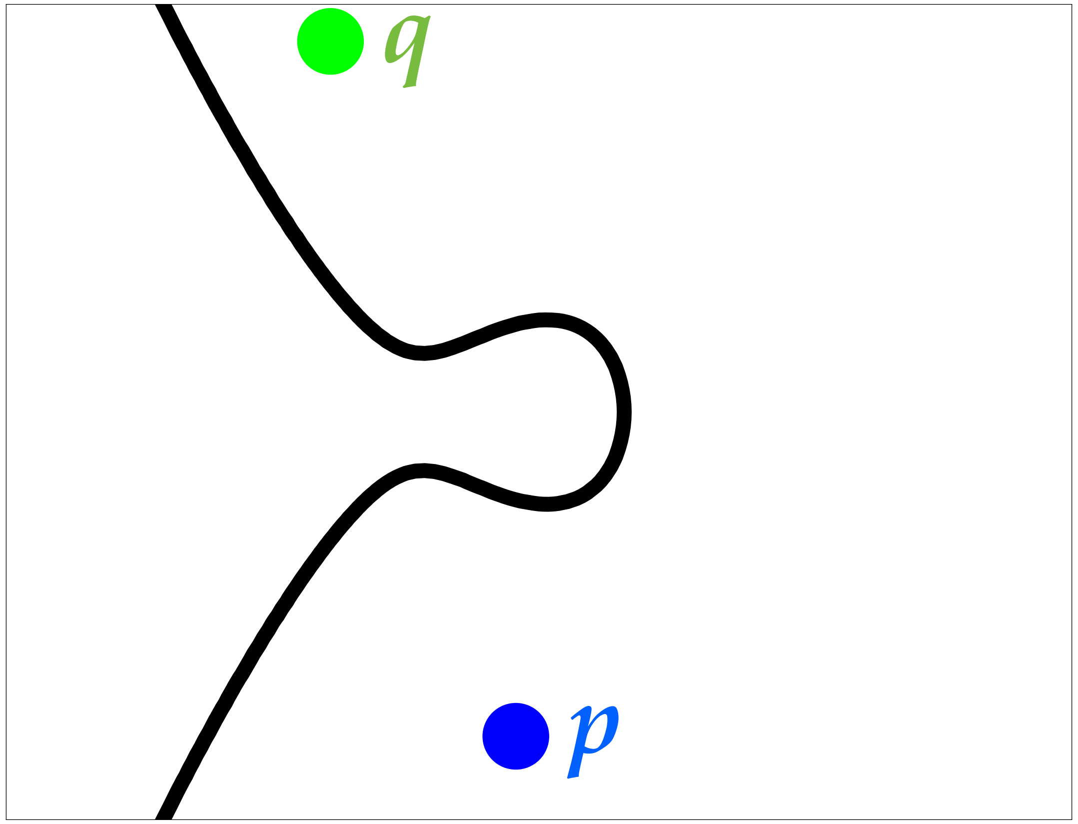

True

Motivations and Previous Works

- Fundamental in computational real algebraic geometry.
- Many important applications in science and engineering.
- Previous work:

1975	Collins
1983	Schwartz, Sharir
1984	Arnon, Collins, McCallum
1987	Canny, Roy
1988	Arnon, McCallum
1989	Alonso, Raimondo
1992	Feng, Grigor'ev, Vorobjov
1993	Hong
1994	Heintz, Roy, Solerno
1996	Basu, Pollack, Roy
2010	Hong, Quinn
2011	Safey El Din, Schost
2012	Basu, Roy, Safey El Din, Schost
2013	Basu, Roy

Method: Overview

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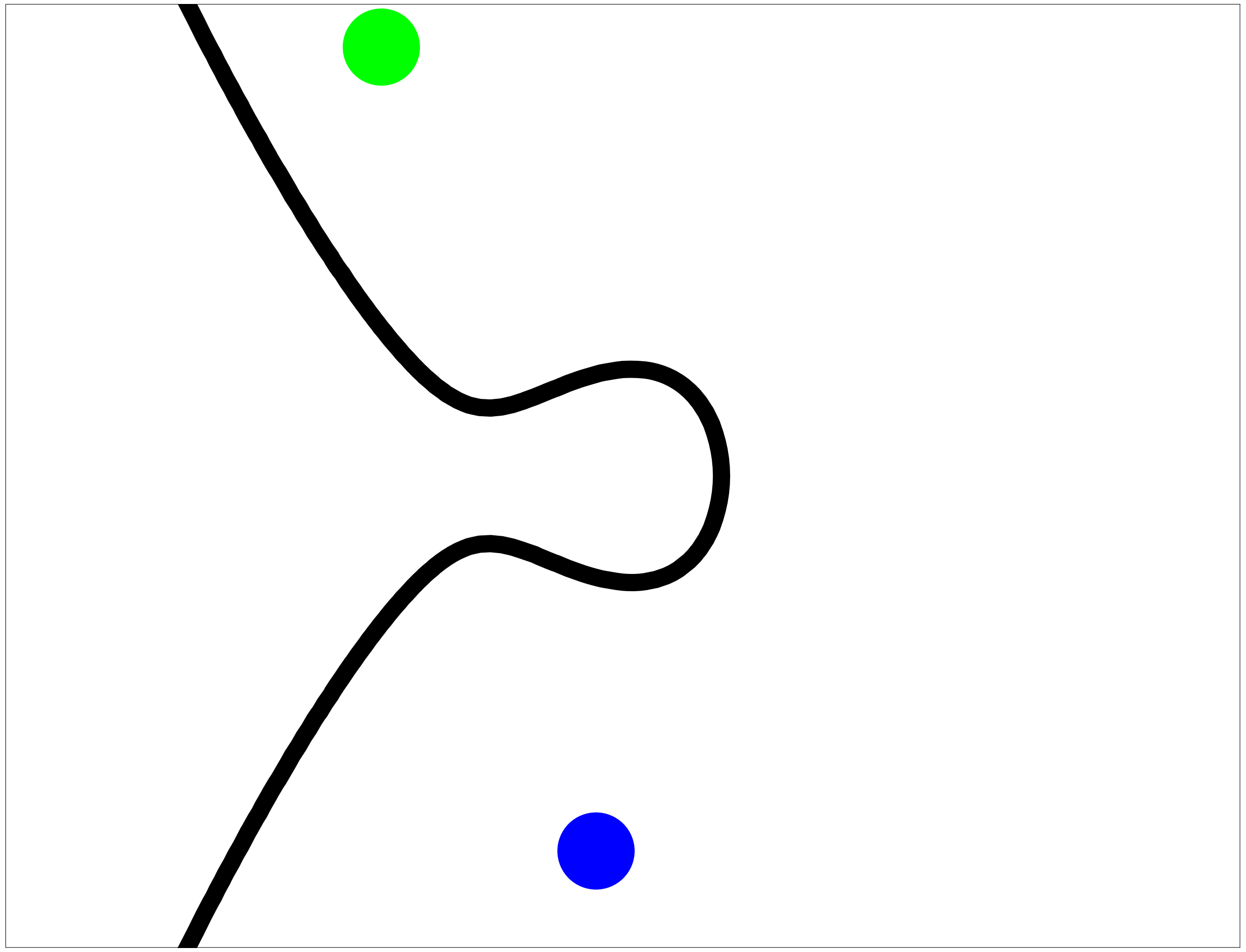
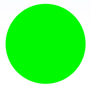
Input: $f(x_1, x_2)$, ,

Method: Overview



Input: $f(x_1, x_2)$,  , 

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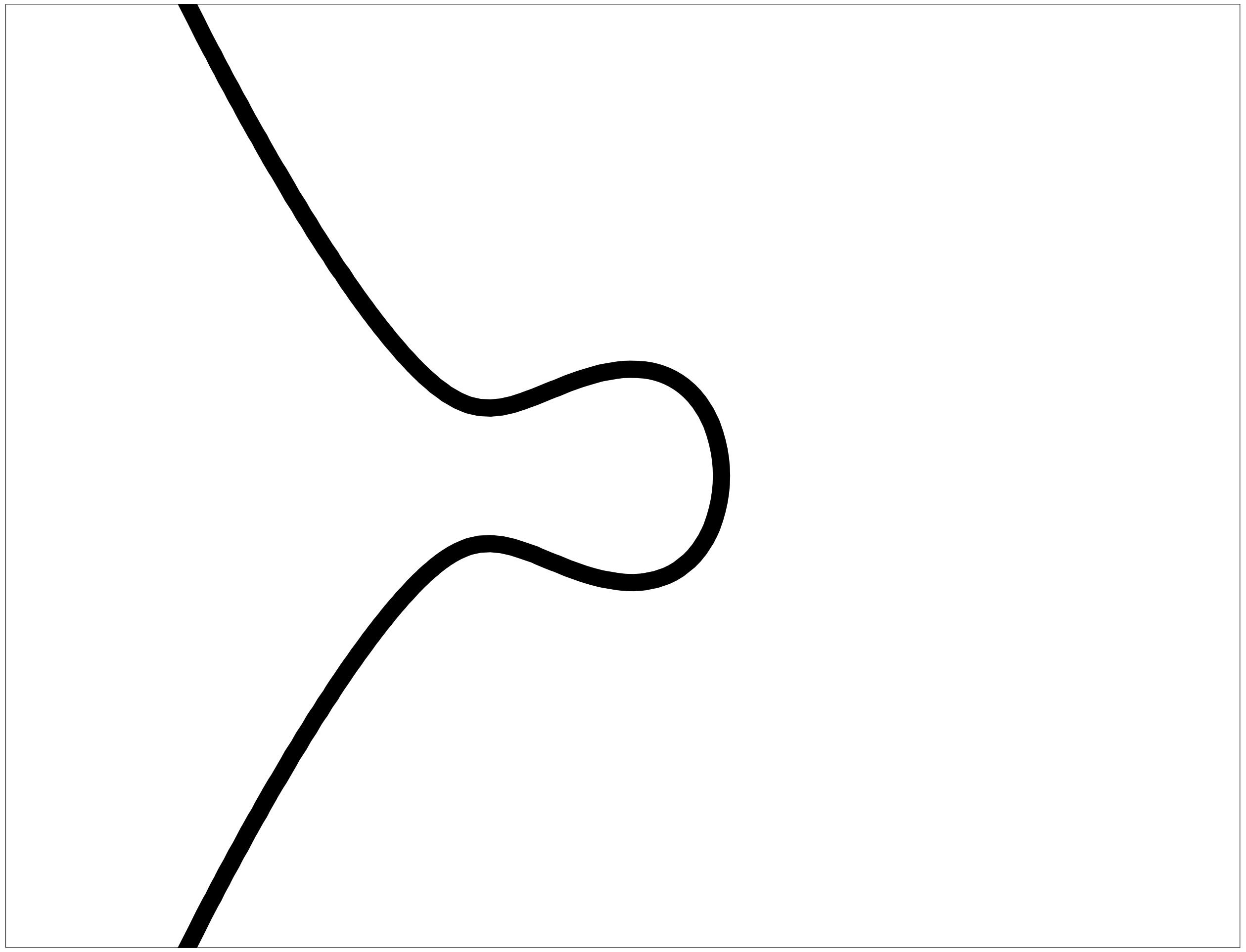
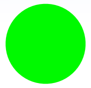


Input: $f(x_1, x_2)$,  , 

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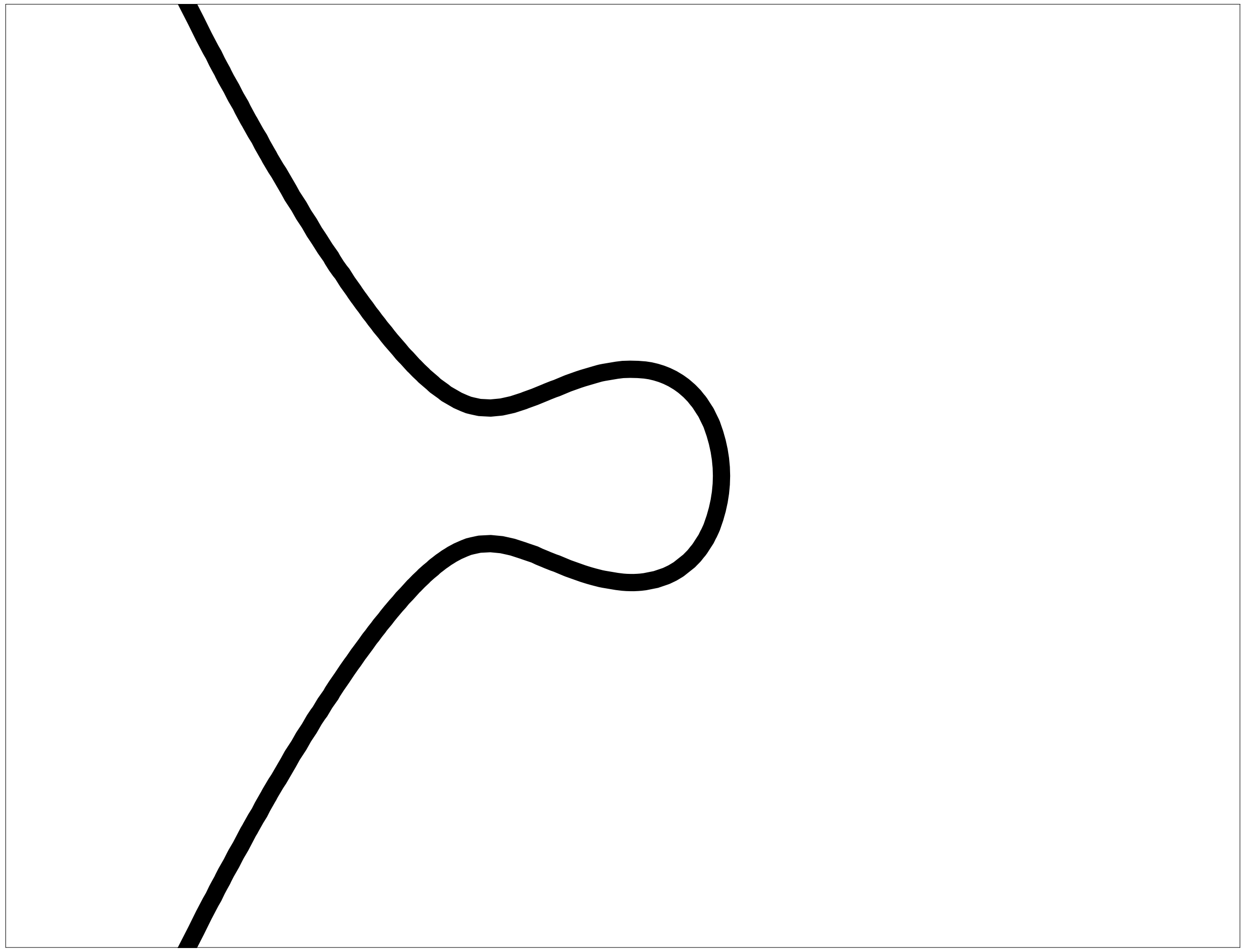
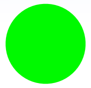
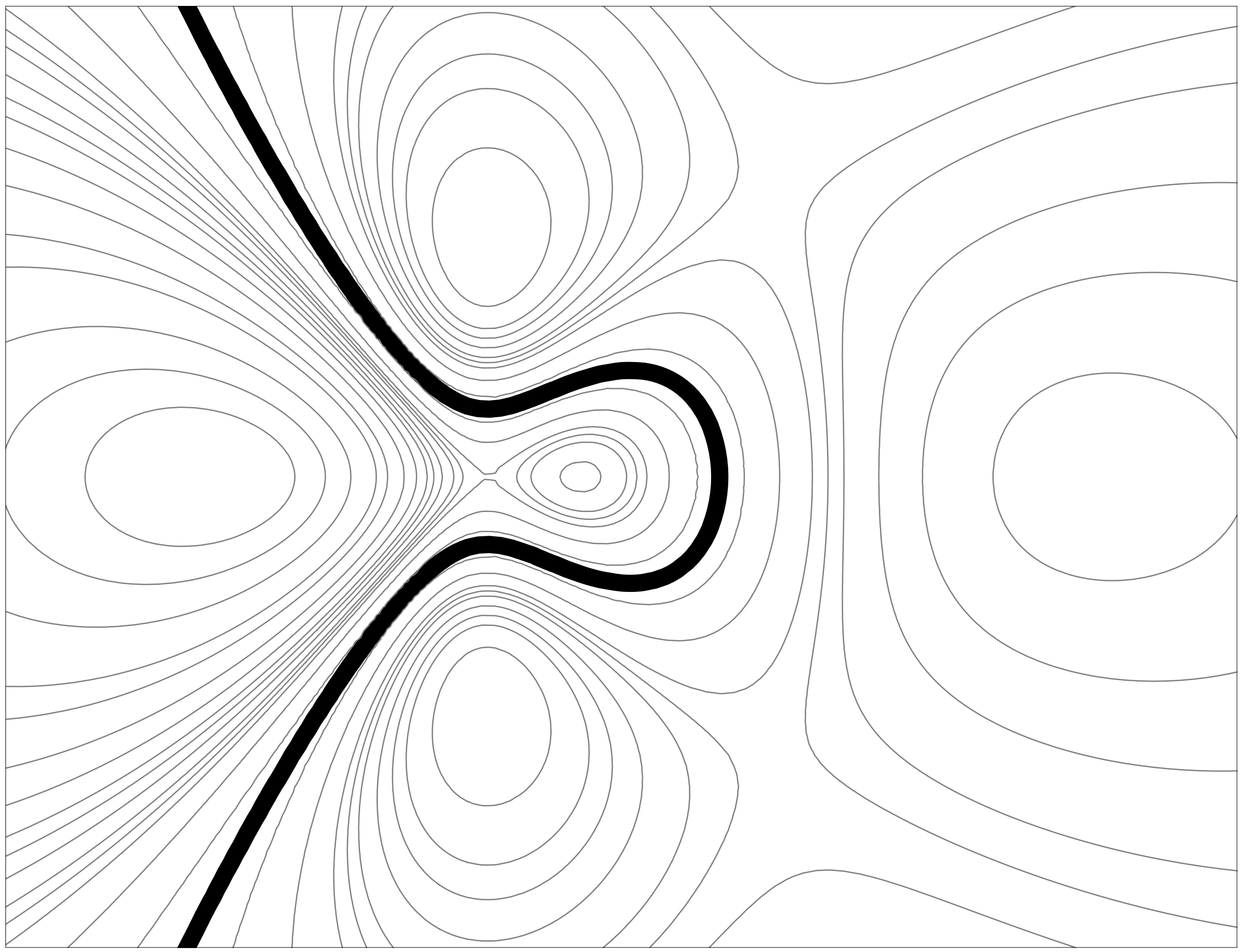


Input: $f(x_1, x_2)$,  , 

1: $g = \dfrac{f^2}{\left(x_1^2 + x_2^2 + 1\right)^{\deg(f)+1}}$

Method: Overview



Input: $f(x_1, x_2)$, ●, ●

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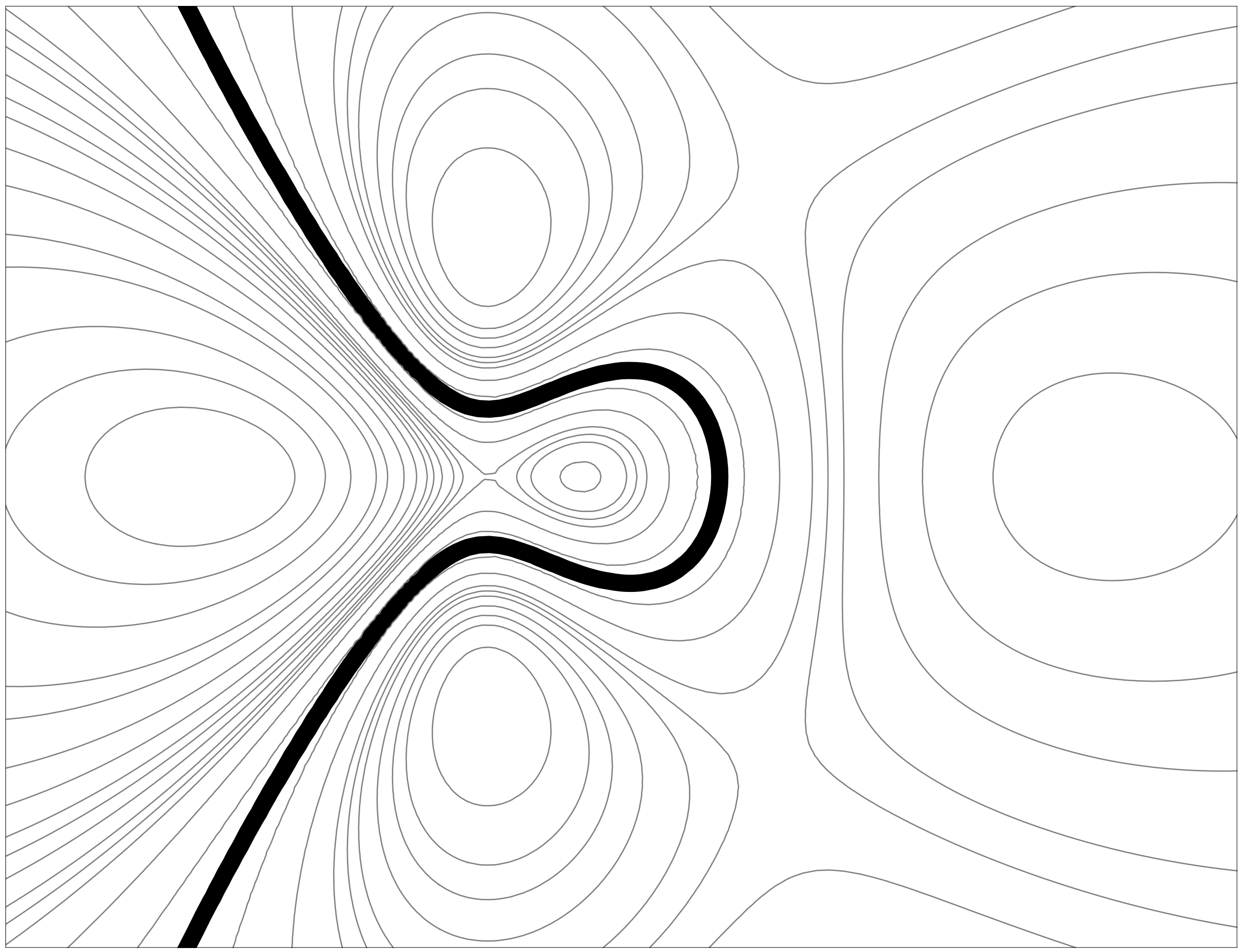
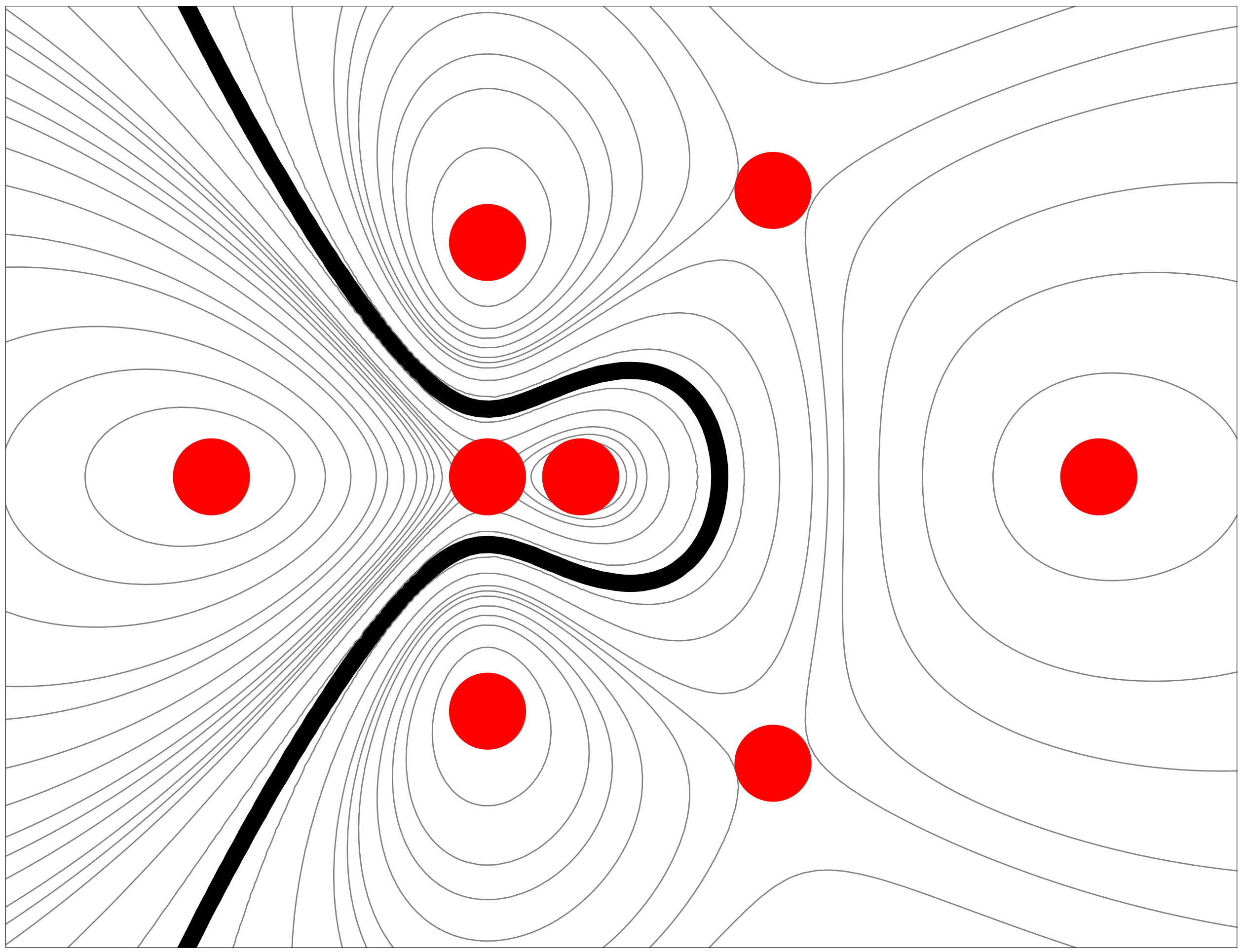


Input: $f(x_1, x_2)$, ●, ●

1: $g = \dfrac{f^2}{\left(x_1^2 + x_2^2 + 1\right)^{\deg(f)+1}}$

2: Solve $\nabla g(x) = 0 \wedge g(x) \neq 0$

Method: Overview



Input: $f(x_1, x_2)$, [blue dot], [green dot]

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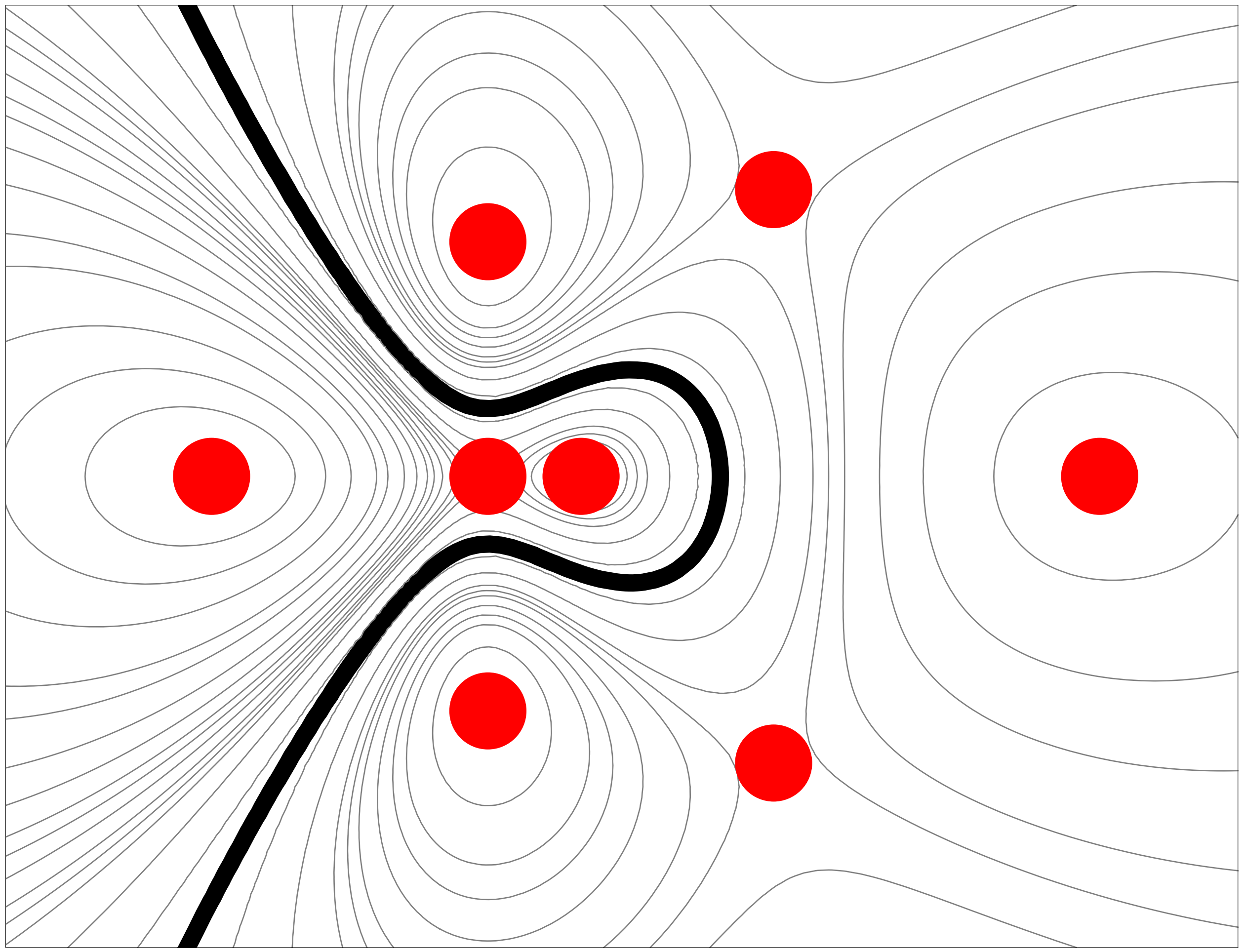

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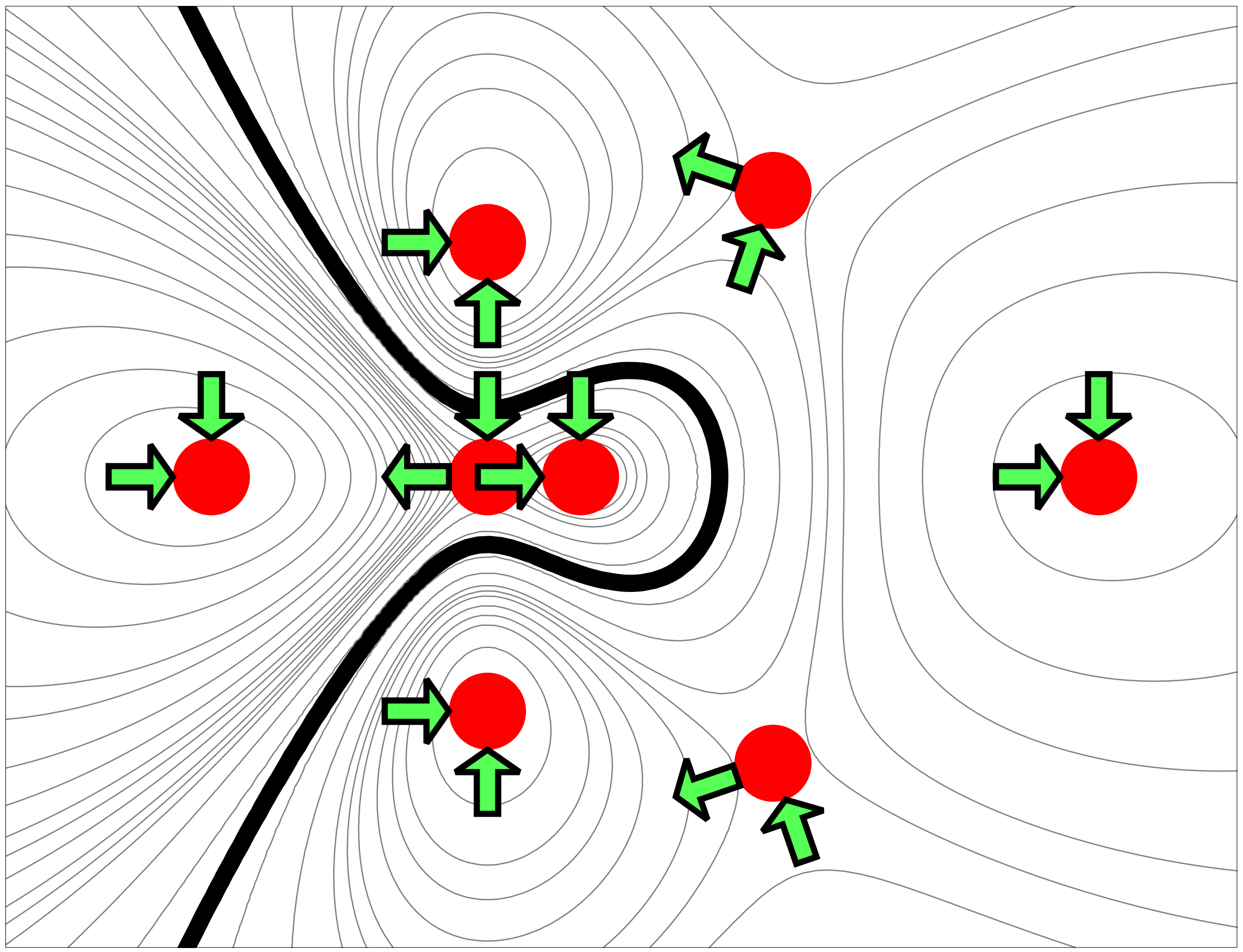
3: Find eigenvectors of (Hess g)(●)

Method: Overview



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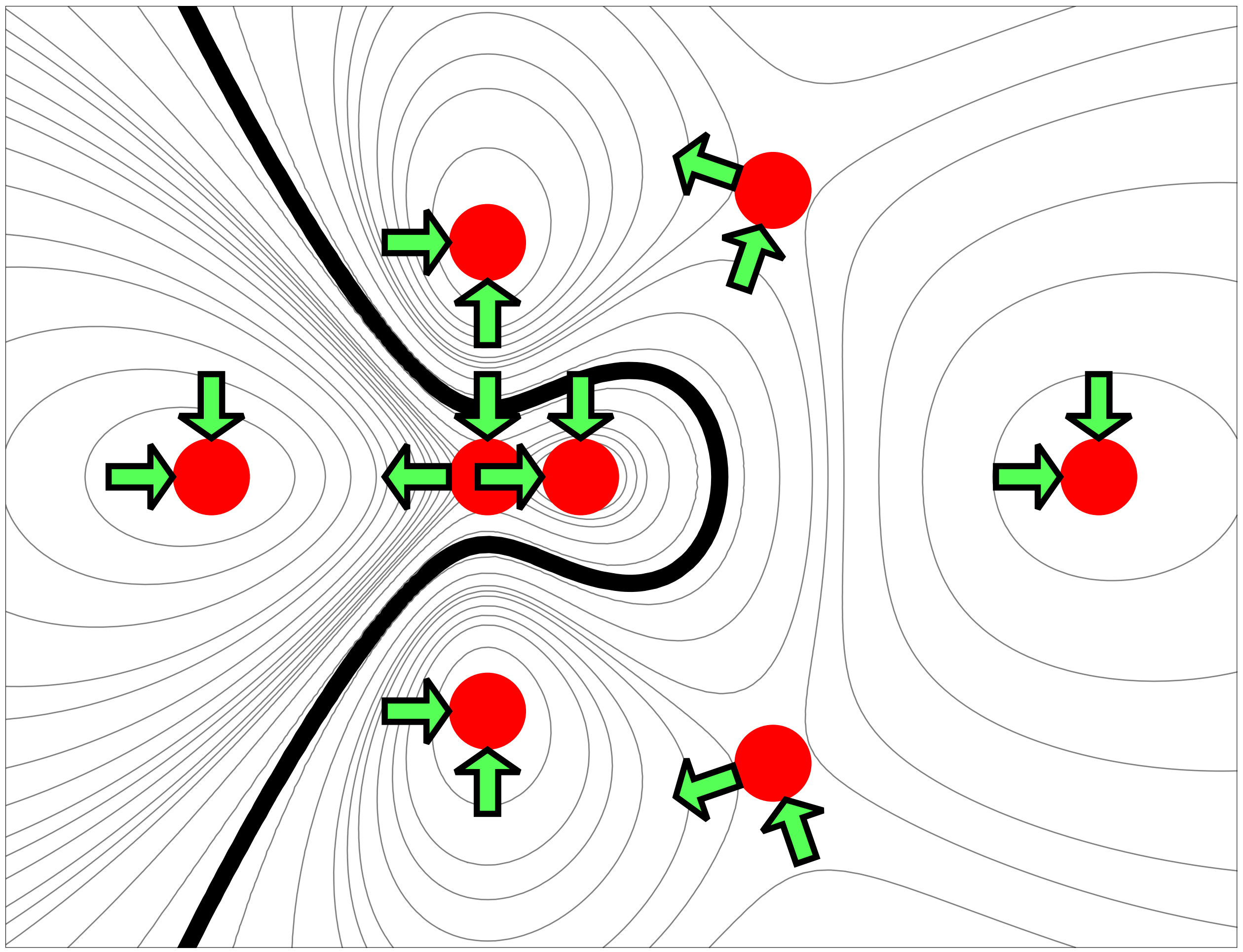

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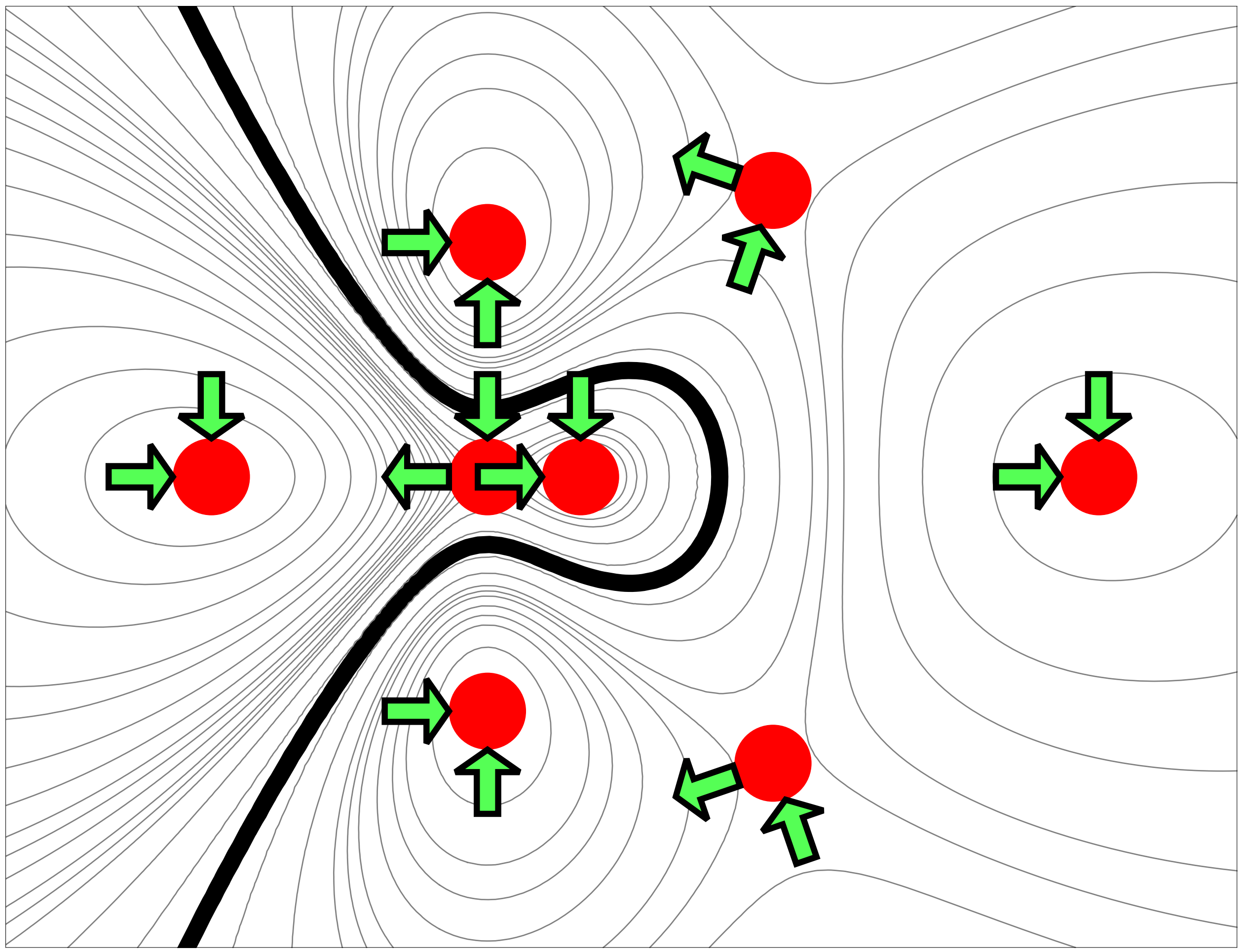
3: Find eigenvectors of (Hess g)(●)

4: Steepest ascent using outgoing eigenvectors

Method: Overview



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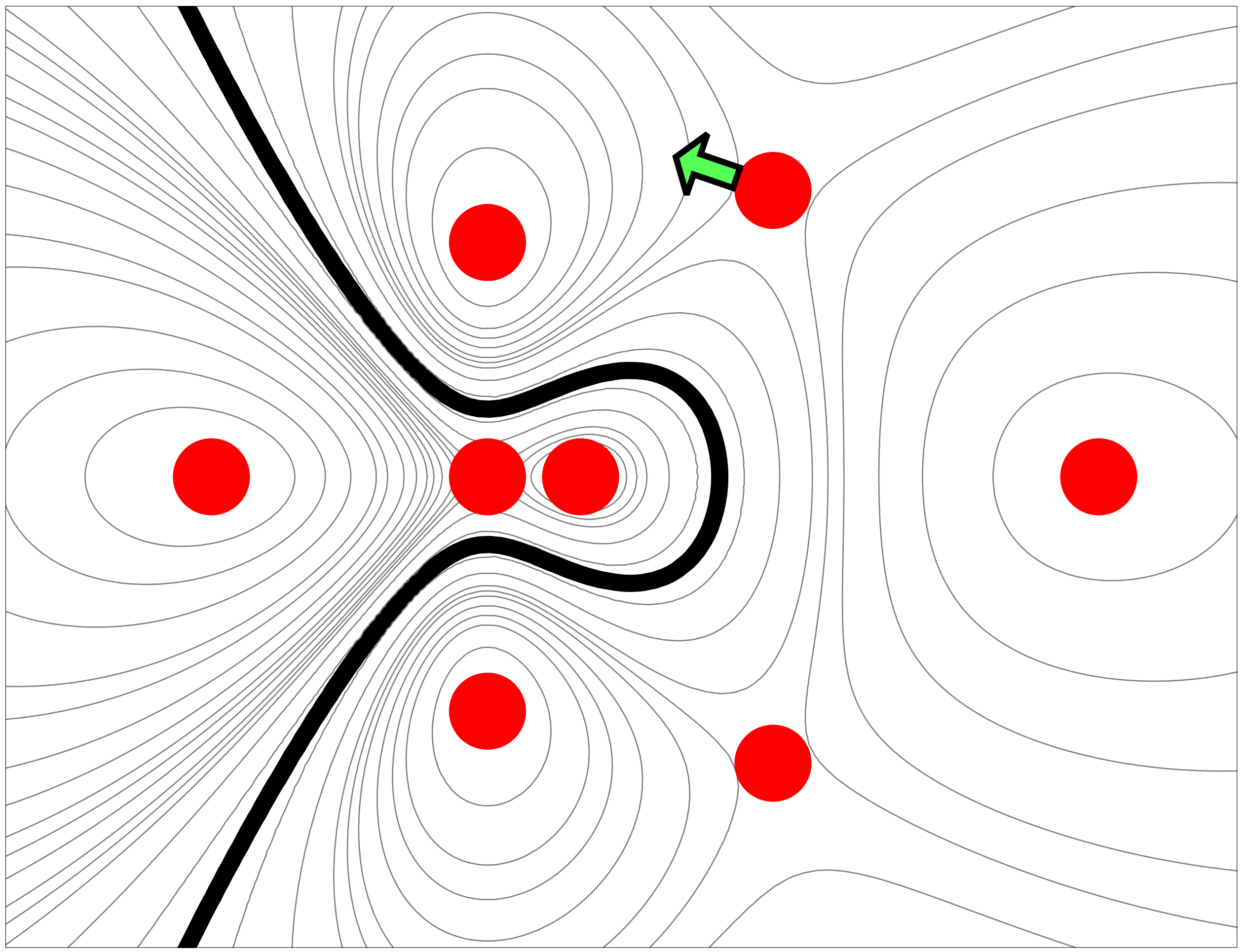
3: Find eigenvectors of (Hess g)()

4: Steepest ascent using outgoing eigenvectors

positive eigenvalue

Method: Overview



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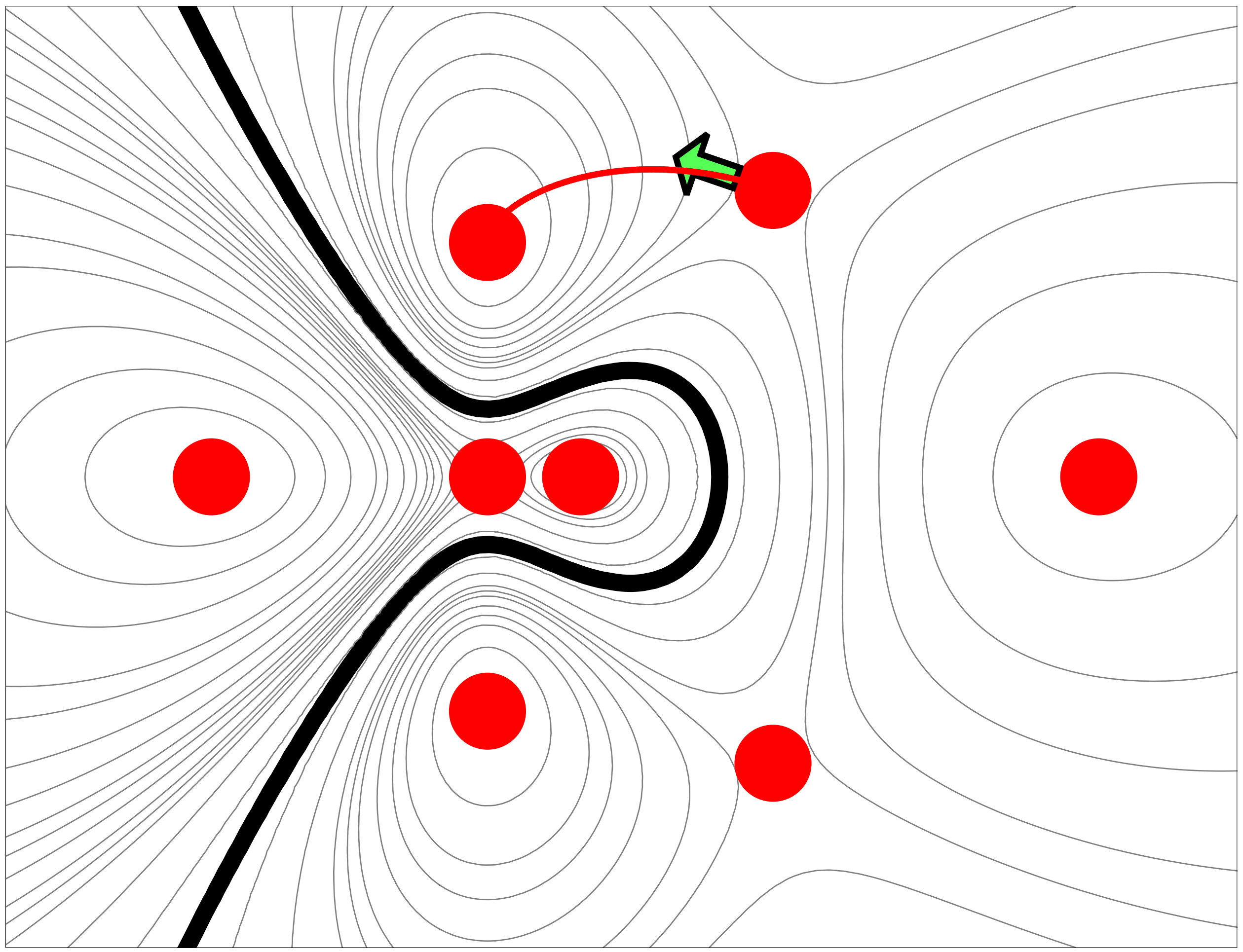

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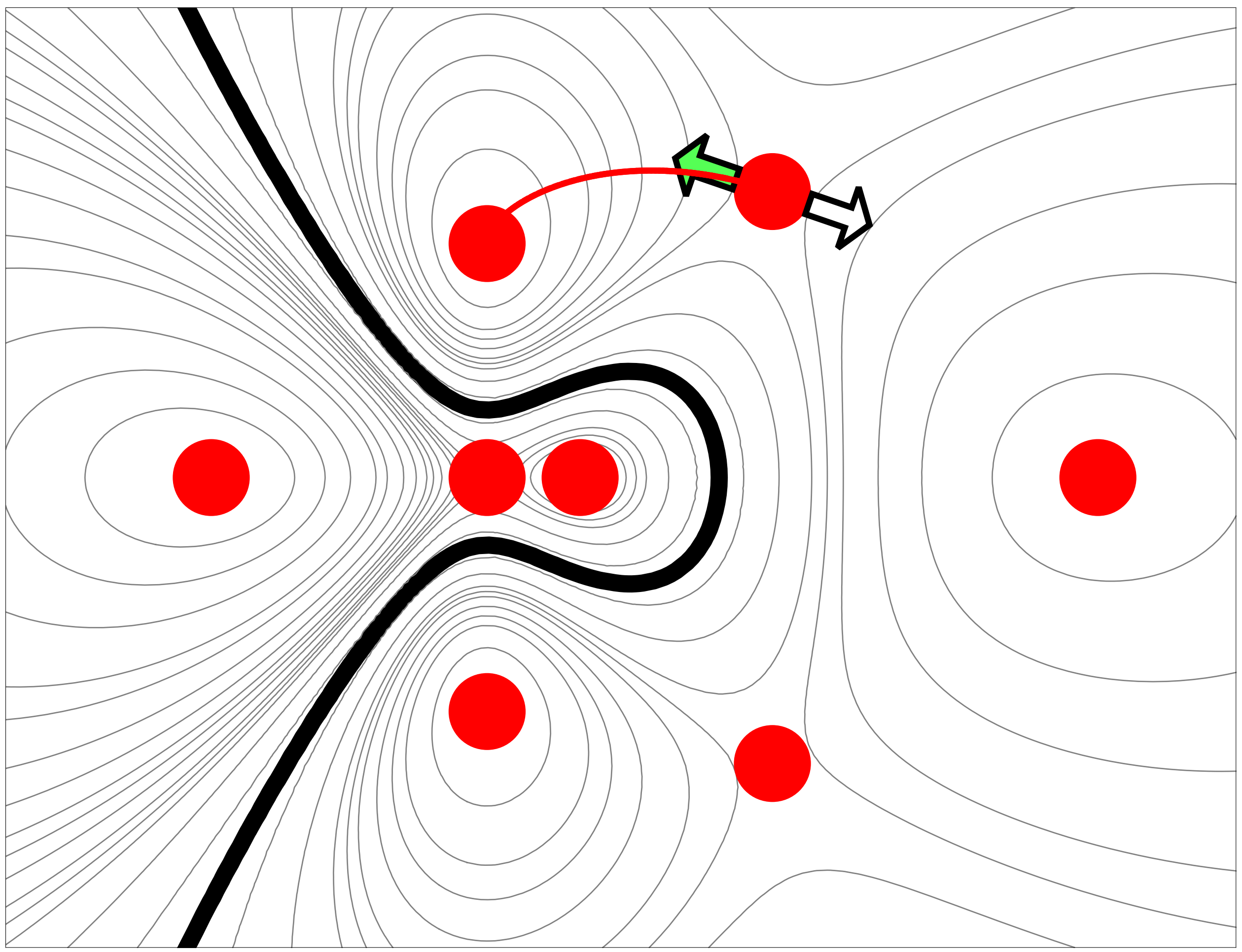

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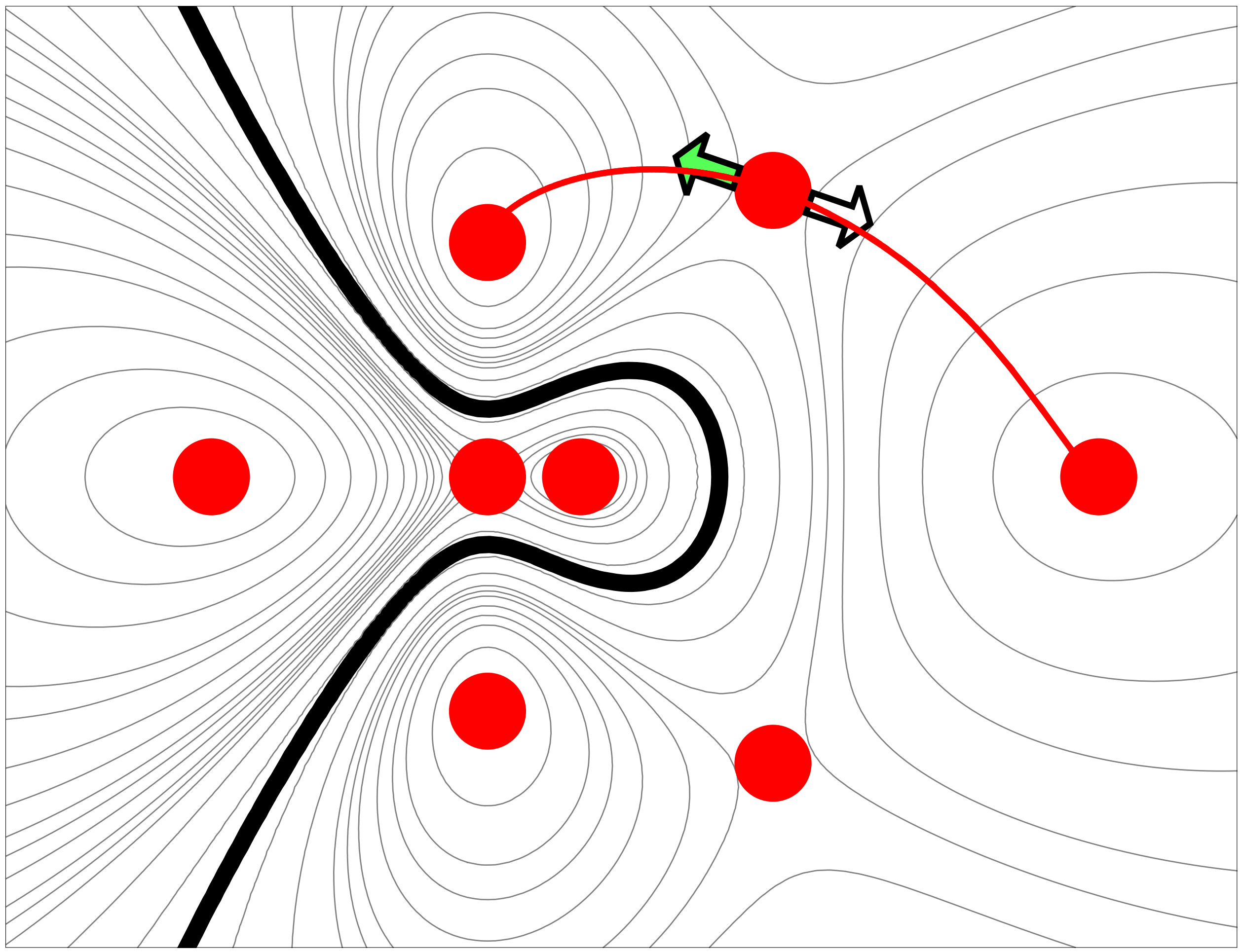

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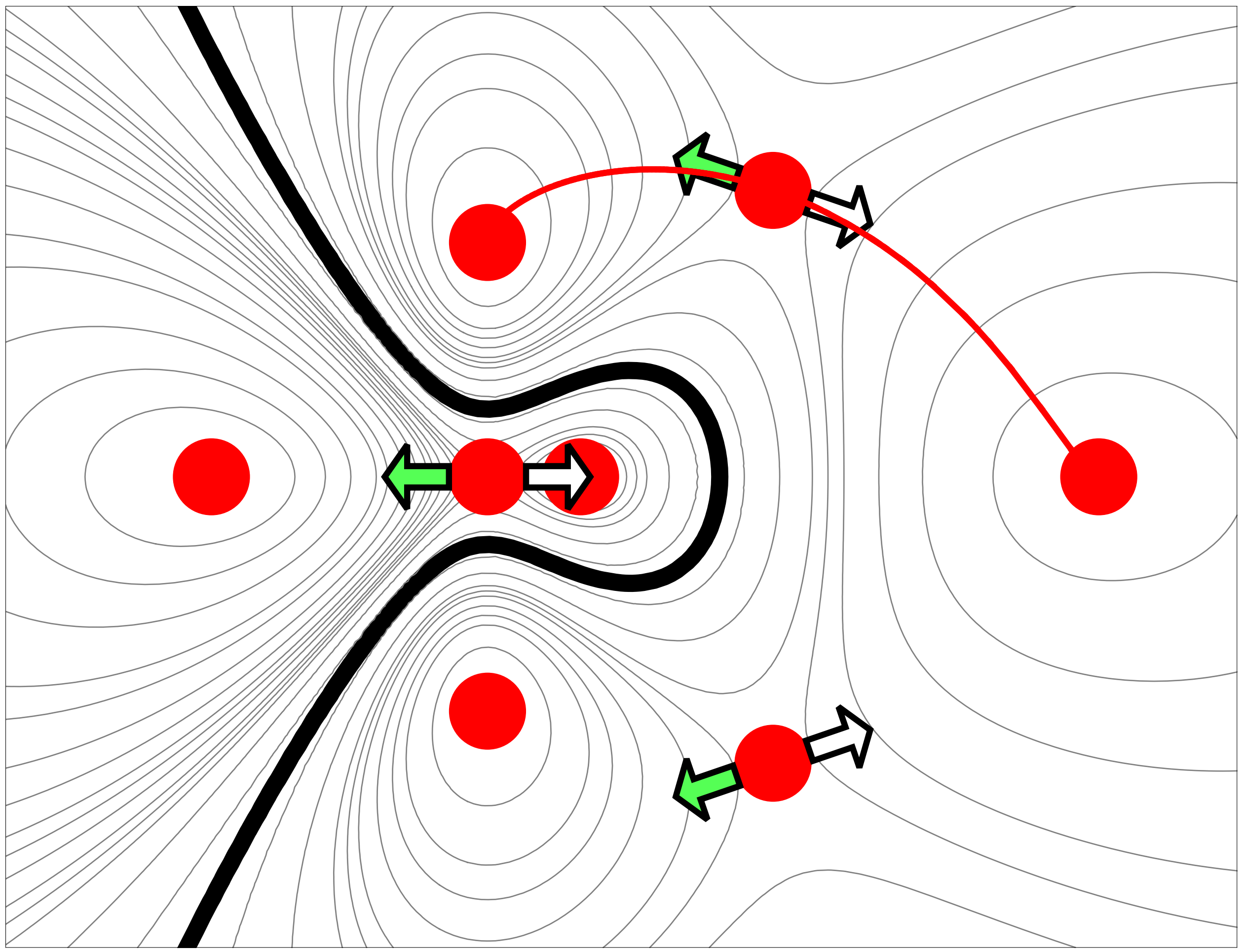
positive eigenvalue

Method: Overview



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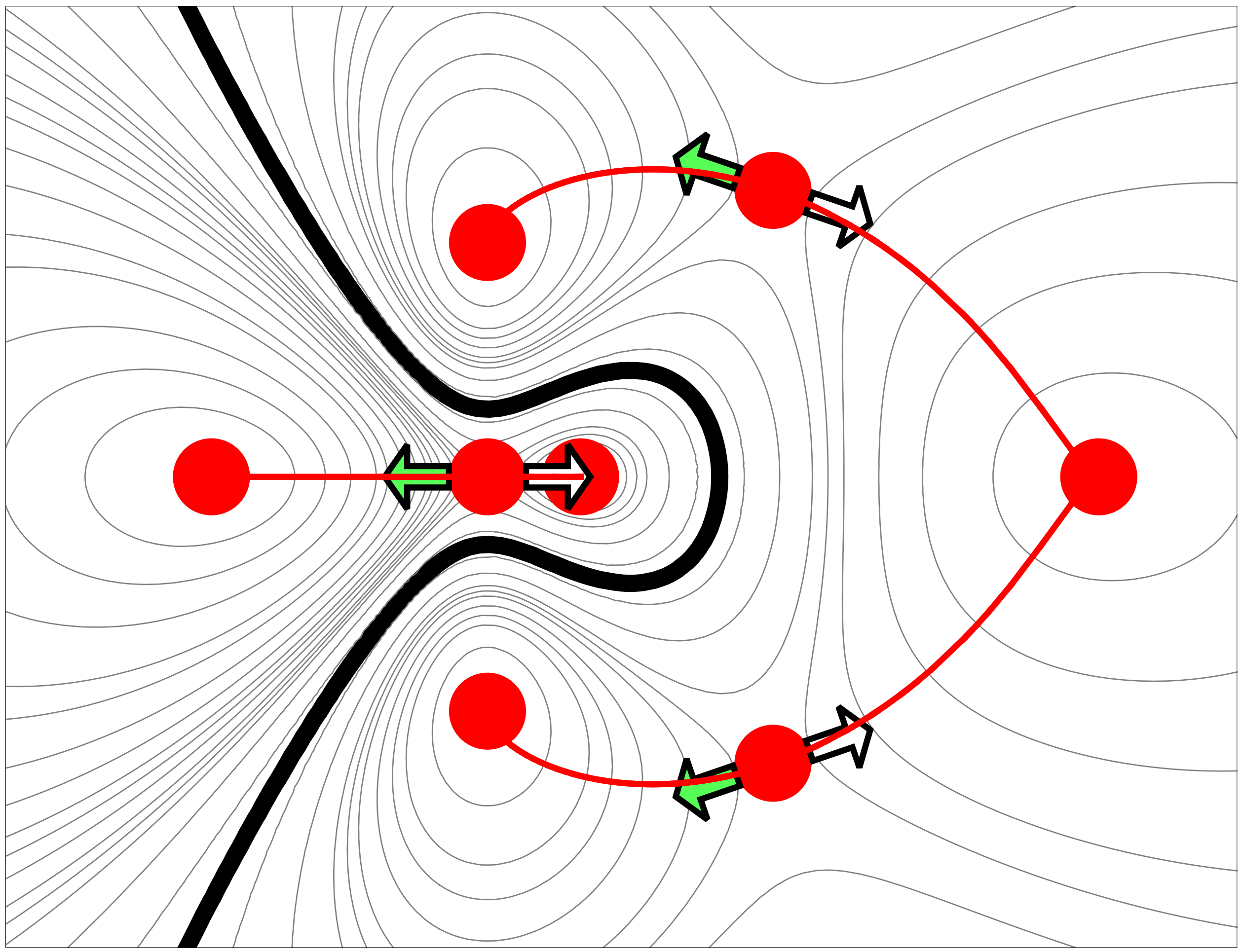
positive eigenvalue

Method: Overview



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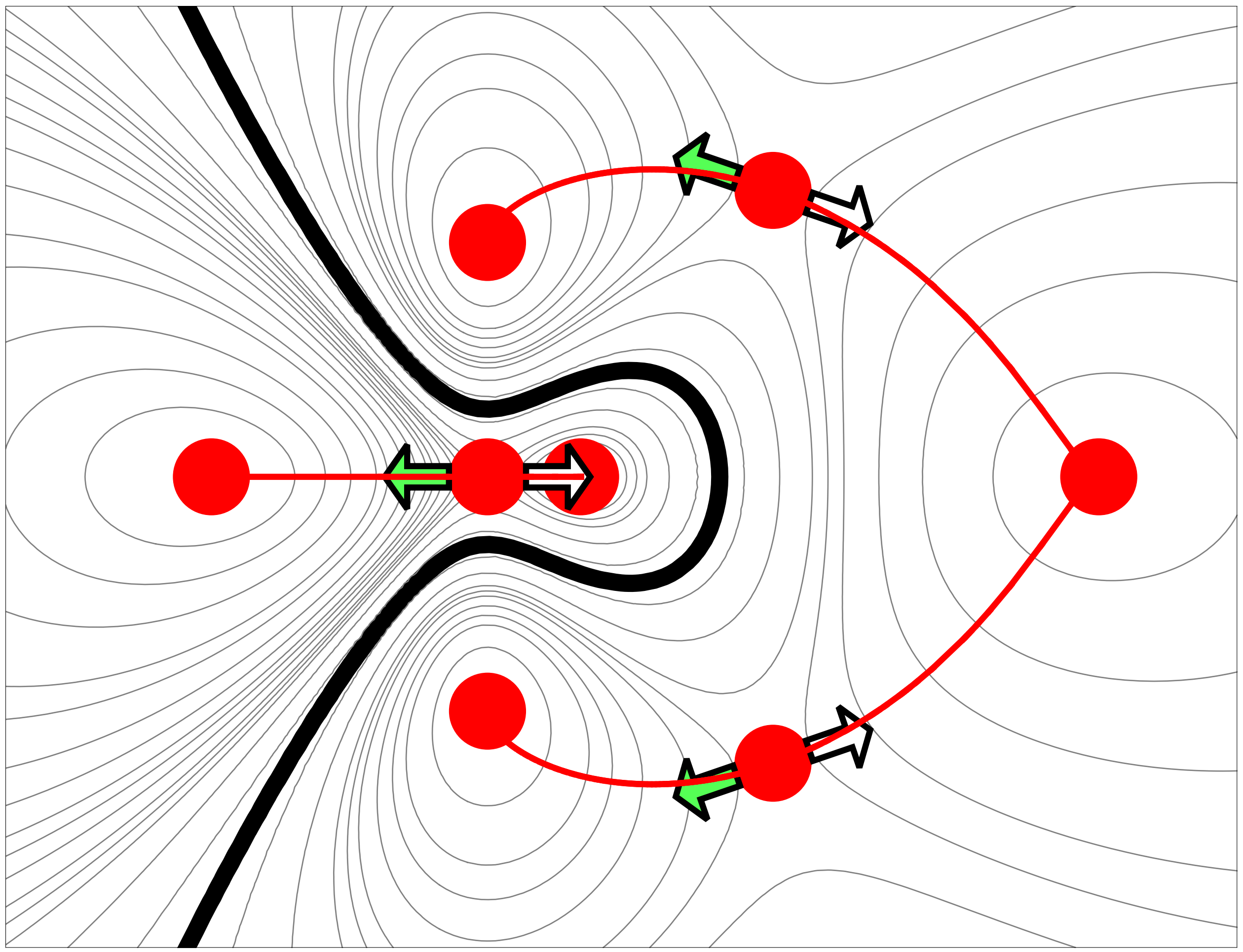
4: Steepest ascent using outgoing eigenvectors *positive eigenvalue*

Method: Overview



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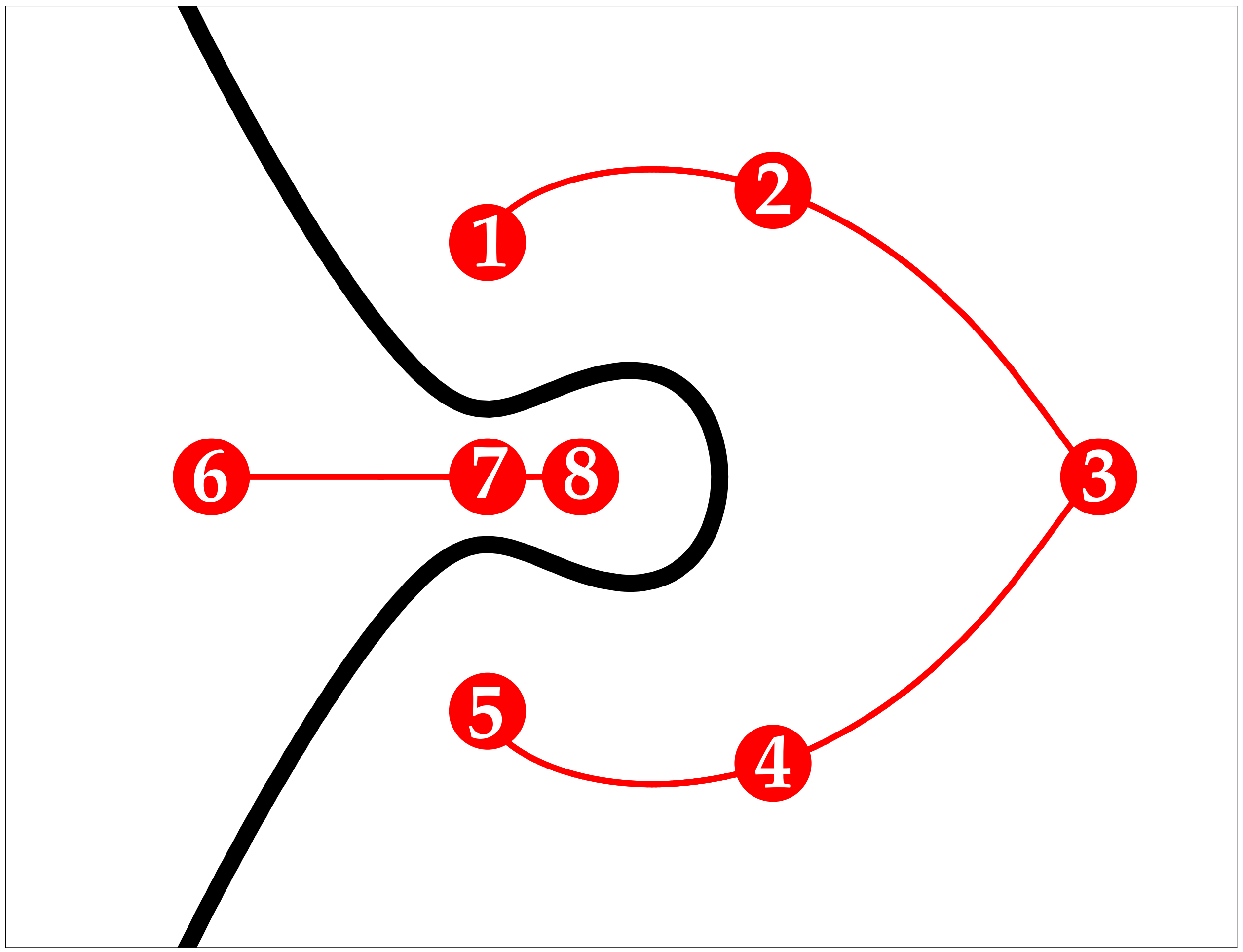
4: Steepest ascent using outgoing eigenvectors ← *positive eigenvalue*

5: Form adjacency matrix

Method: Overview



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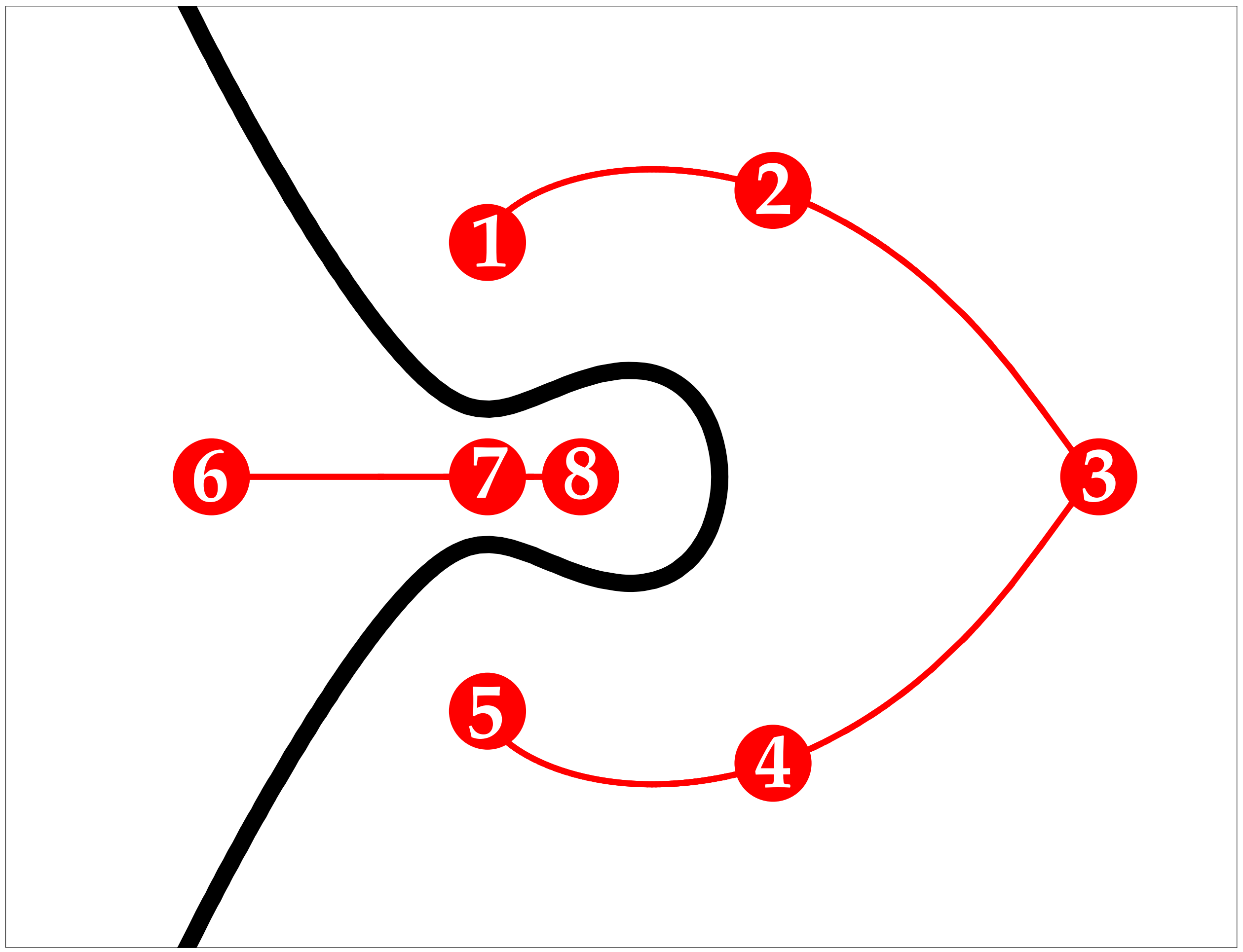


$$
\begin{array}{c|c}
 & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} &
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\end{array}
$$

Input: $f(x_1, x_2)$, ,

1: $g = \dfrac{f^2}{\left(x_1^2 + x_2^2 + 1\right)^{\deg(f)+1}}$

2: Solve $\nabla g(x) = 0 \wedge g(x) \neq 0$

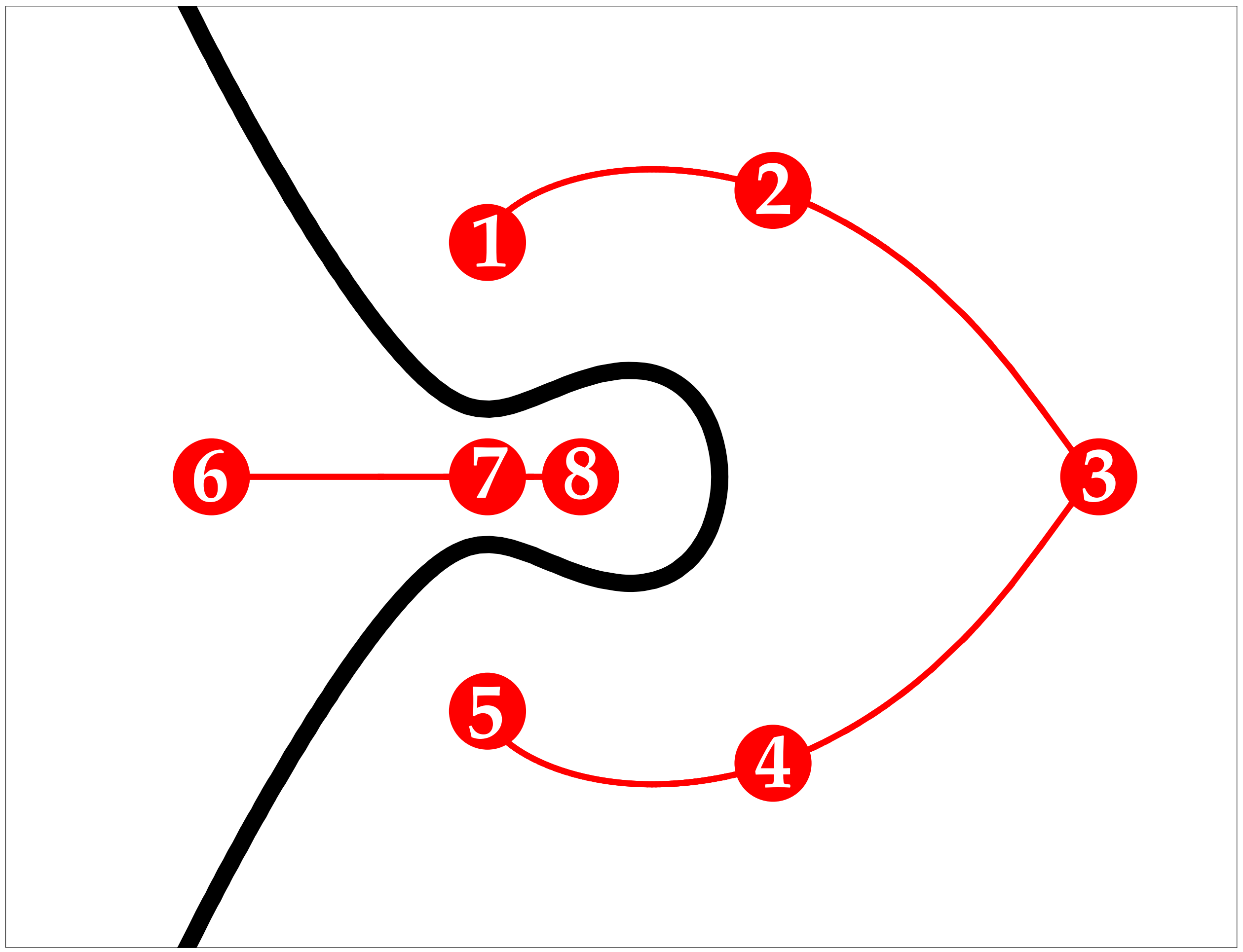
3: Find eigenvectors of (Hess g)()

4: Steepest ascent using outgoing eigenvectors

positive eigenvalue

5: Form adjacency matrix

Method: Overview



$$\begin{array}{cc} & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

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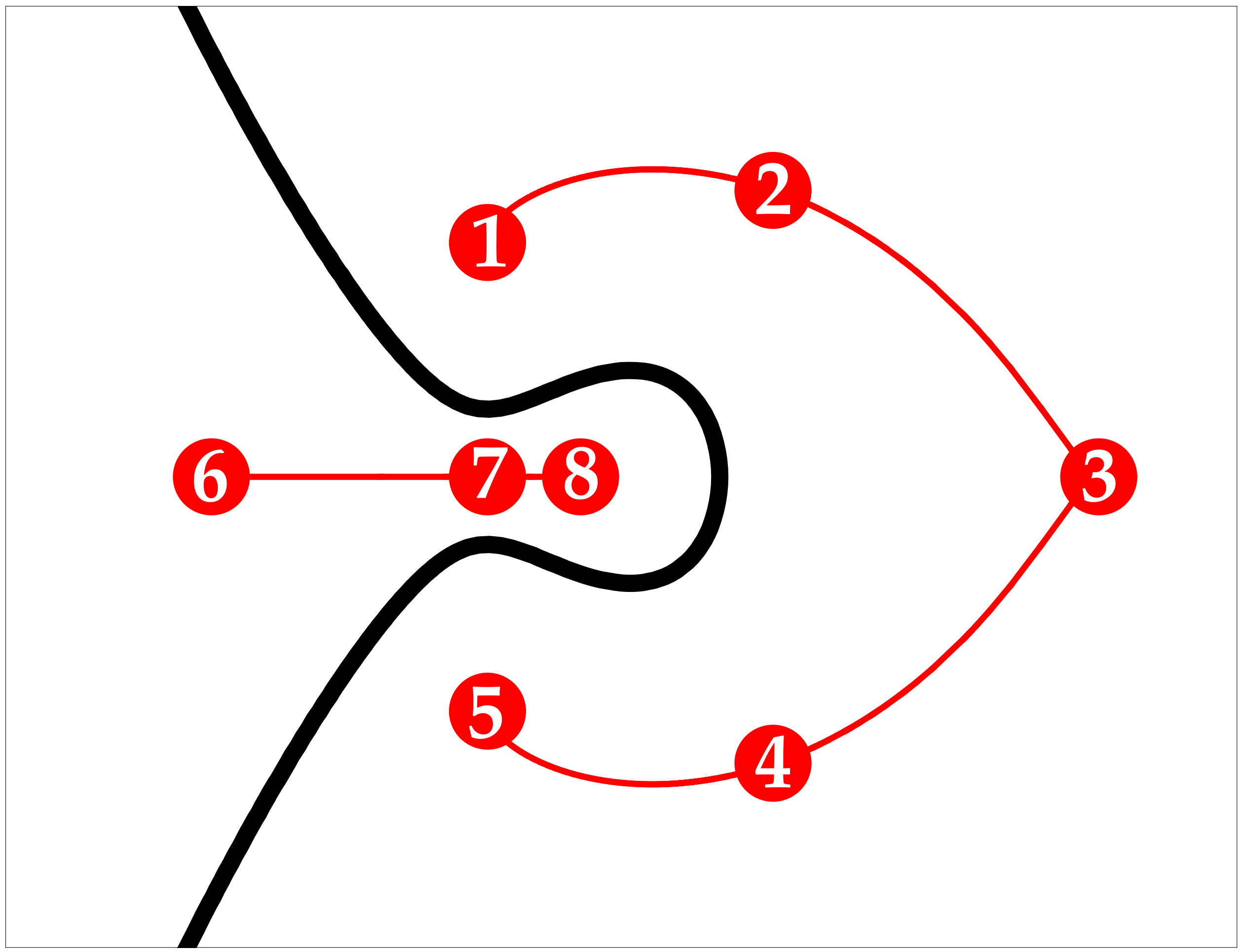
6: Closure of adjacency matrix

Method: Overview



$$
\begin{array}{c|cccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
2 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
3 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
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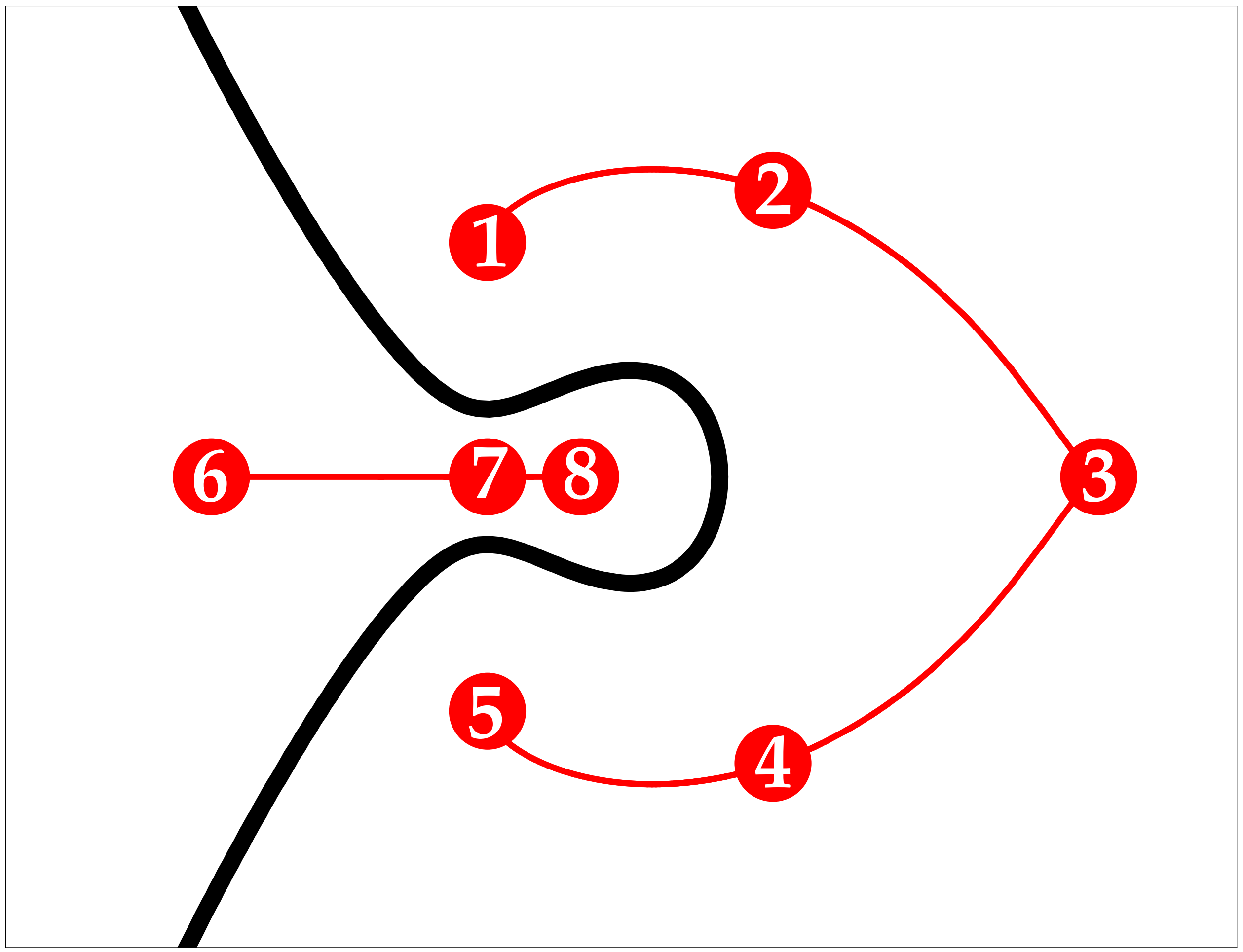

$$
\begin{array}{c|c}
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4: Steepest ascent using outgoing eigenvectors *positive eigenvalue*

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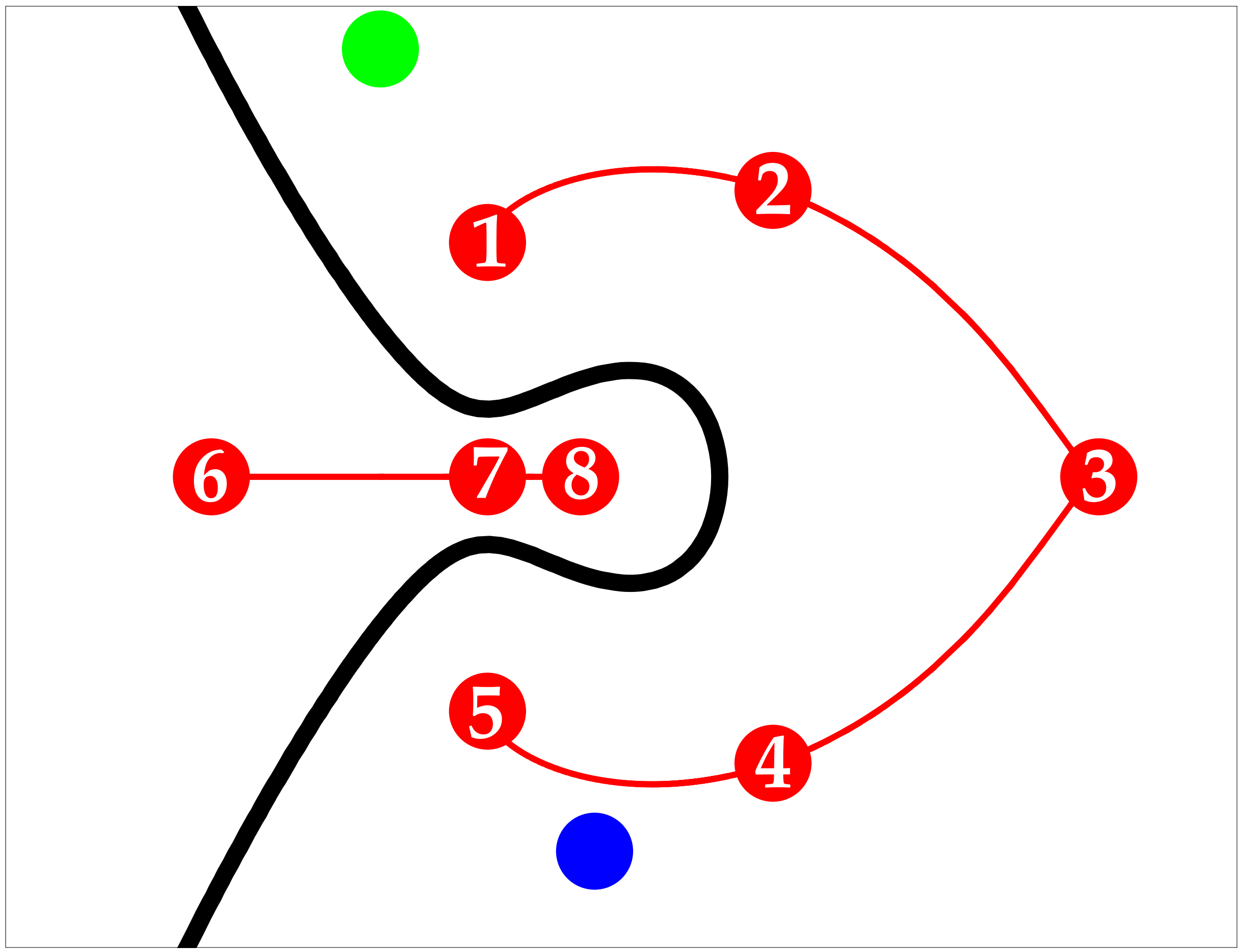
6: Closure of adjacency matrix

7: Steepest ascent from

Method: Overview



$$\begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8 \end{array}\begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

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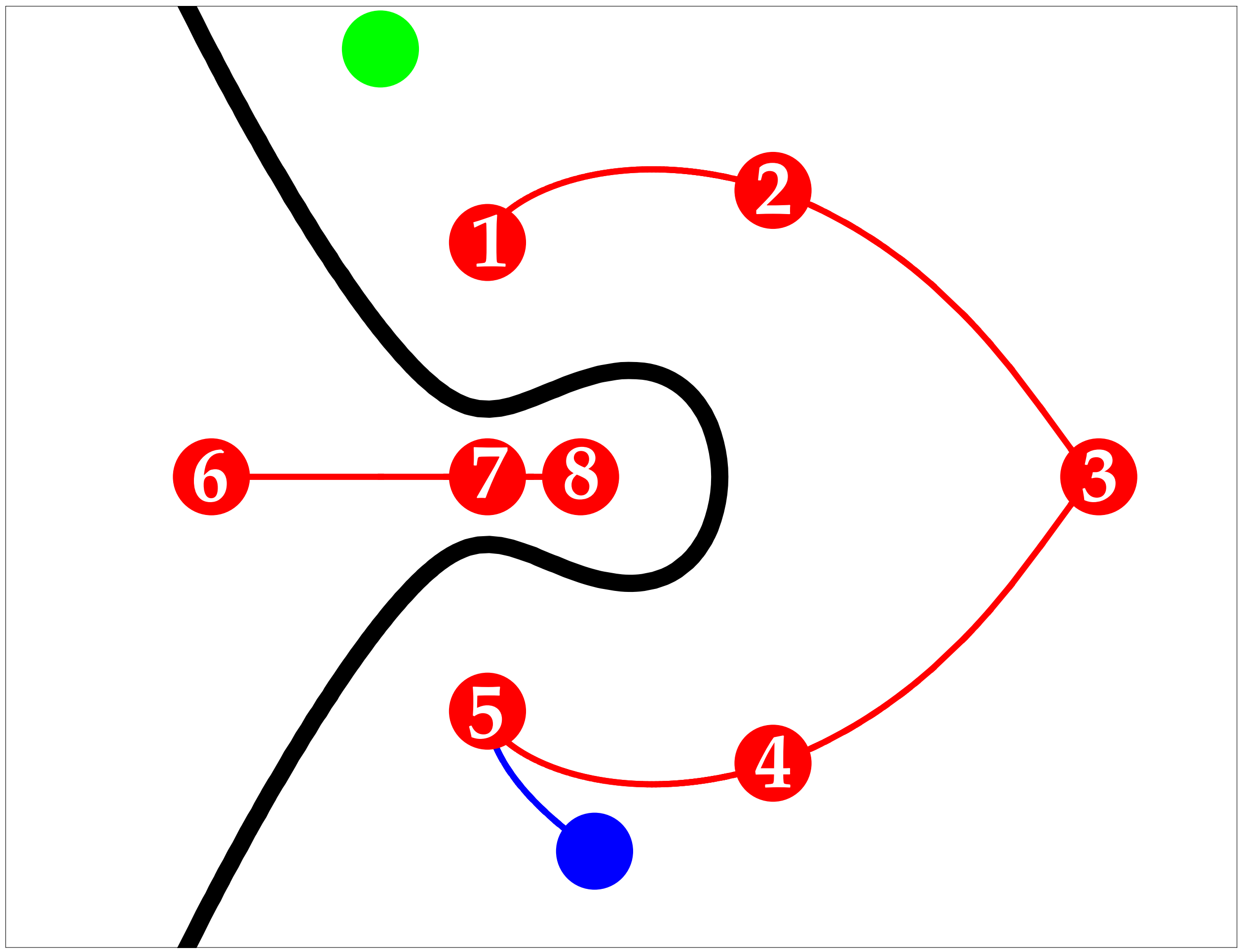

$$
\begin{array}{c|cccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
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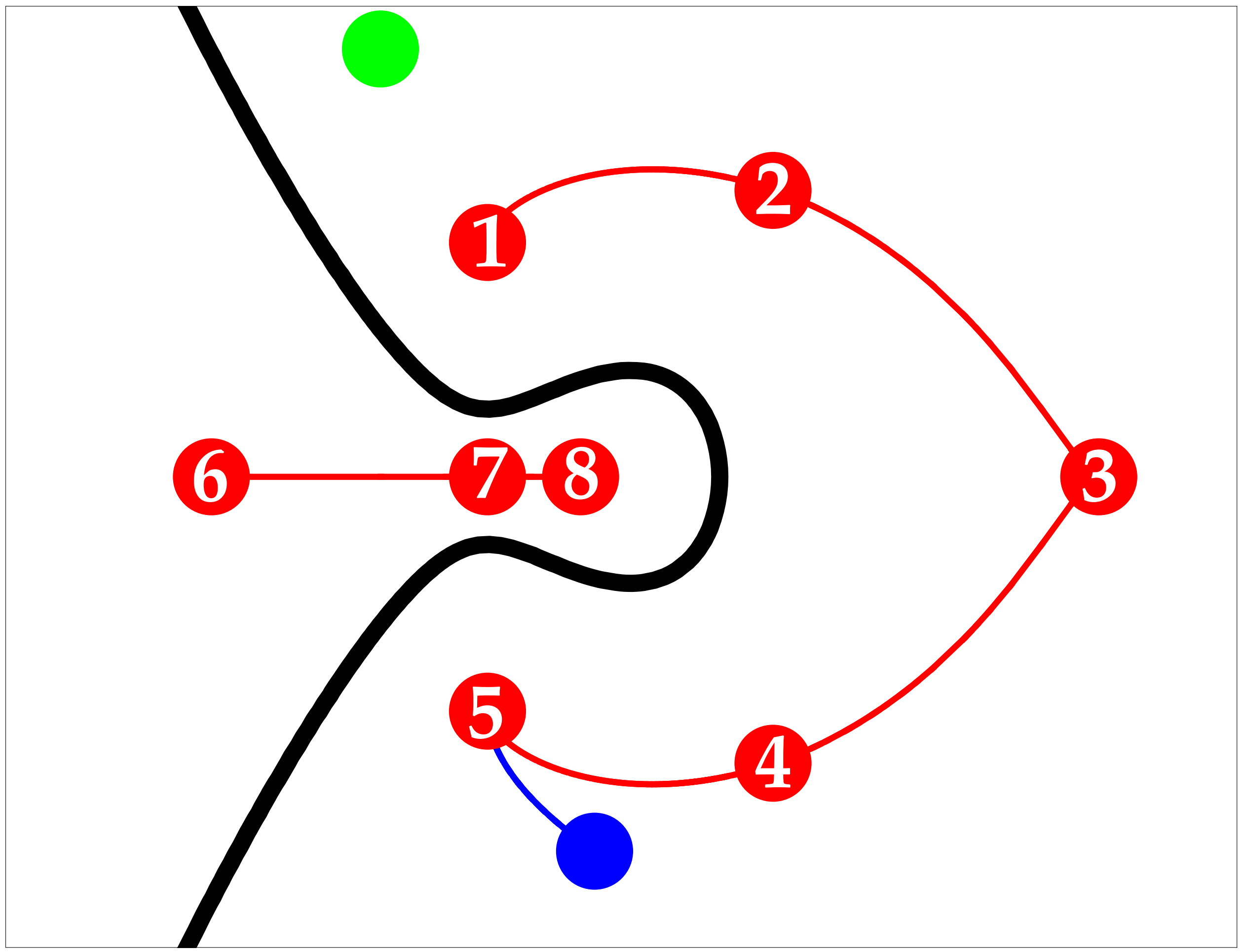

$$
\begin{array}{c|c}
 & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} \\
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix} &
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
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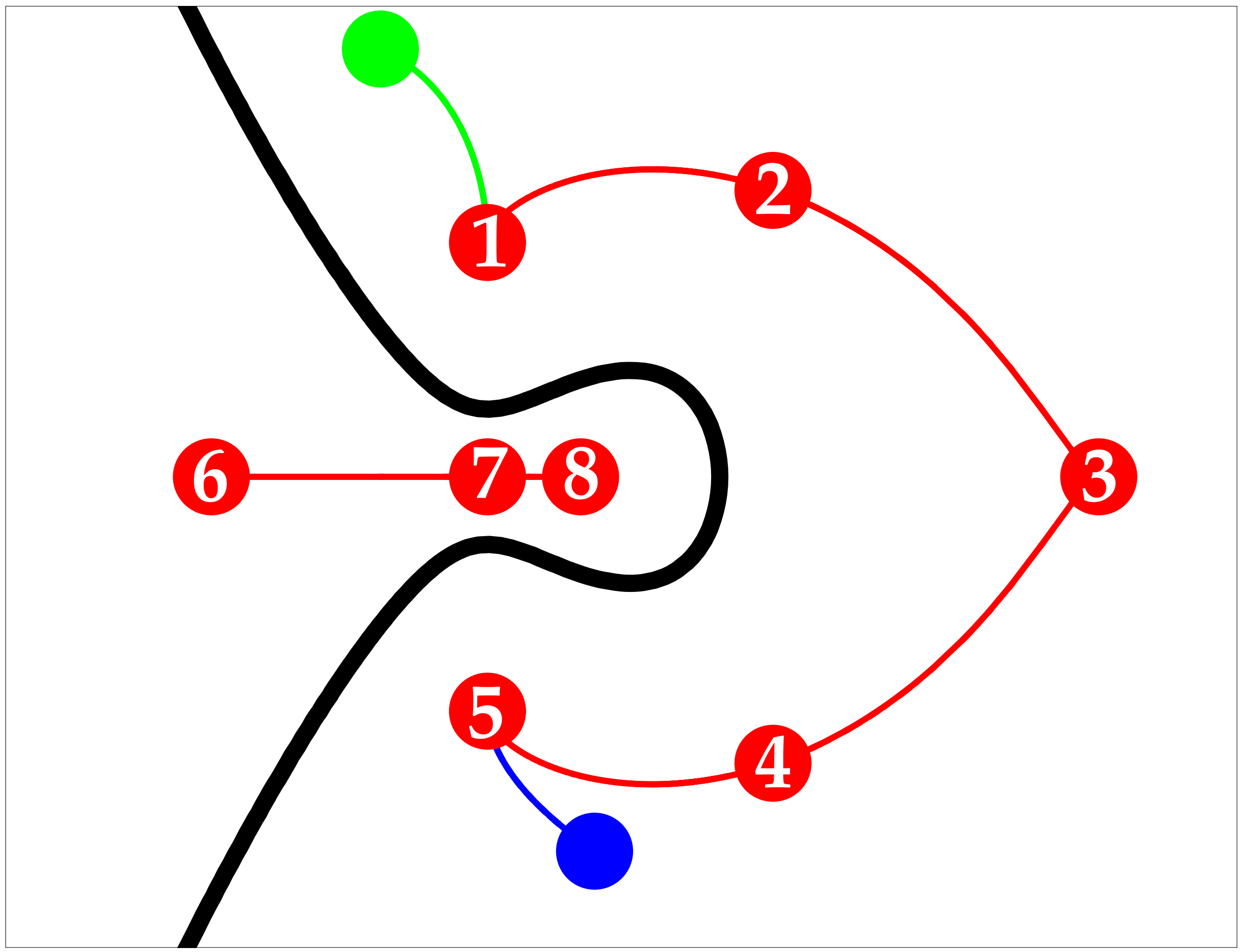

$$\begin{array}{c|cccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
\end{array}$$

$$\begin{array}{c}1\\2\\3\\4\\5\\6\\7\\8\end{array}
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
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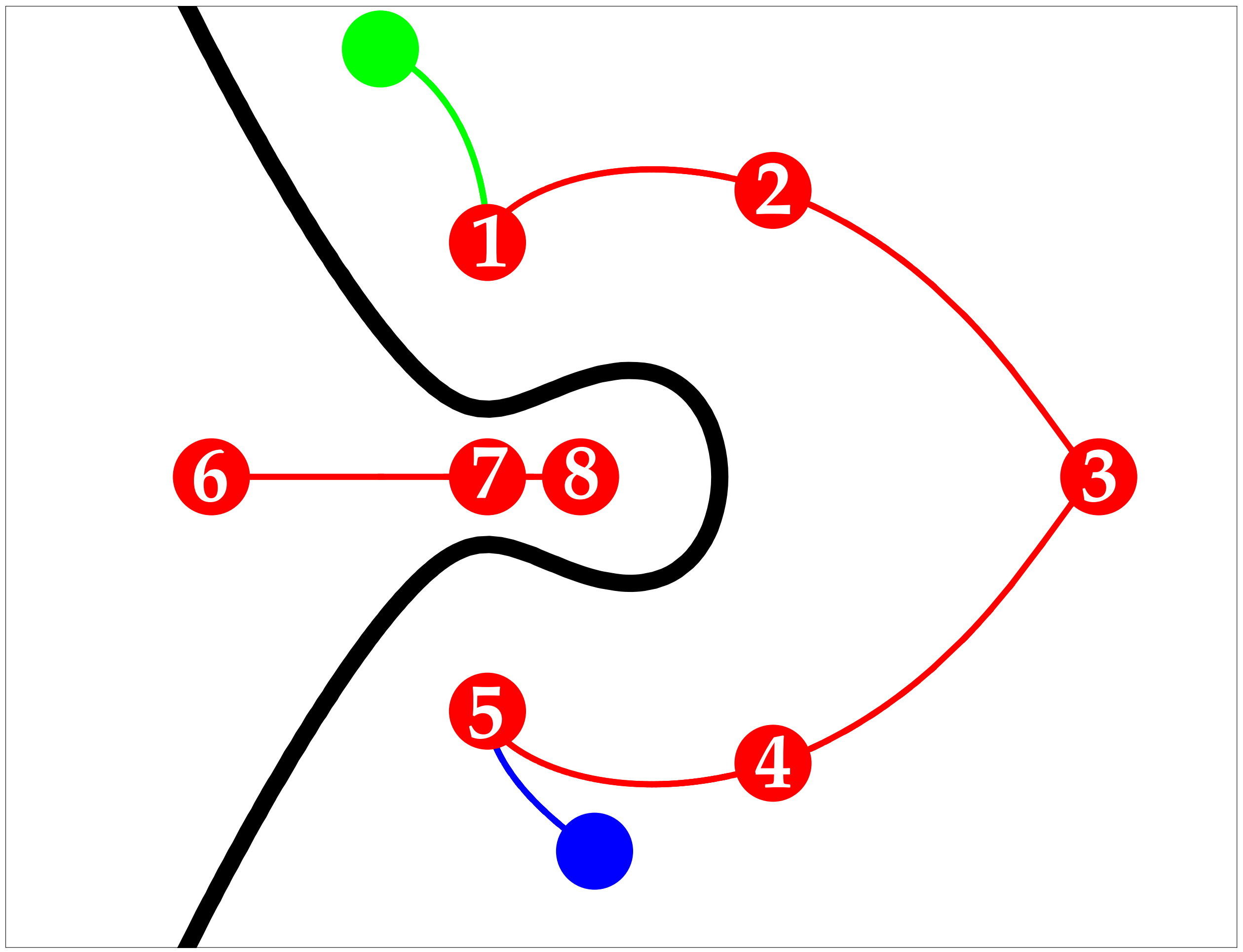

$$\begin{array}{c c}
 & \begin{matrix} \color{red}1 & \color{red}2 & \color{red}3 & \color{red}4 & \color{red}5 & \color{red}6 & \color{red}7 & \color{red}8 \end{matrix} \\
\begin{matrix} \color{red}1 \\ \color{red}2 \\ \color{red}3 \\ \color{red}4 \\ \color{red}5 \\ \color{red}6 \\ \color{red}7 \\ \color{red}8 \end{matrix} &
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
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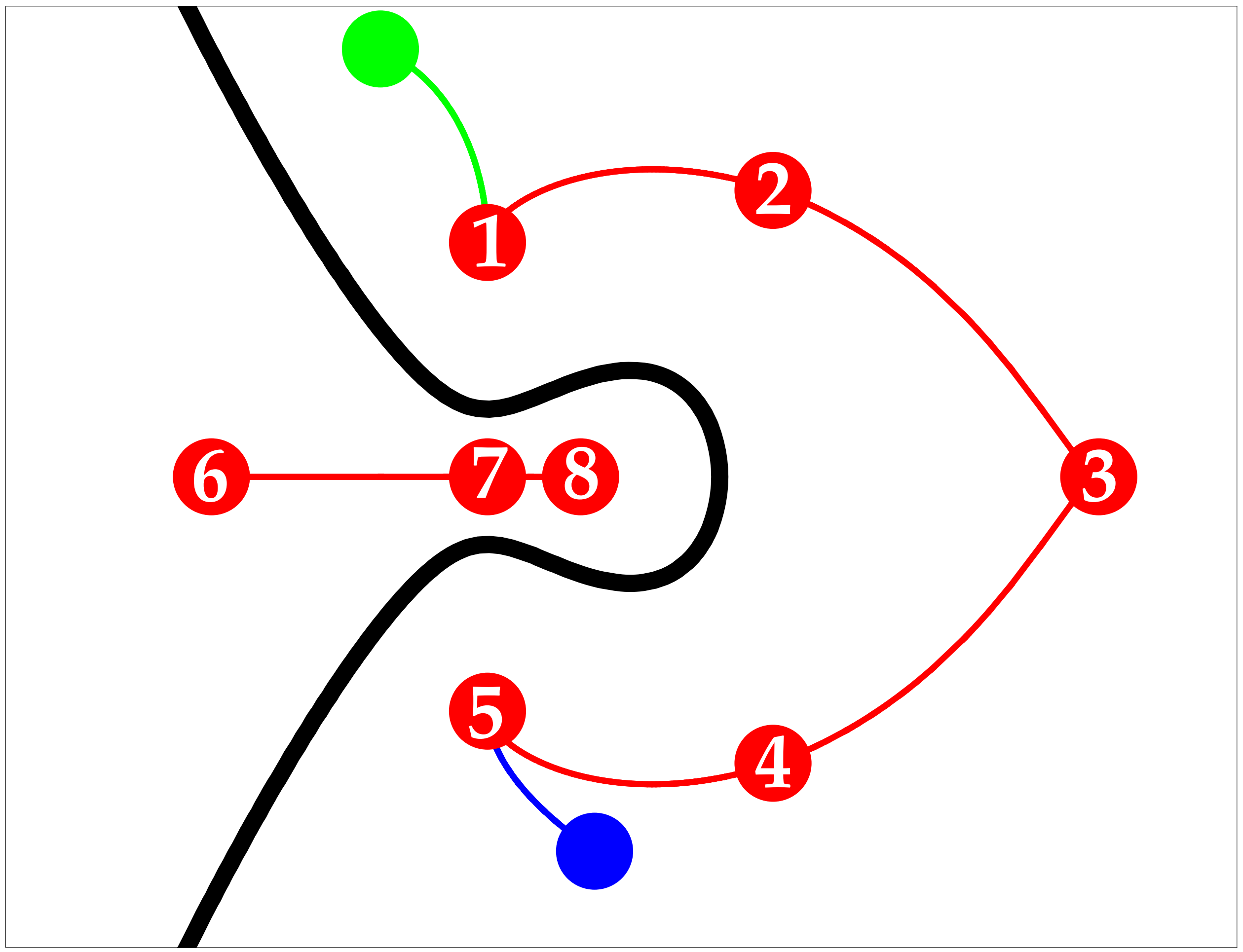
7: Steepest ascent from

8: Steepest ascent from

9: Read matrix

Method: Overview



$$
\begin{array}{c|c}
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1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{bmatrix}
\end{array}
$$

Input: $f(x_1, x_2)$, ●, ●

1: $g = \dfrac{f^2}{\left(x_1^2 + x_2^2 + 1\right)^{\deg(f)+1}}$

2: Solve $\nabla g(x) = 0 \wedge g(x) \neq 0$

3: Find eigenvectors of (Hess g)(●)

4: Steepest ascent using outgoing eigenvectors

positive eigenvalue

5: Form adjacency matrix

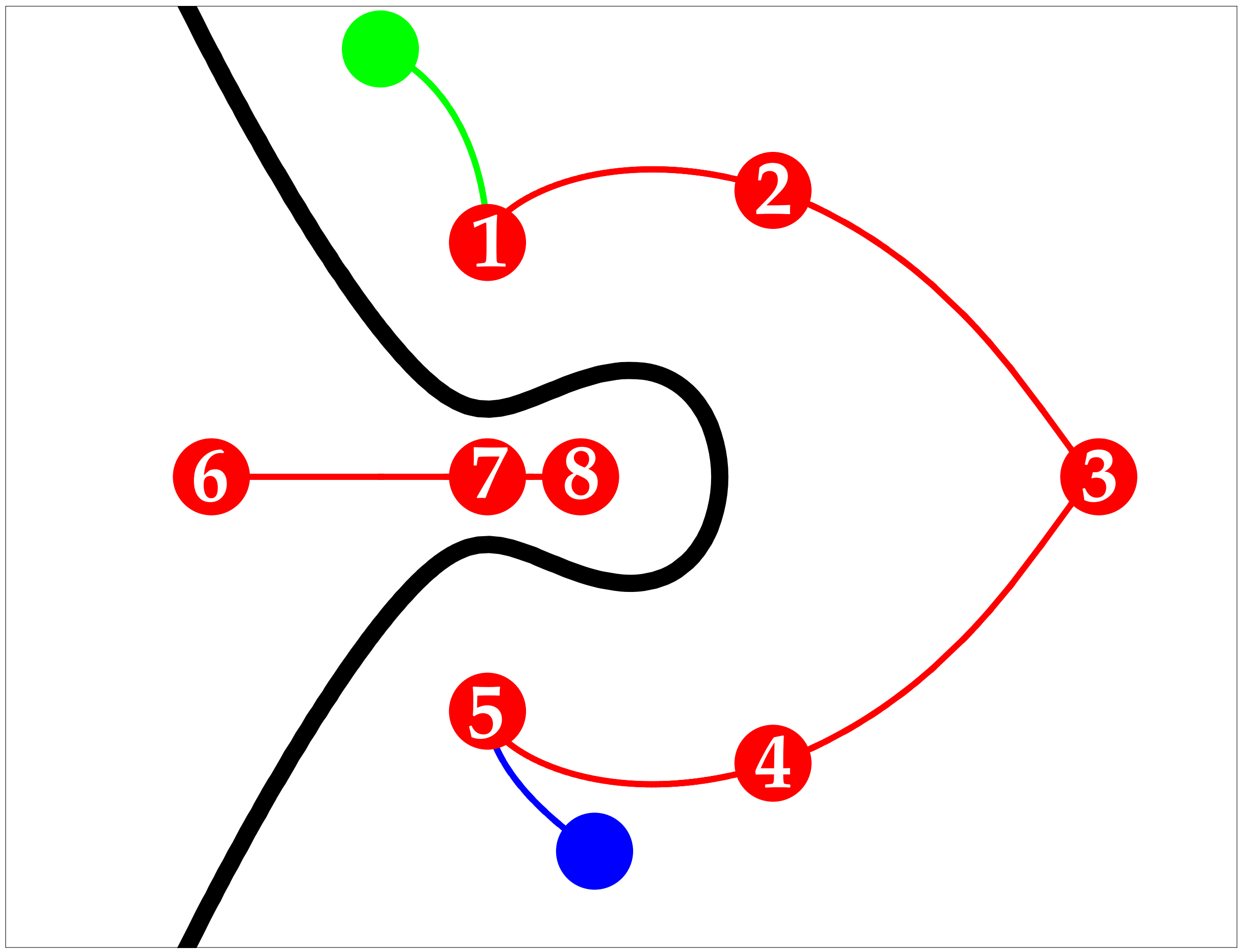
6: Closure of adjacency matrix

7: Steepest ascent from ●

8: Steepest ascent from ●

9: Read matrix

Method: Overview



$$\begin{array}{c|c}
 & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} \\
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\end{array}$$

Input: $f(x_1, x_2)$, ,

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5: Form adjacency matrix

6: Closure of adjacency matrix

7: Steepest ascent from

8: Steepest ascent from

9: Read matrix

Output: True

Method: Demo

Research Challenges

Research Challenges

1. Correctness

Research Challenges

1. Correctness
2. Termination

Research Challenges

1. Correctness
2. Termination
3. Length Bound

1. Correctness: Theorem

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Method Summary:

1. Correctness: Theorem

Method Summary:

1. Using f, form a function g having “good” properties.

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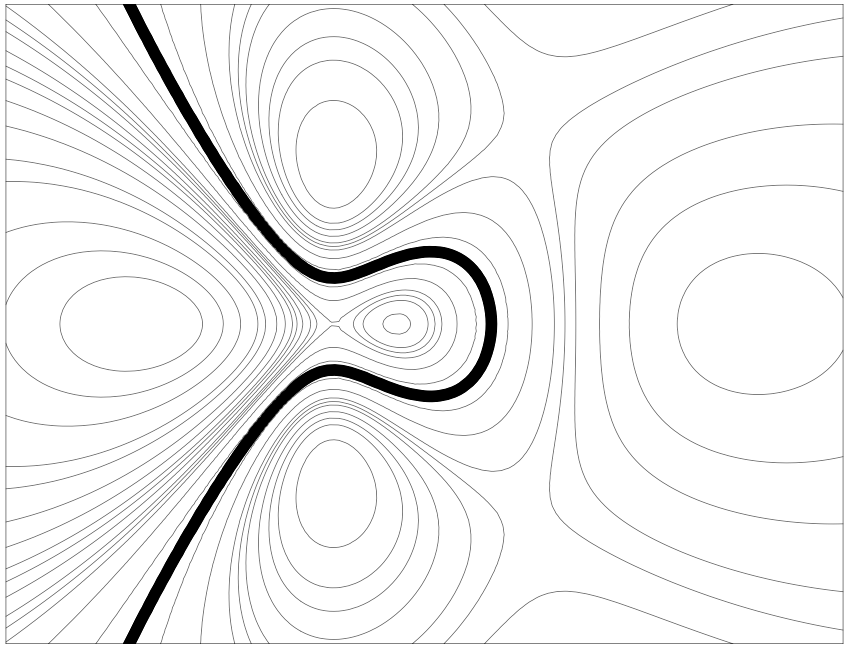

1. Correctness: Theorem

Method Summary:

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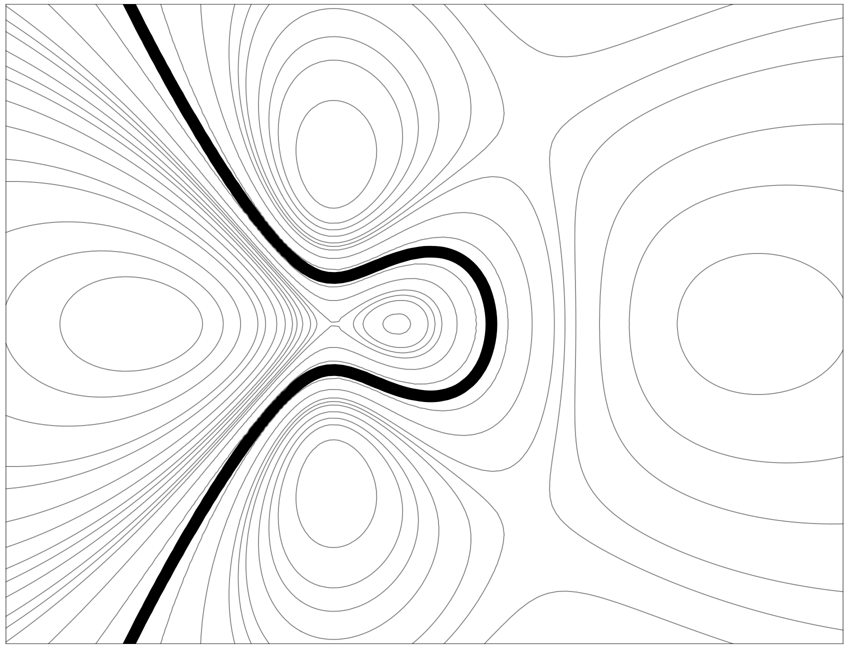
2. Find critical points of g where $g \neq 0$,



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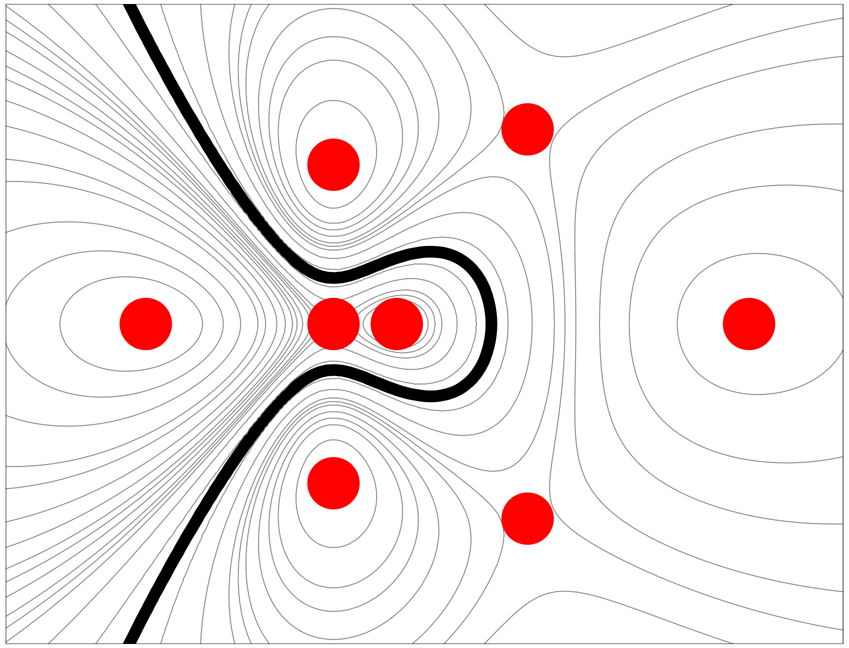

1. Correctness: Theorem

Method Summary:

1. Using f, form a function g having "good" properties.
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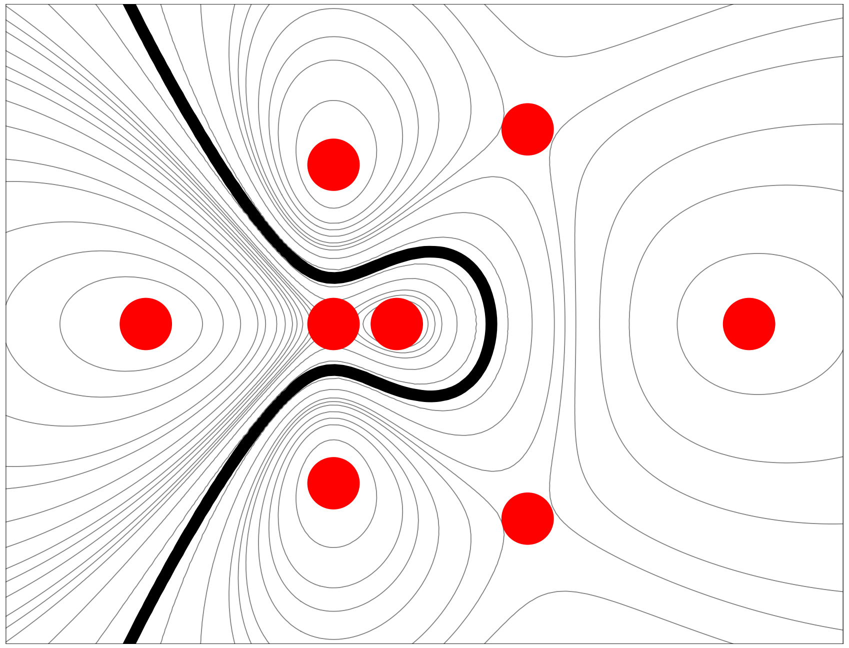

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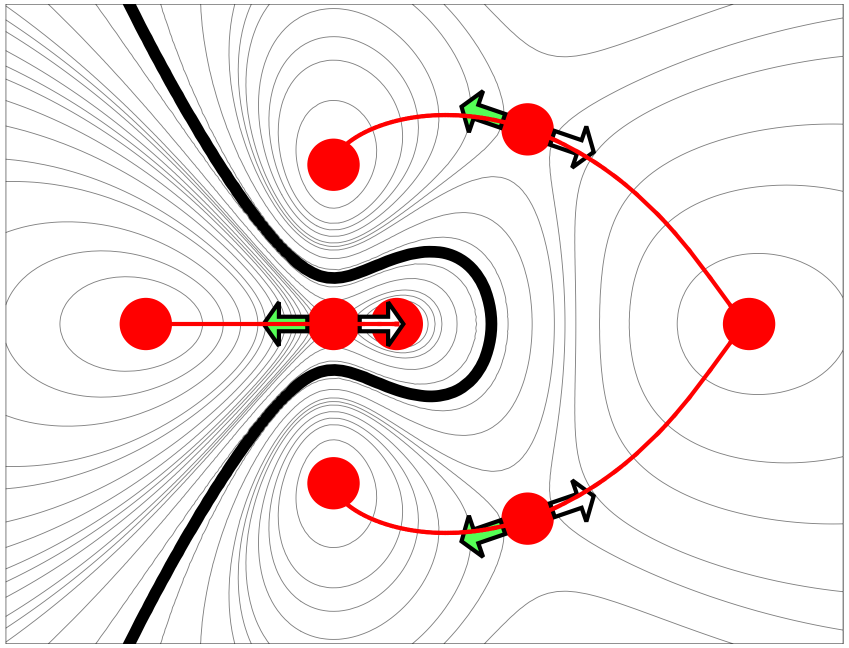

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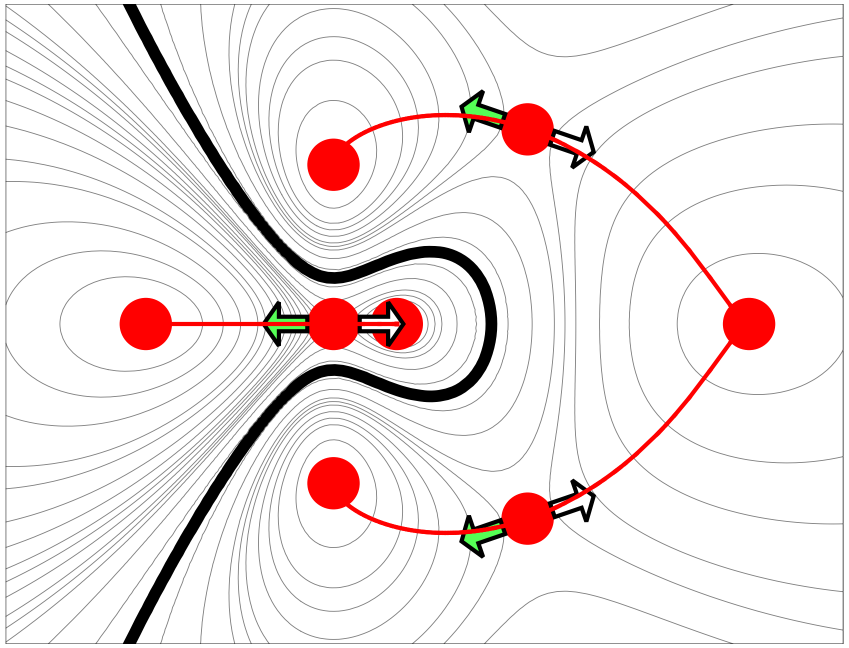
   by steepest ascent paths of ∇g.



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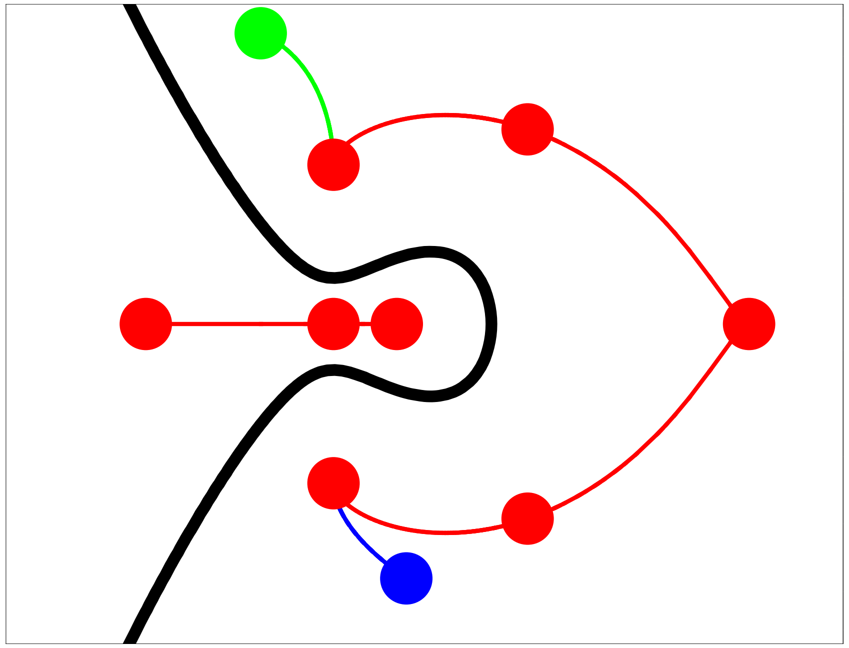

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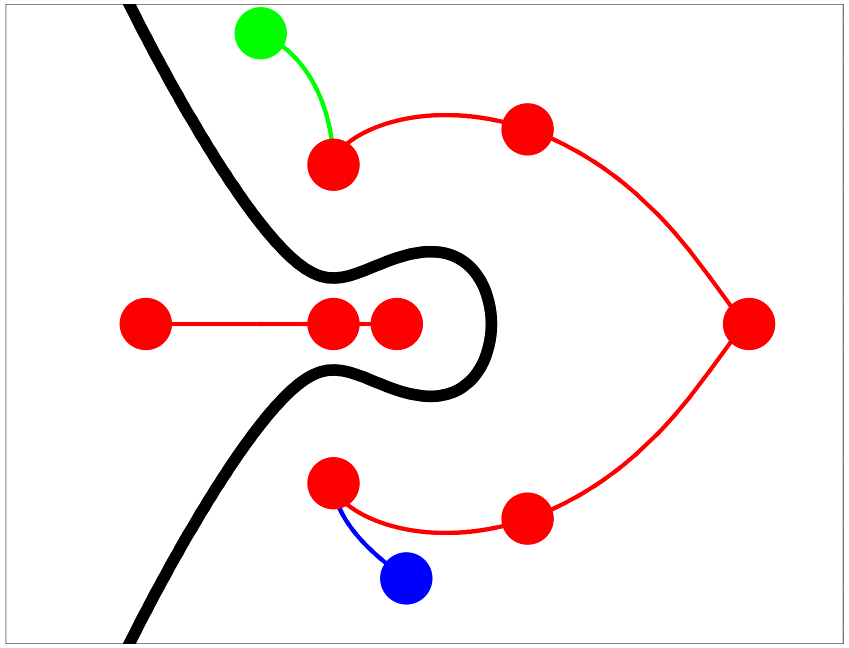

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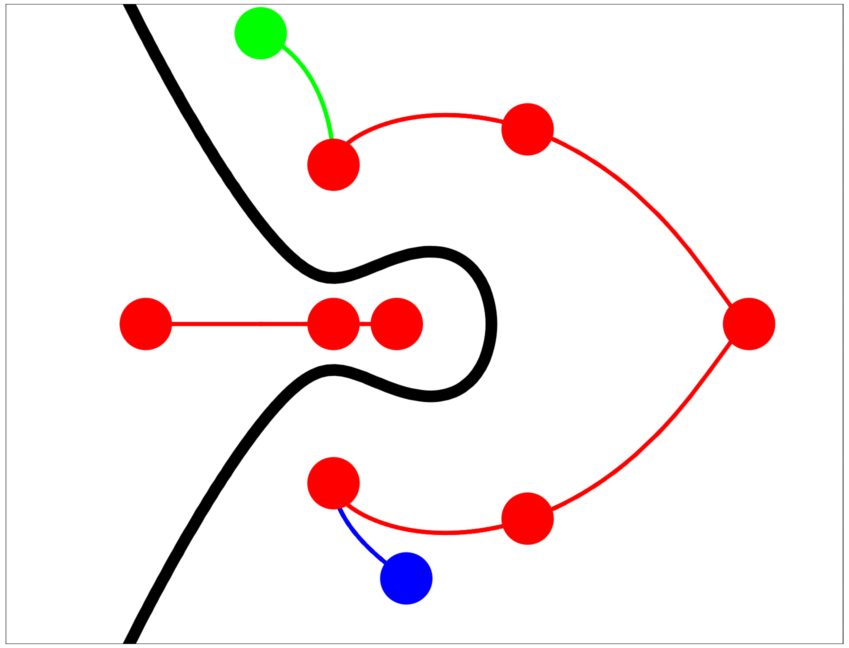
Let g be a "good" function.



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Theorem:

Let g be a "good" function.
Any two critical points of g in a same connected component of $\{g \neq 0\}$
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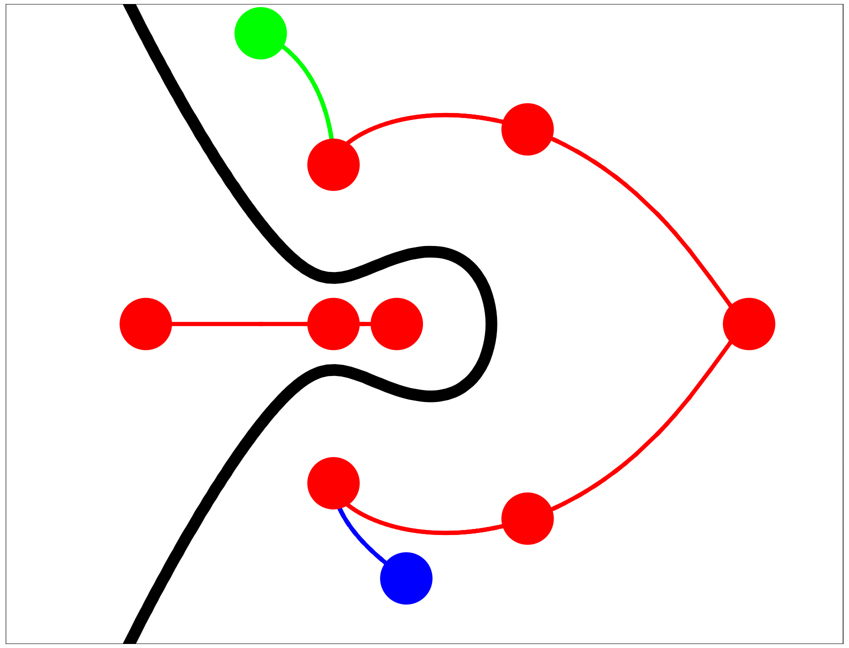

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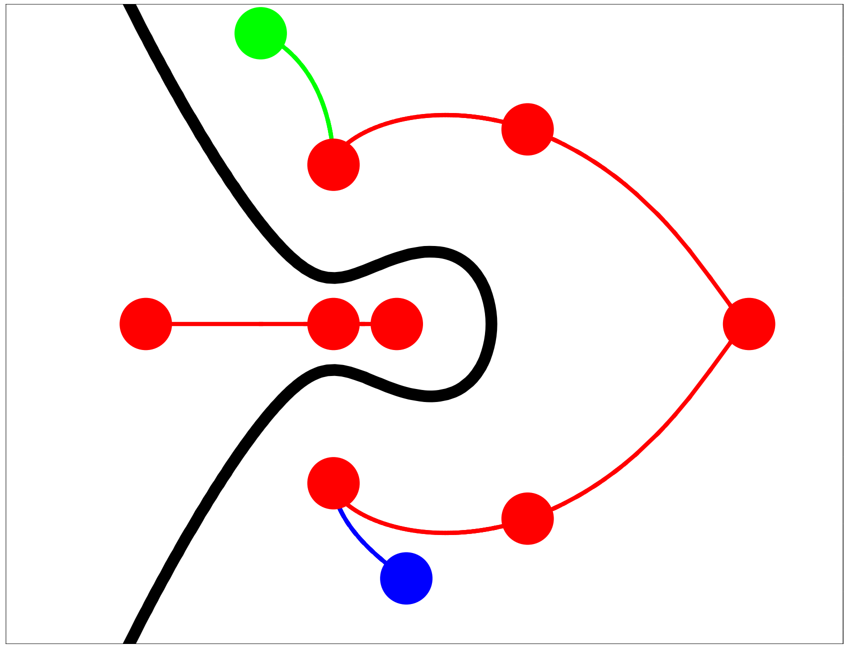

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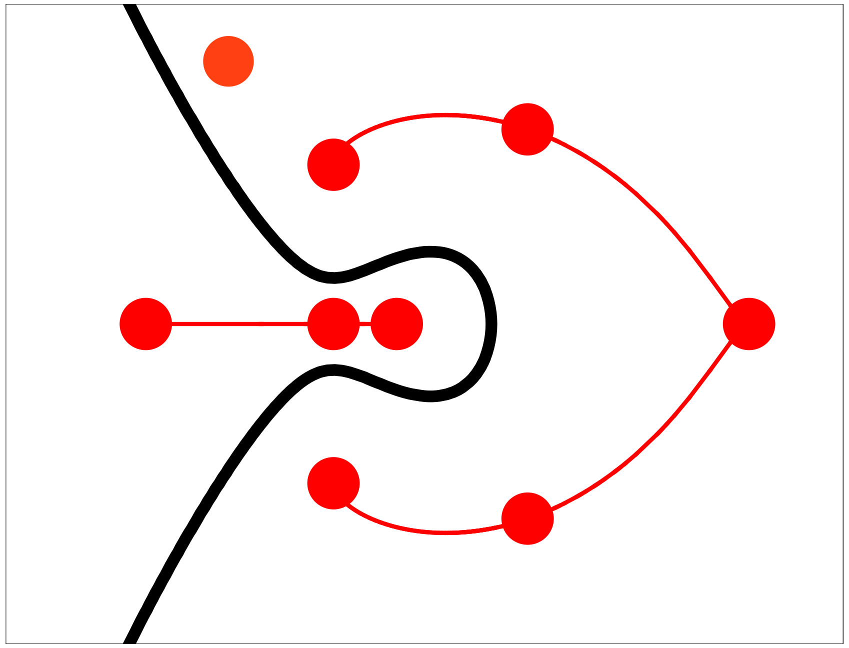

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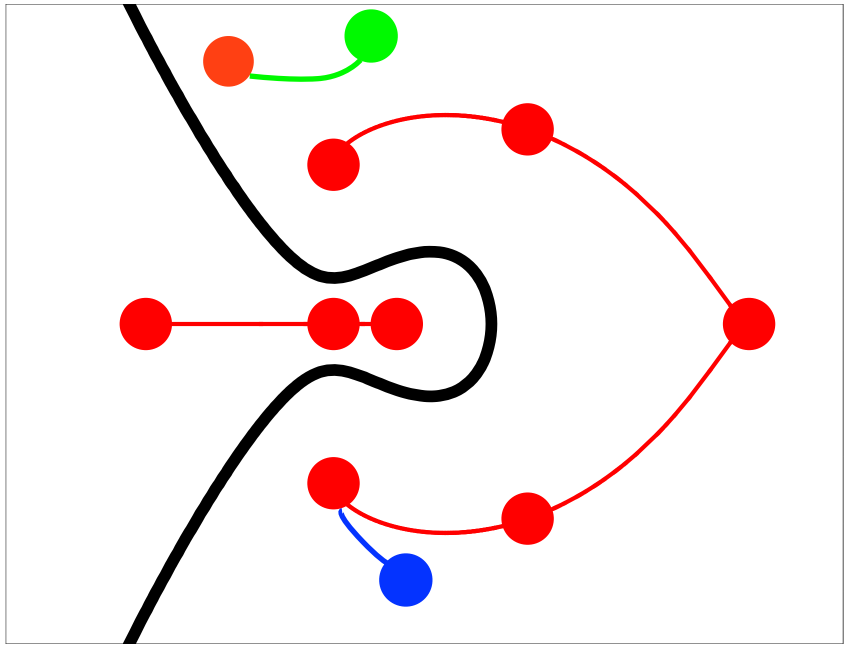


1. Correctness: “Good” Properties

1. Correctness: "Good" Properties

GOOD PROPERTIES

BAD PROPERTIES

1. Correctness: “Good” Properties

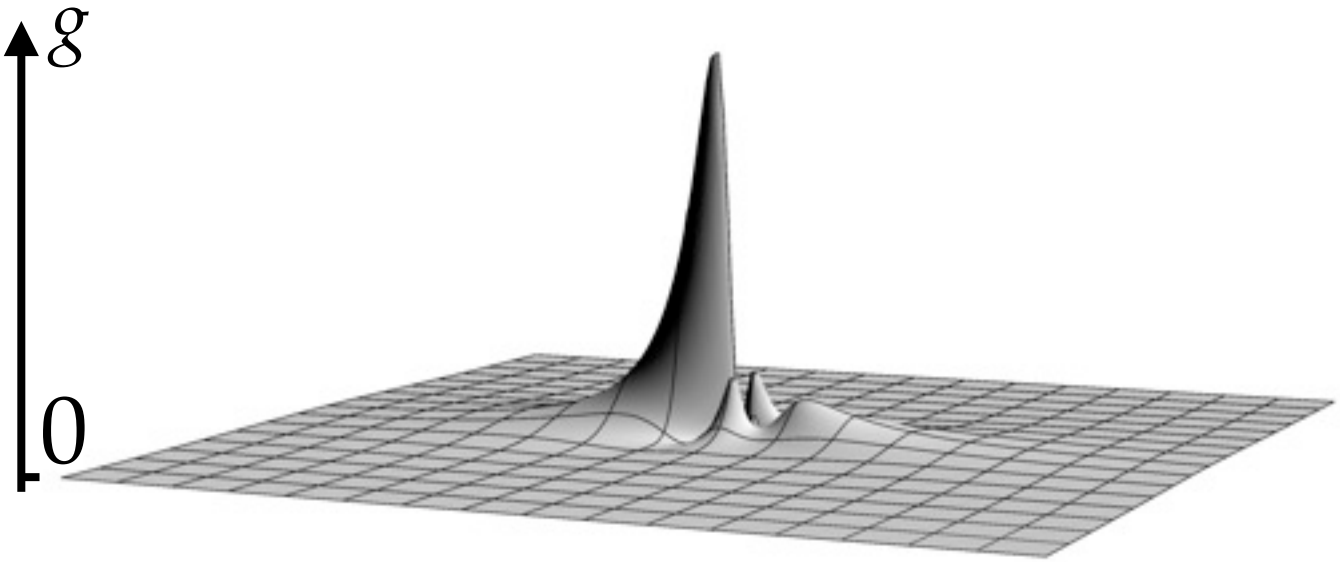
GOOD PROPERTIES

BAD PROPERTIES

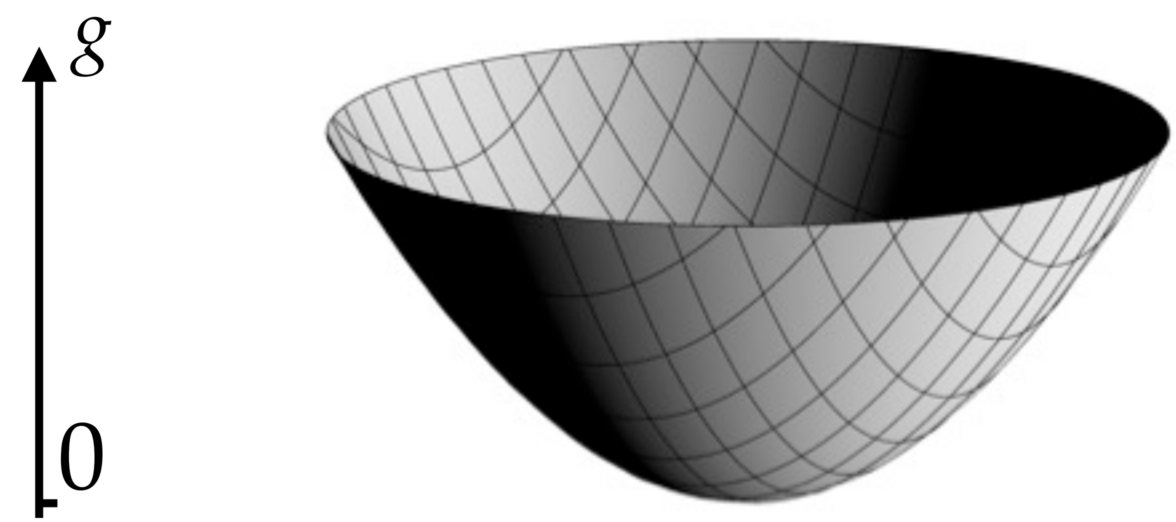
- $g(x) \to 0$ as $\|x\| \to \infty$

1. Correctness: "Good" Properties

GOOD PROPERTIES



BAD PROPERTIES



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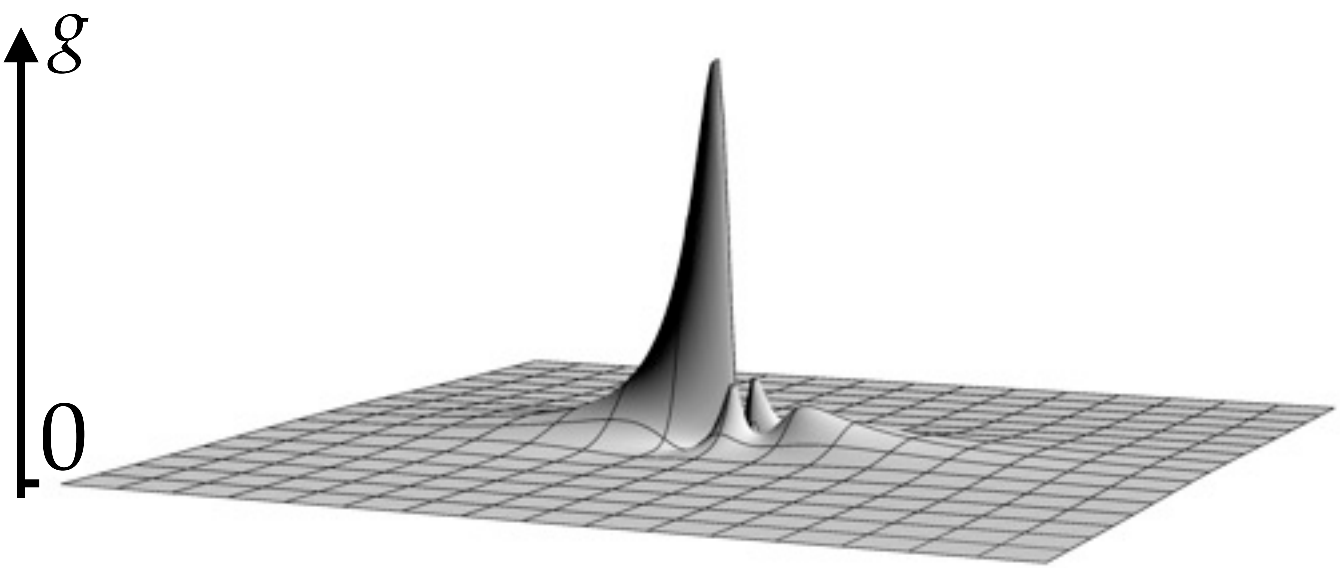
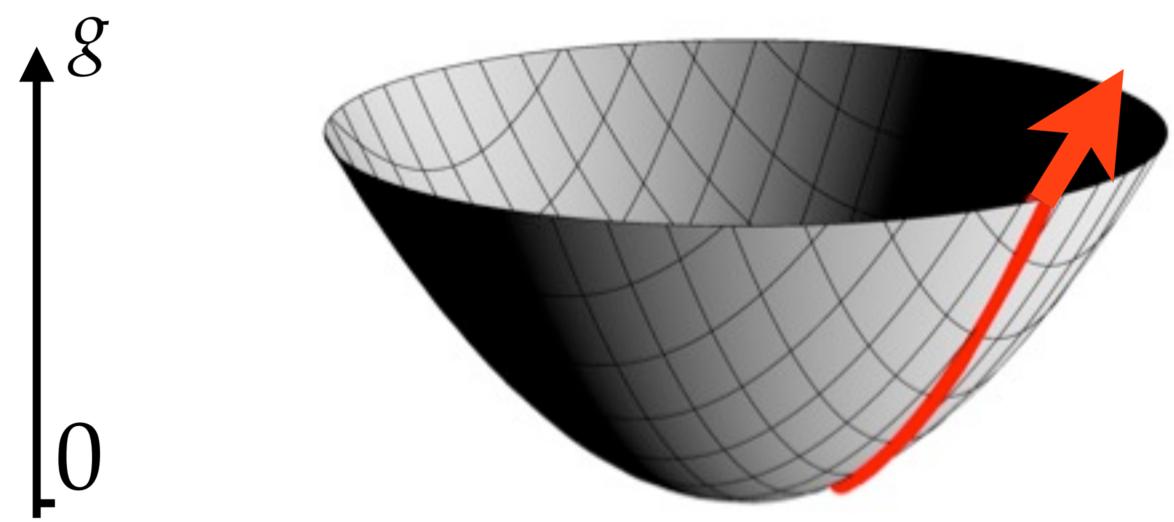
1. Correctness: "Good" Properties

GOOD PROPERTIES **BAD PROPERTIES**





- $g(x) \to 0$ as $\|x\| \to \infty$

1. Correctness: “Good” Properties

GOOD PROPERTIES

BAD PROPERTIES

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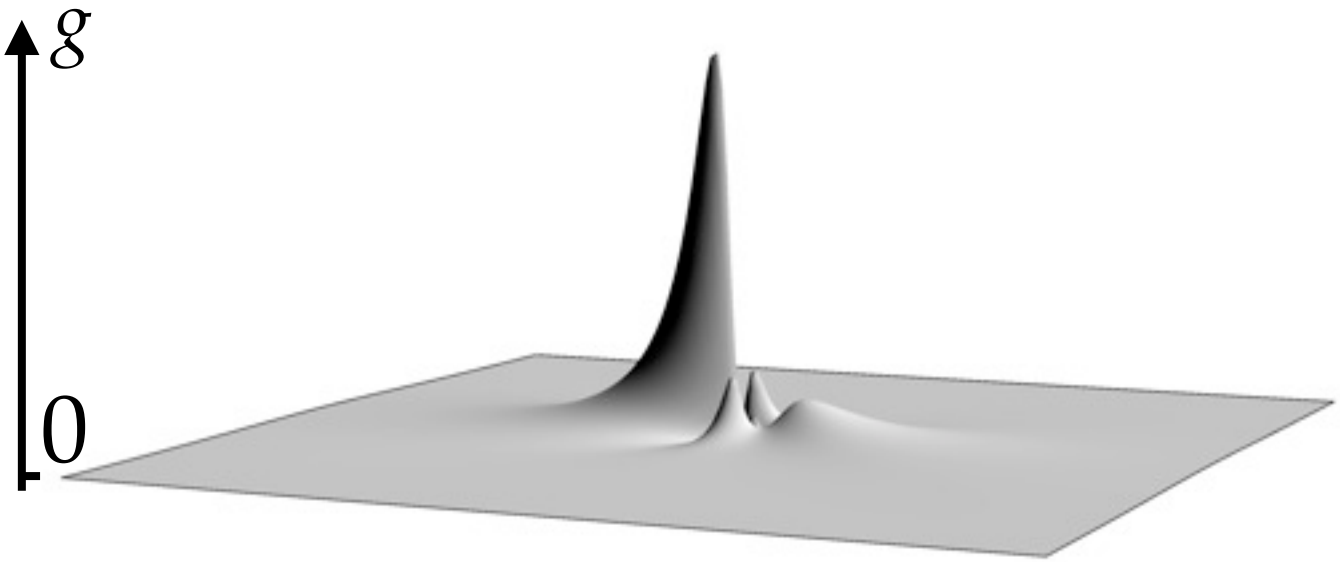

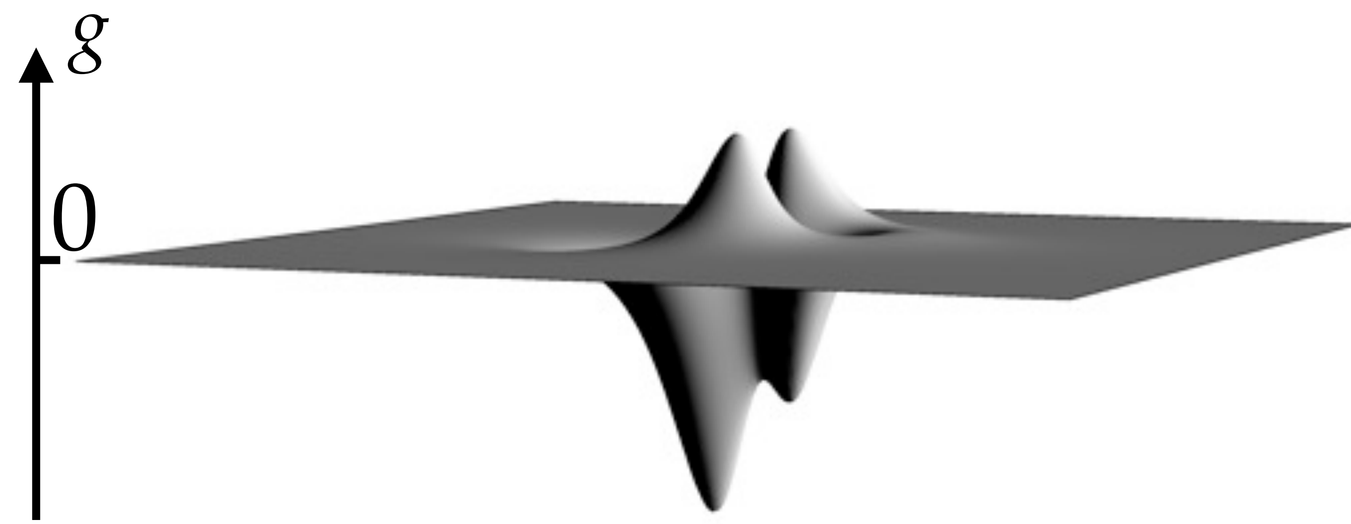

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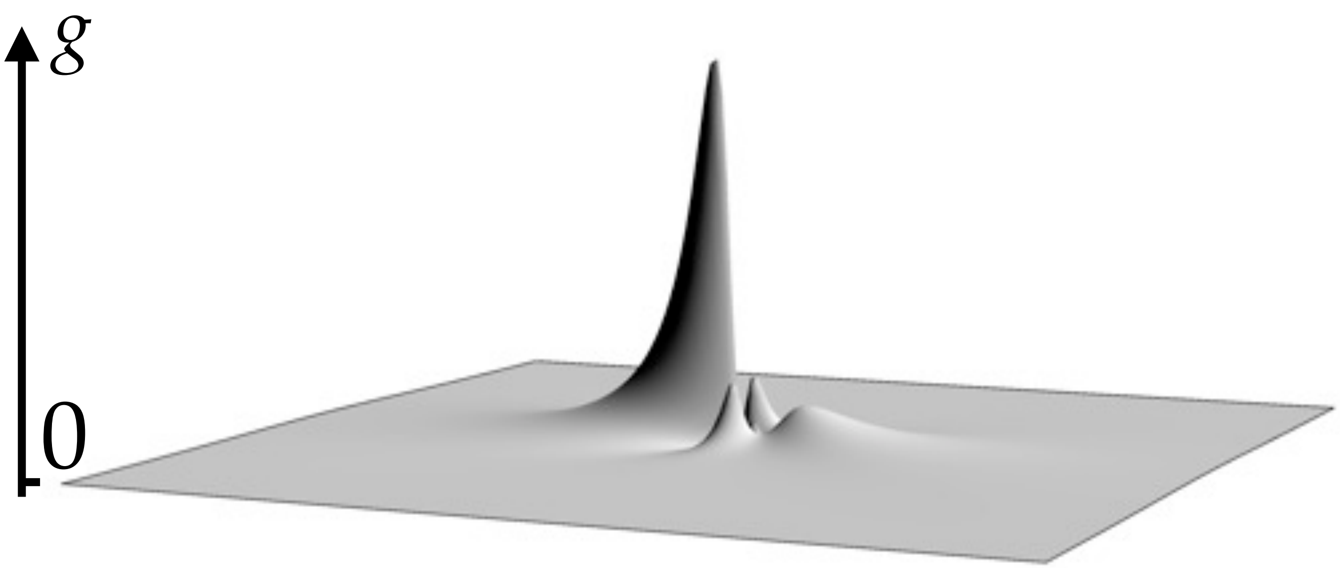

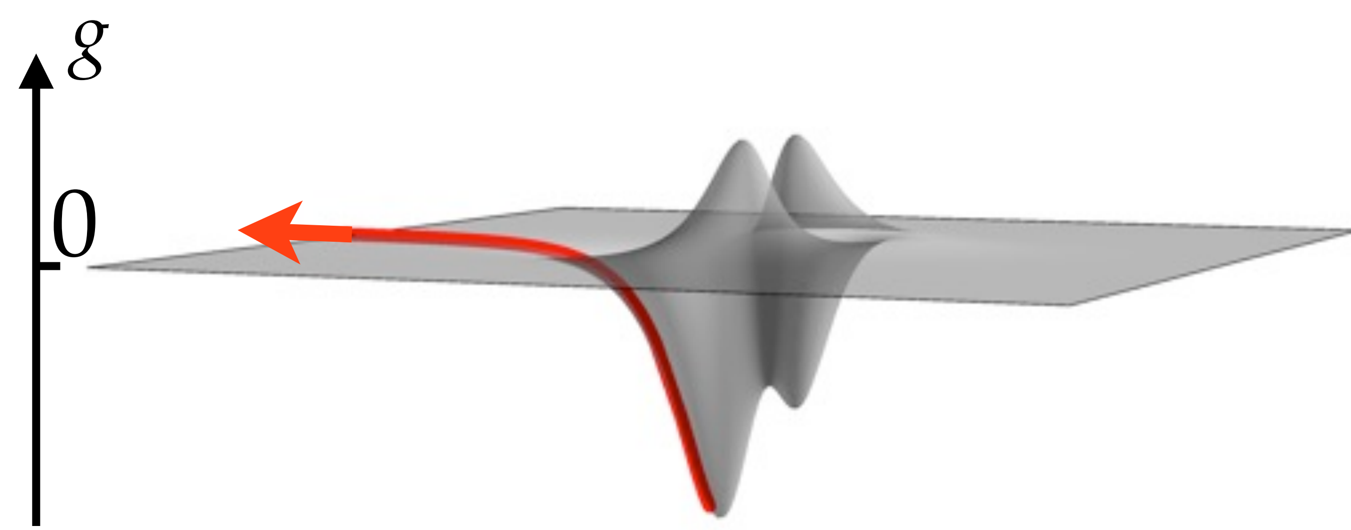

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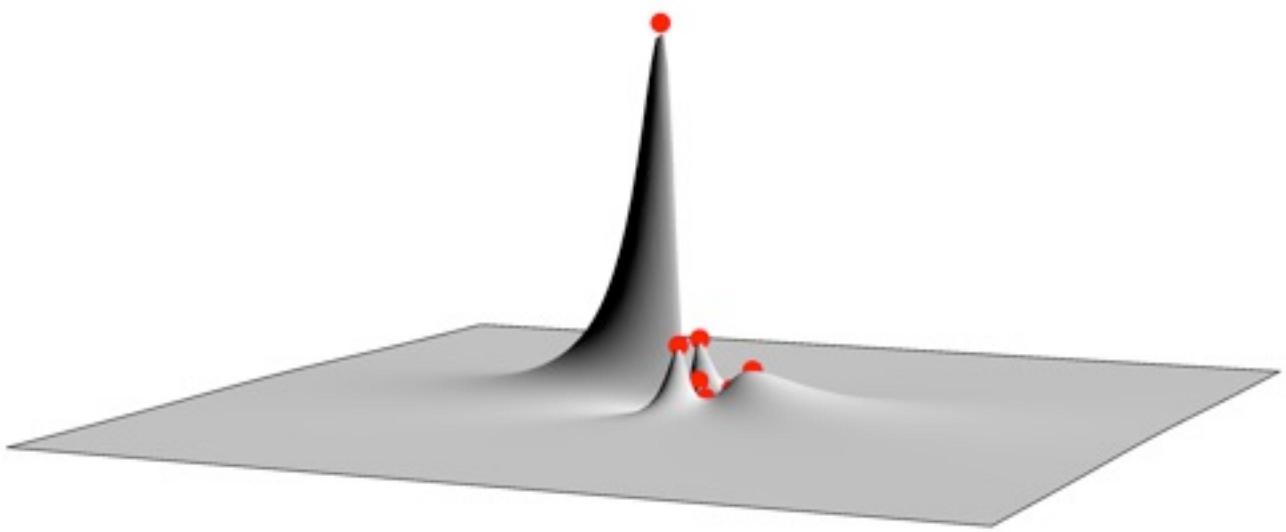

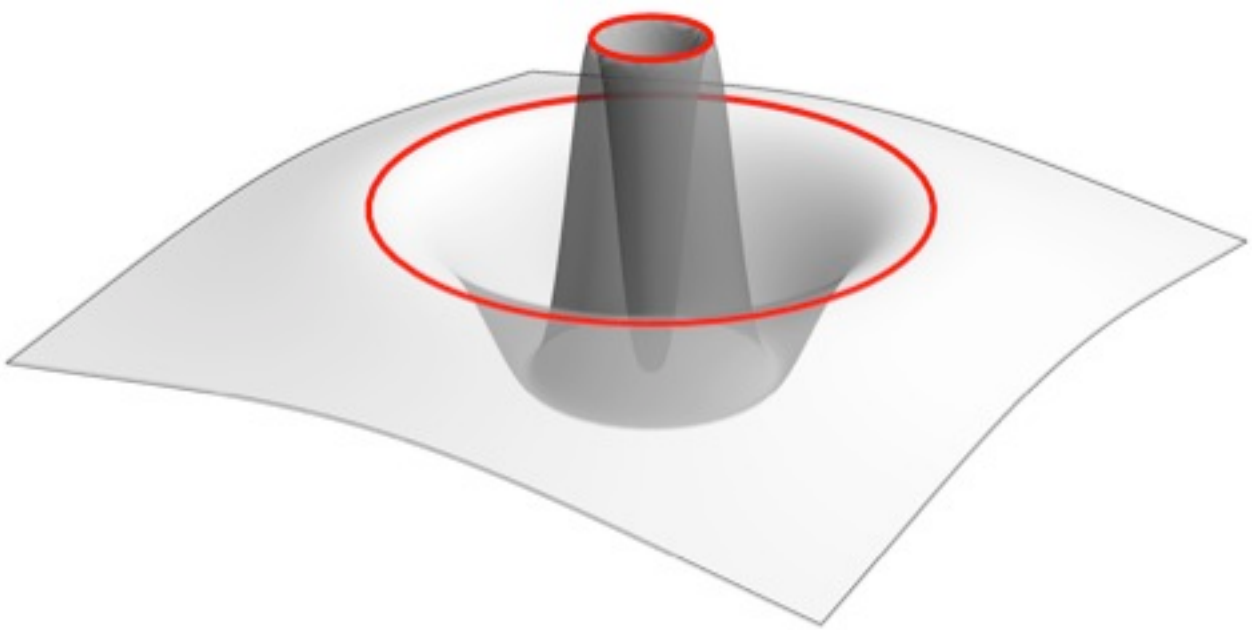

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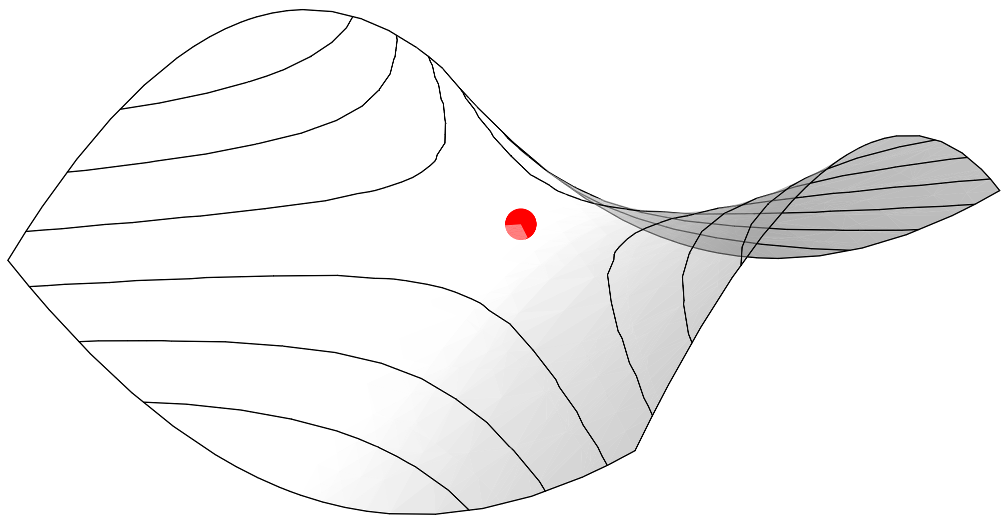

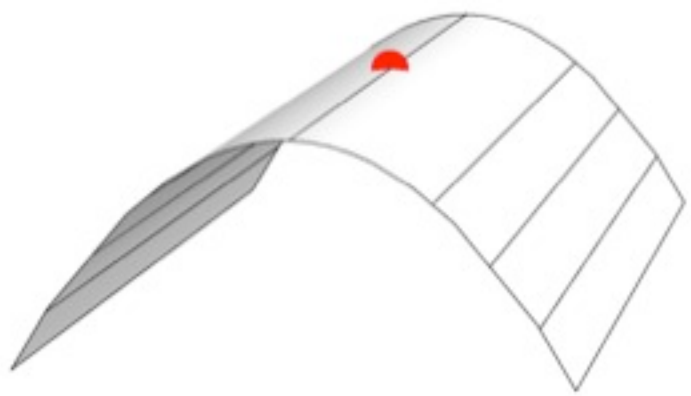

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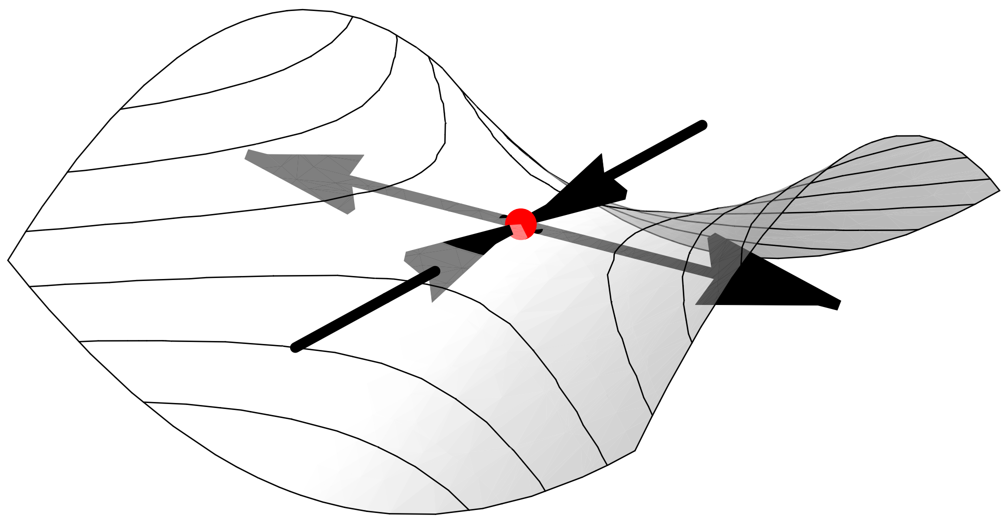

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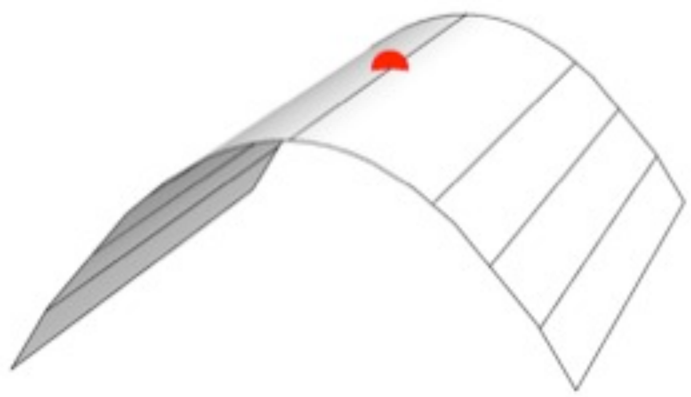

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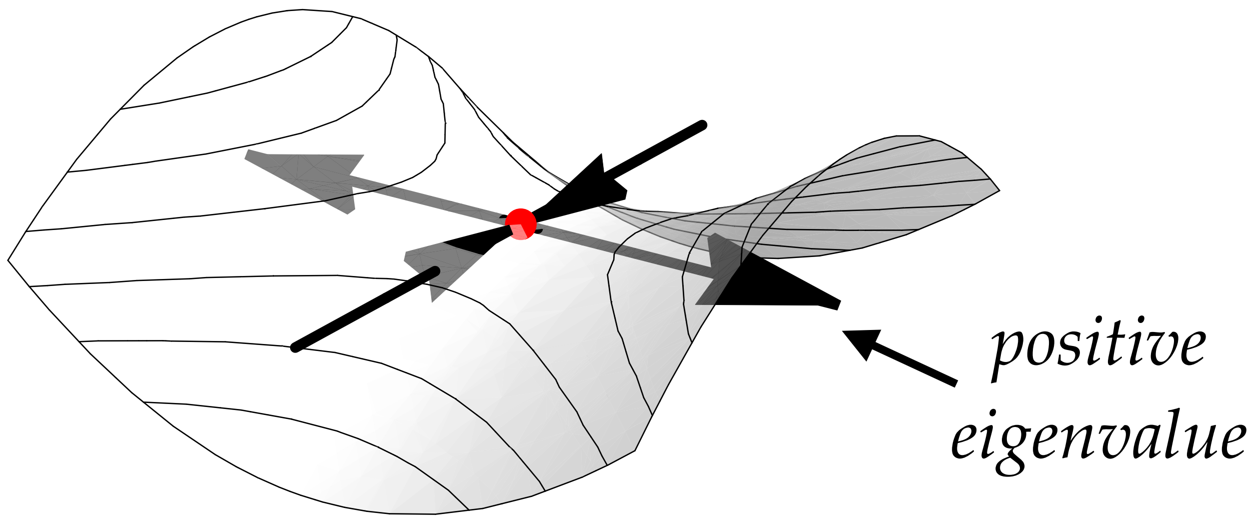

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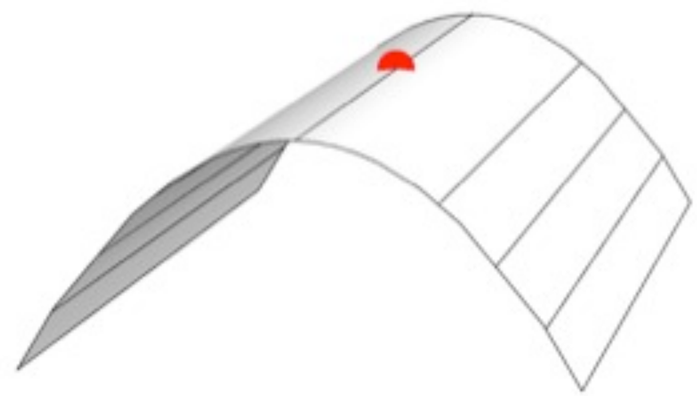

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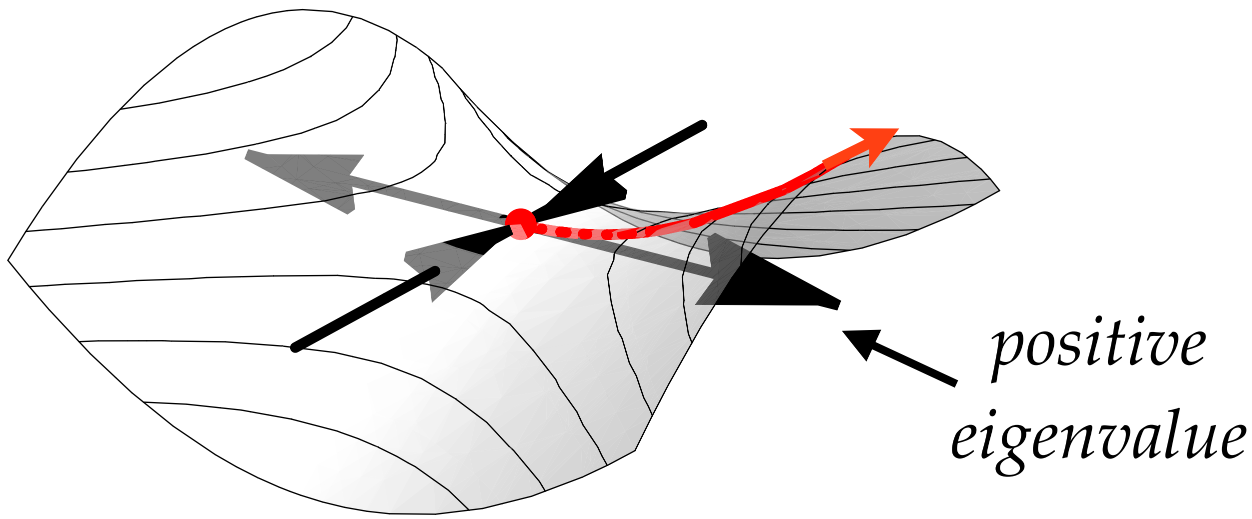

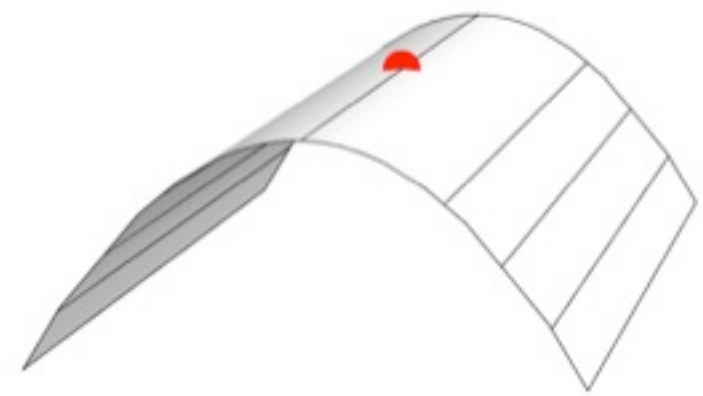

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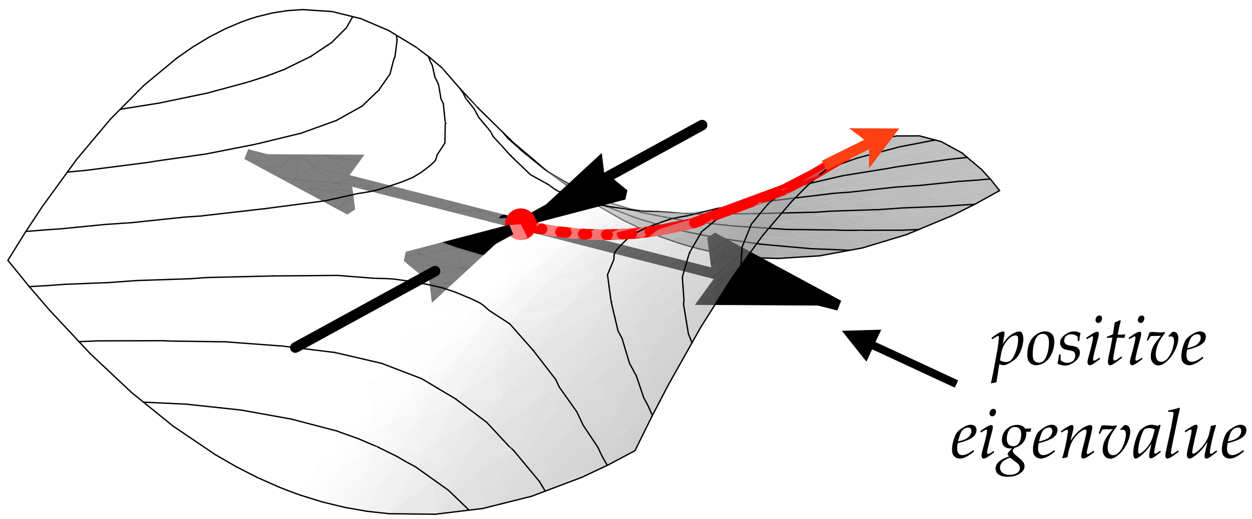

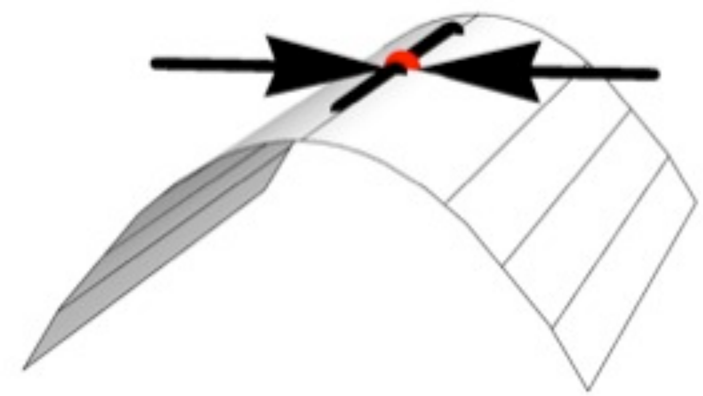

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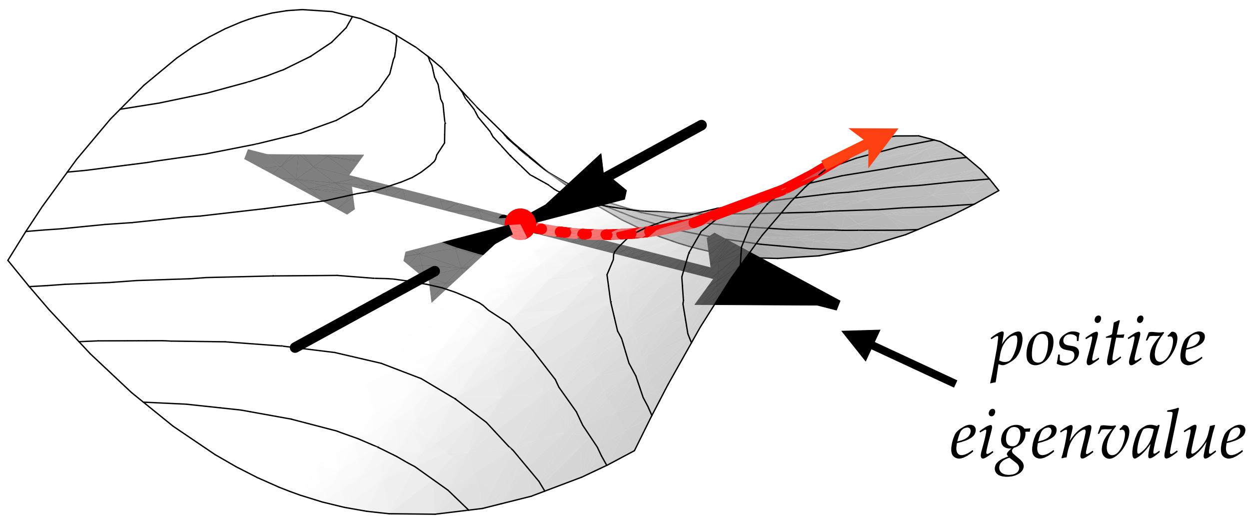

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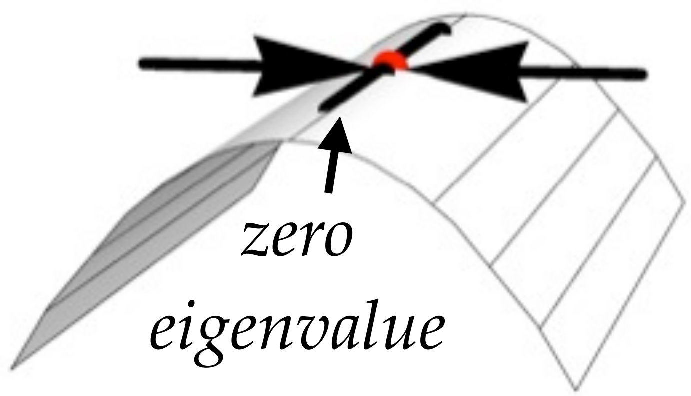

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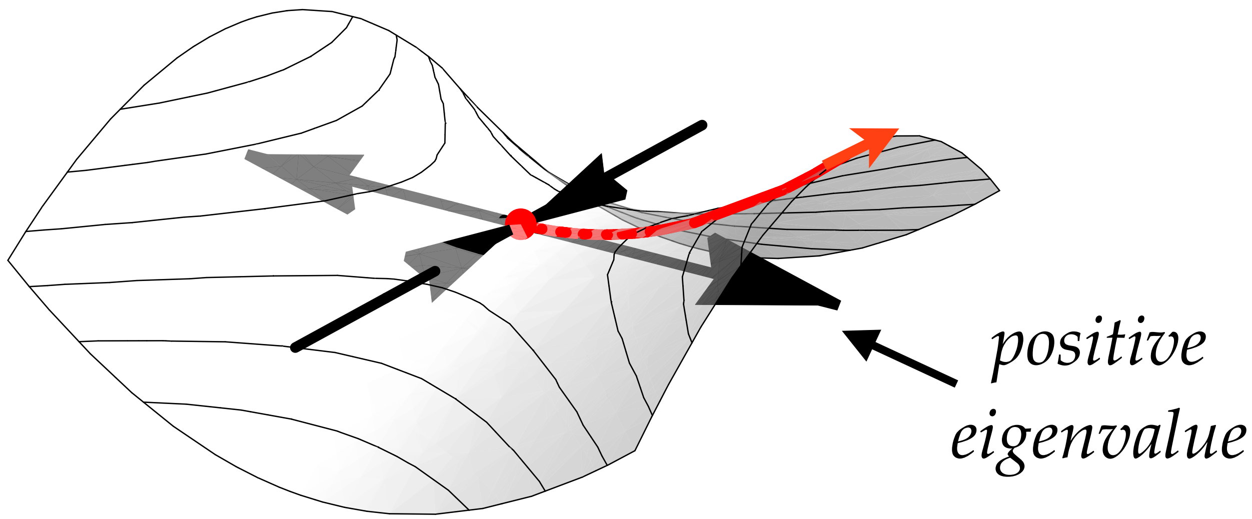

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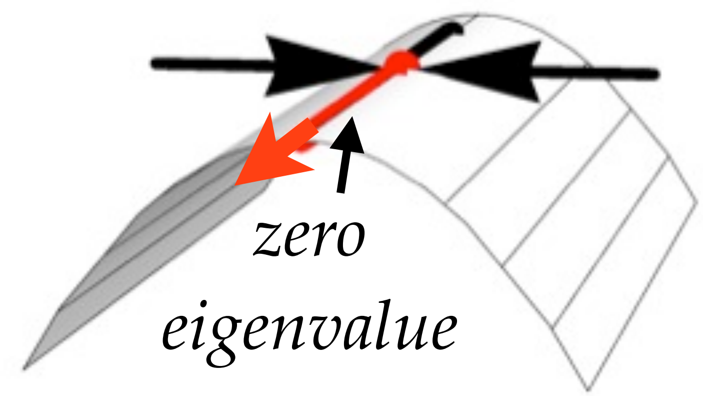

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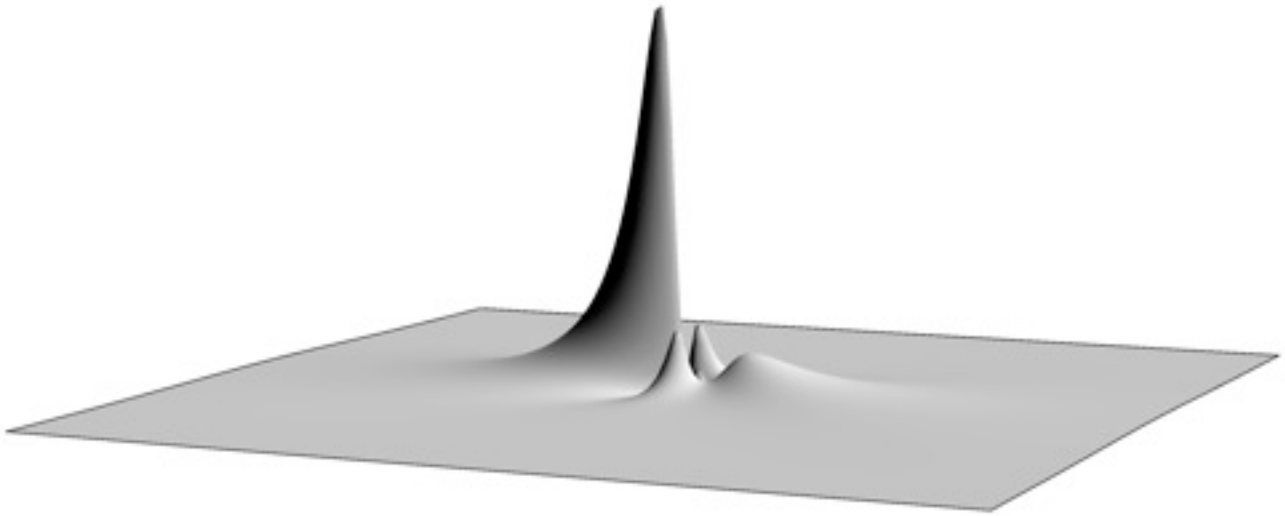

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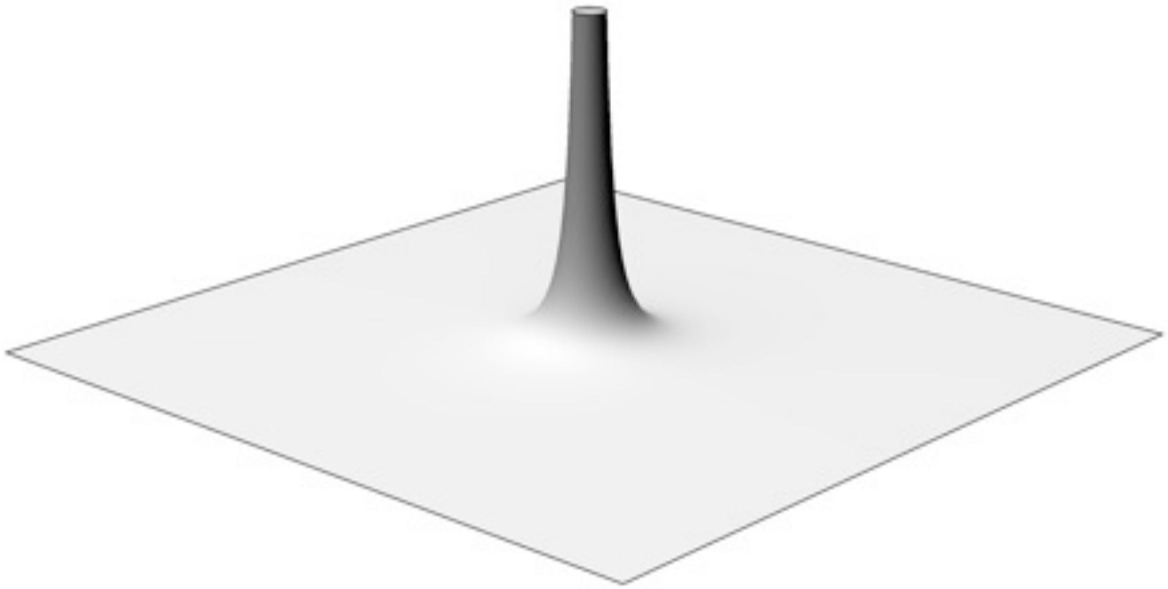

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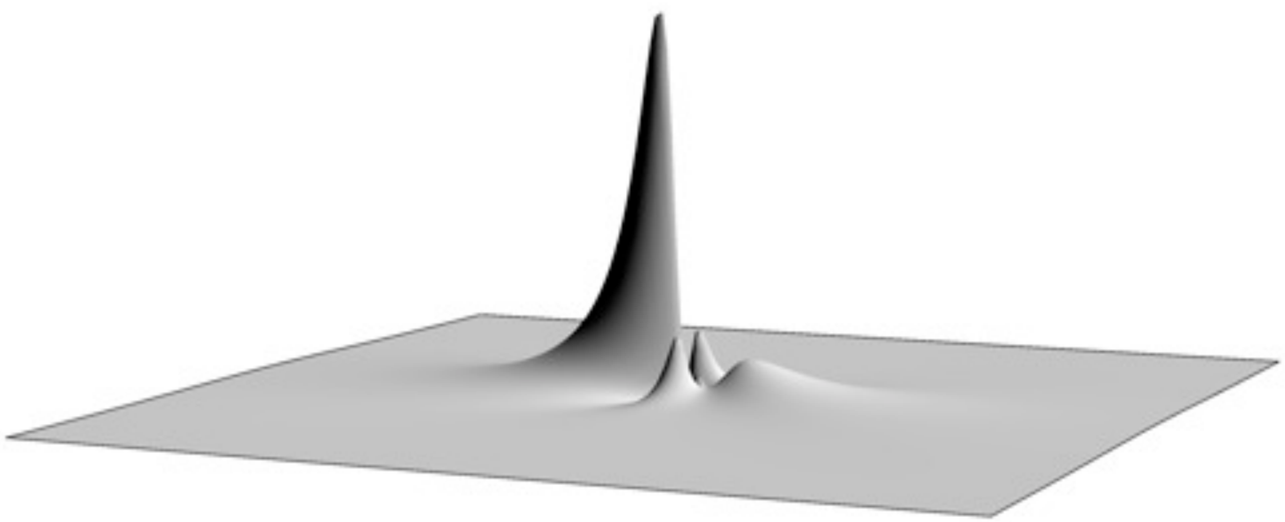

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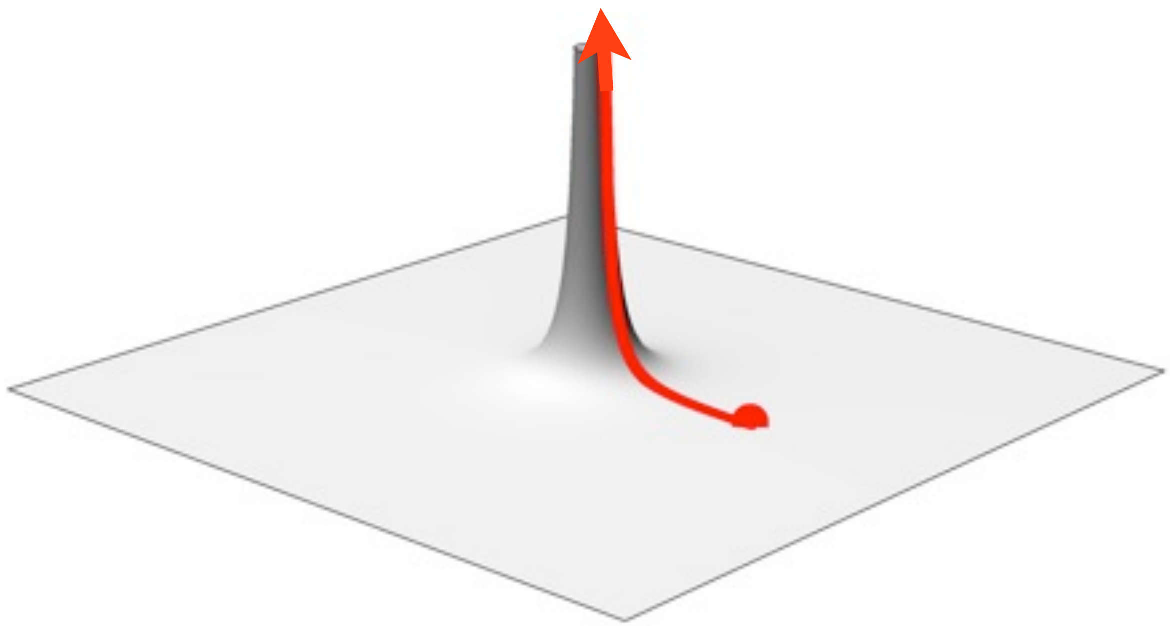

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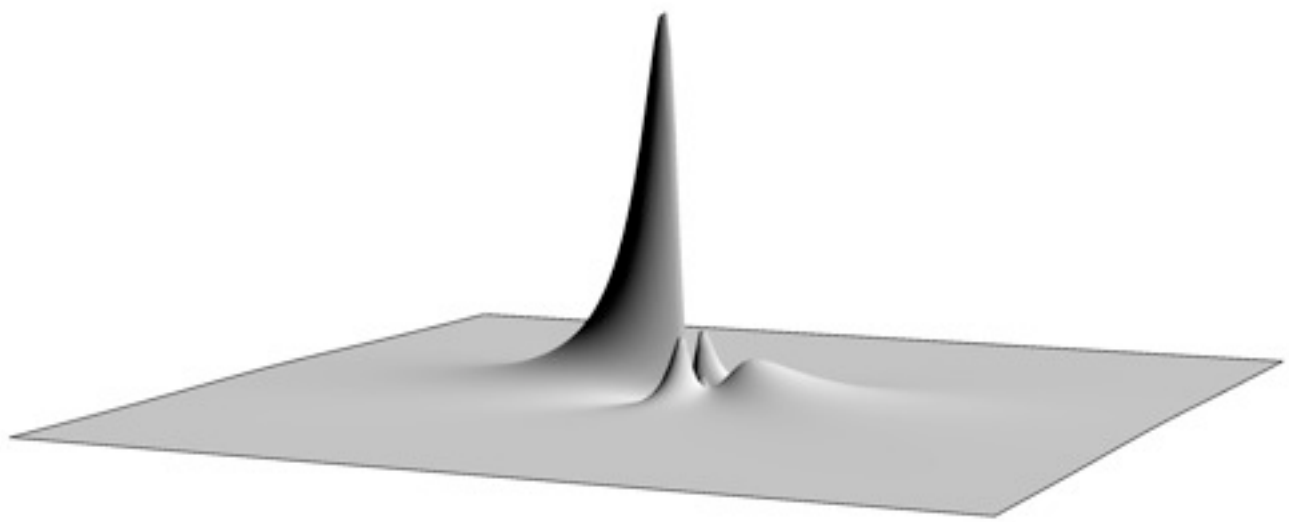

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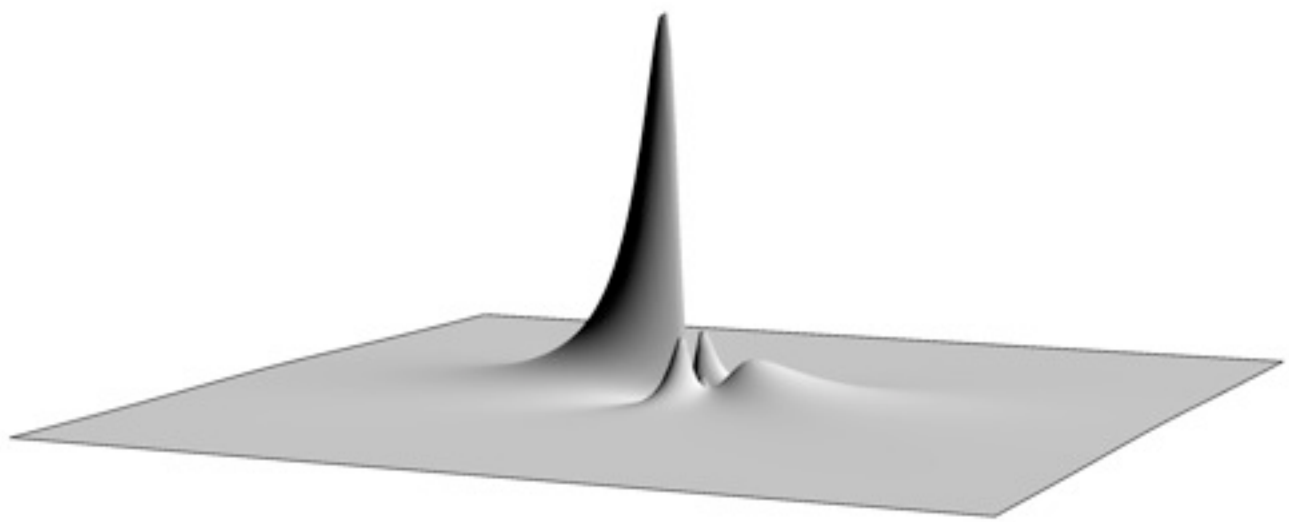

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Example

$$g = \frac{f^2}{\left(x_1^2 + x_2^2 + 1\right)^{\deg(f)+1}}$$

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Trajectory of ∇g through ●

$$\phi'(t) = \nabla g\big(\phi(t)\big)$$
$$\phi(0) = \bullet$$

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Trajectory of ∇g through

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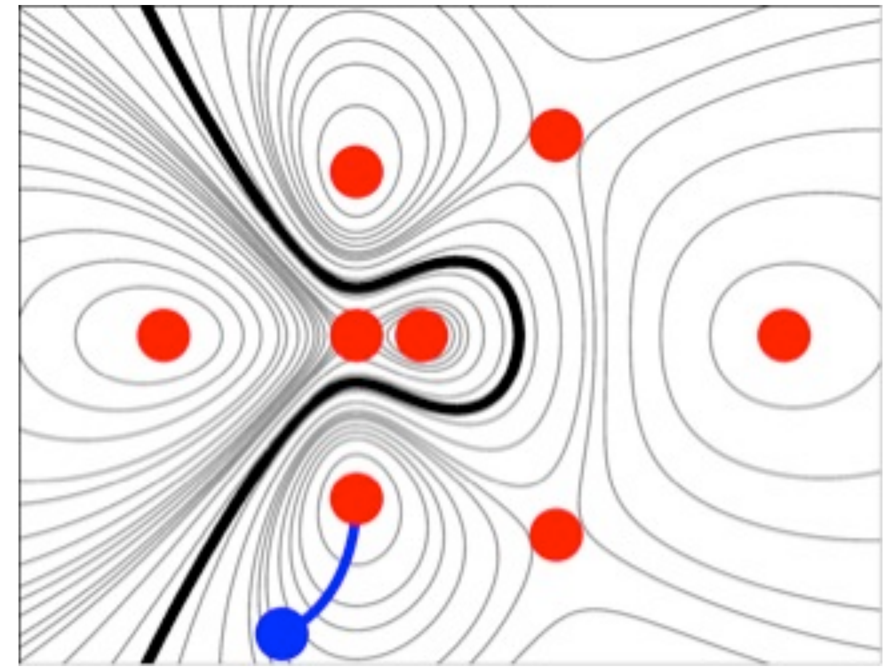
$(0, \infty) \xrightarrow{\phi}$



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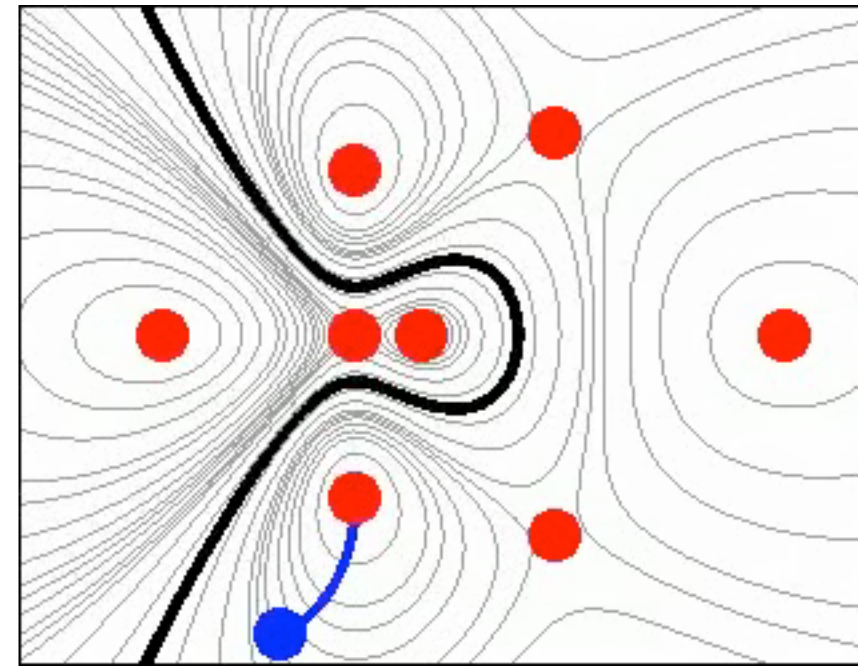

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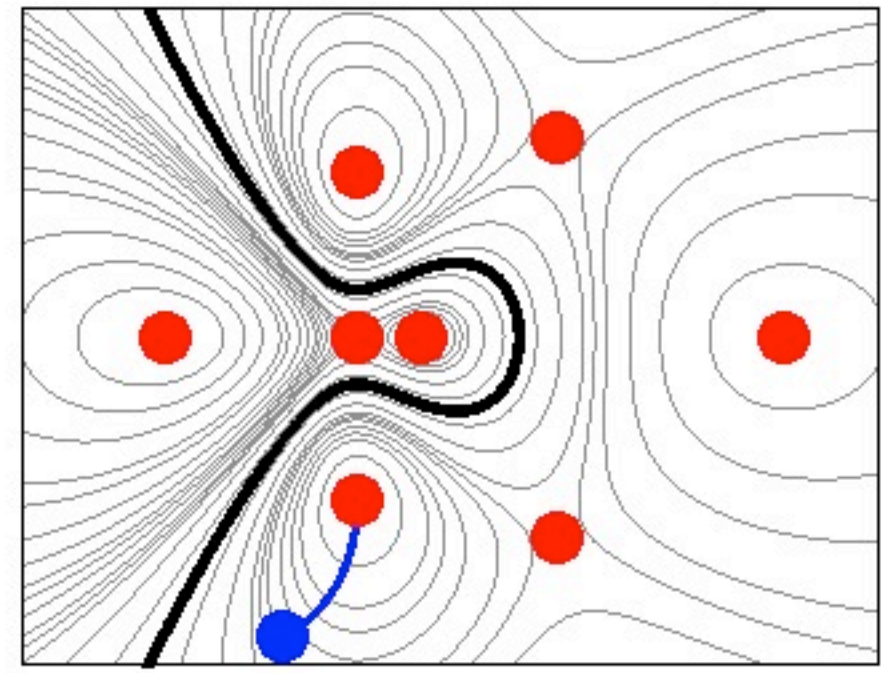

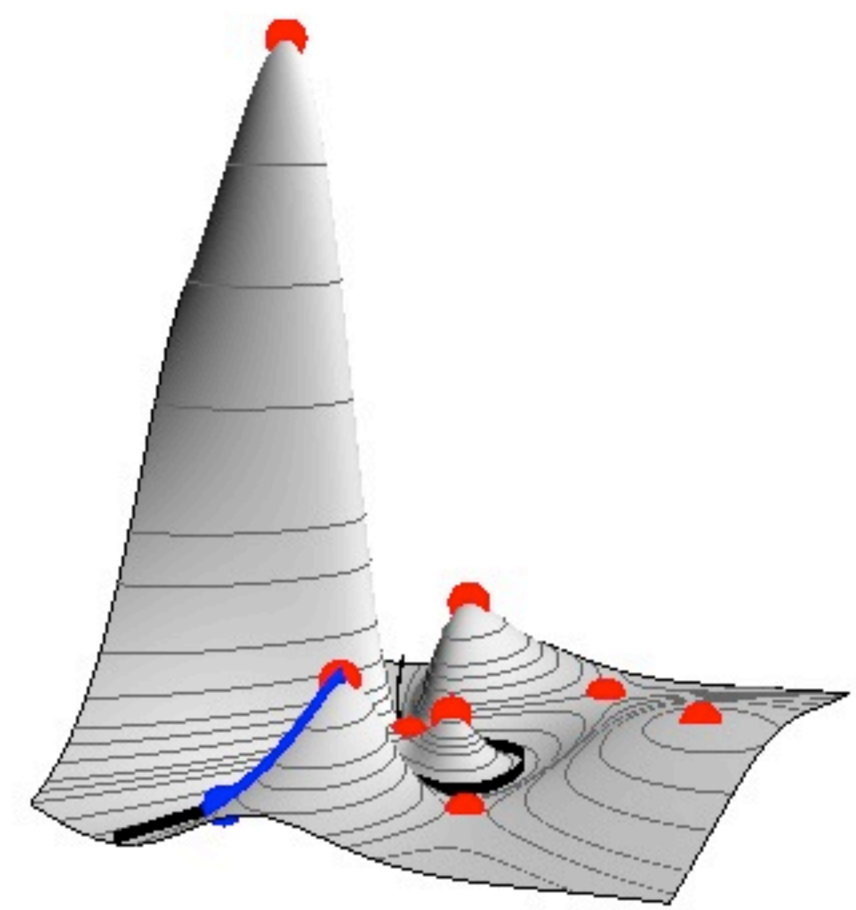

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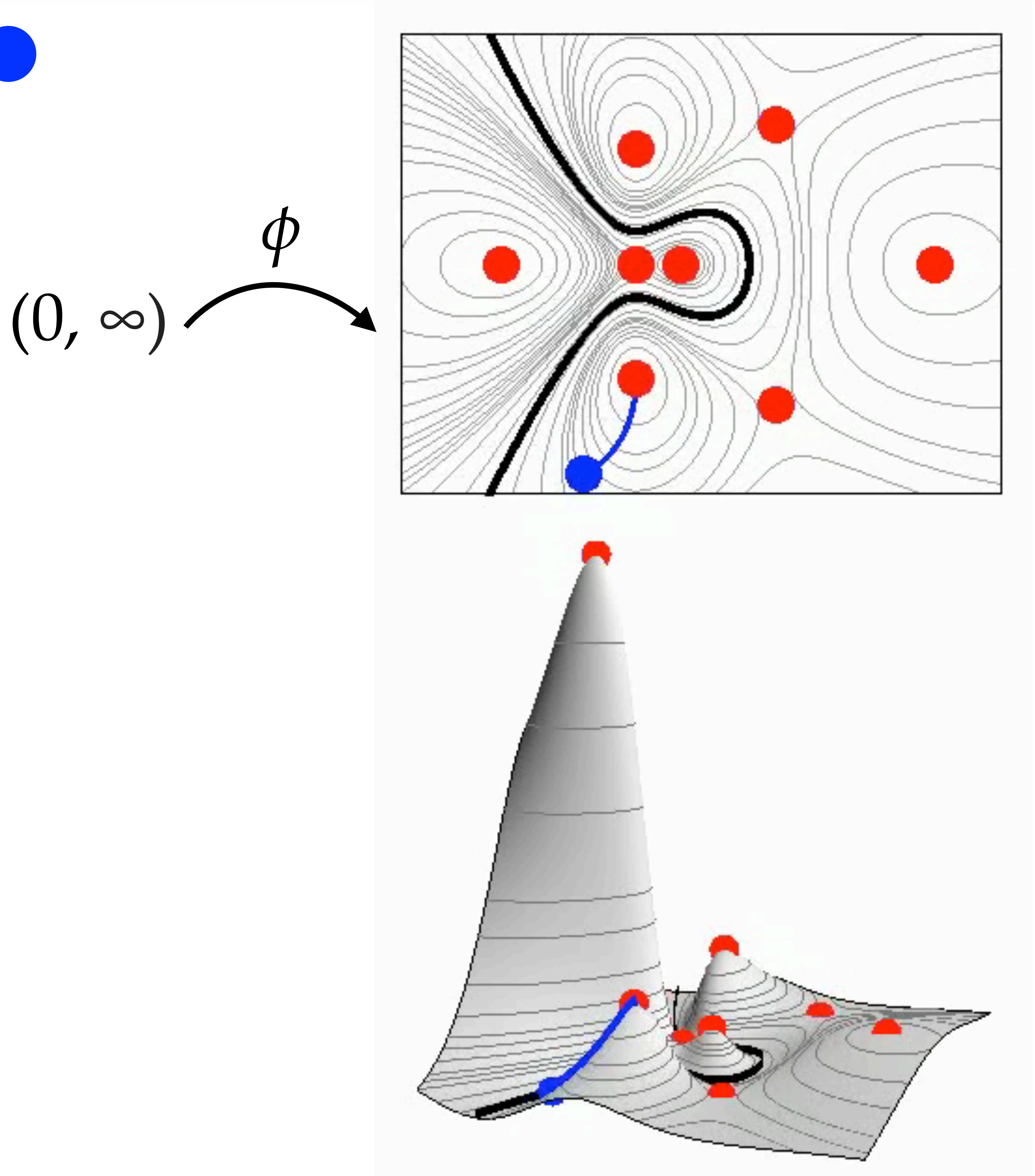

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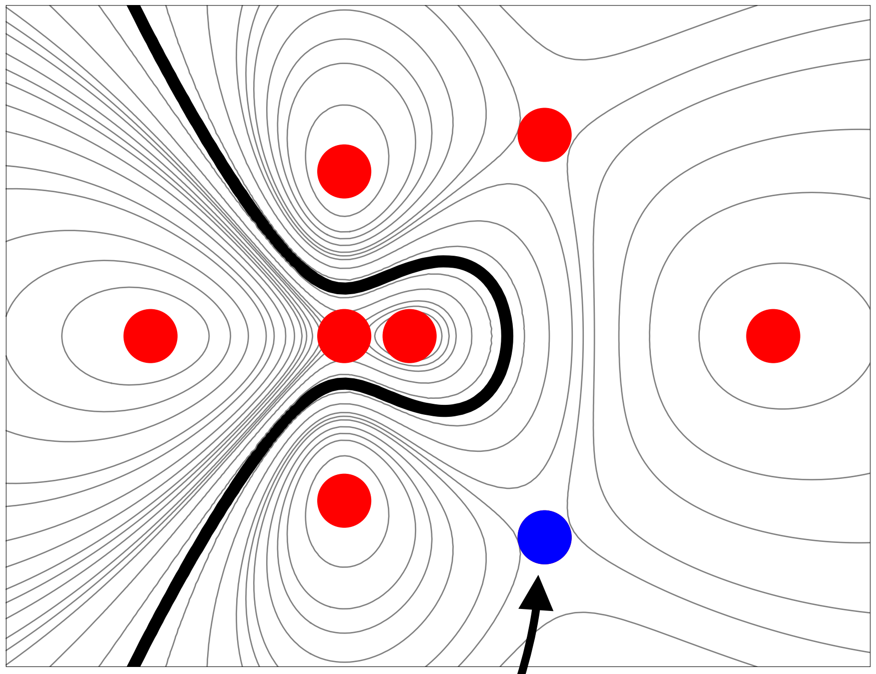

$\nabla g = 0$

1. Correctness: Steepest Ascent

Trajectory of ∇g through using

$$\phi'(t) = \nabla g(\phi(t)) \text{ when } t \neq 0$$

and

$$\lim_{t \to 0^+} \phi(t) =$$

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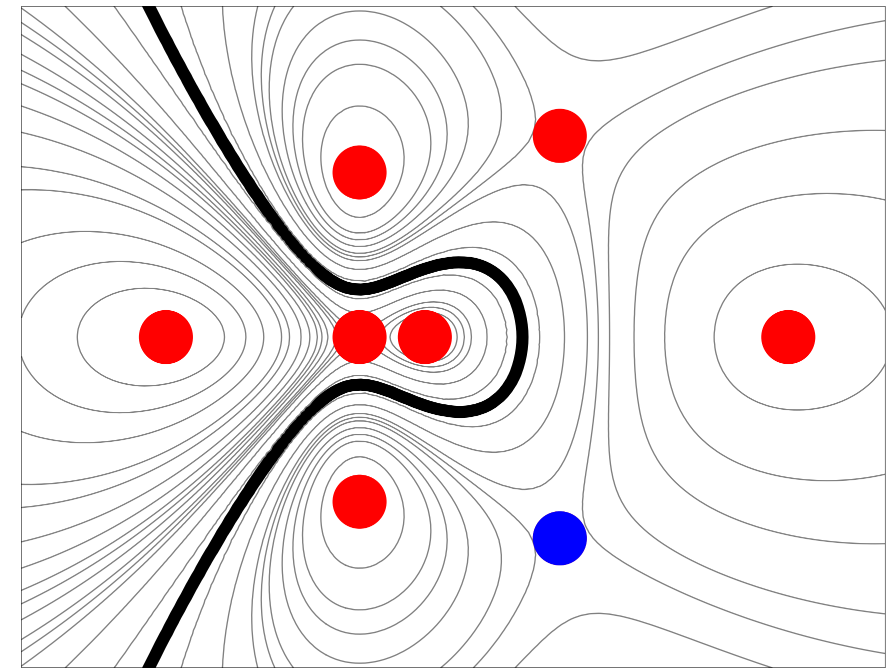
$$\lim_{t \to 0^+} \frac{\phi'(t)}{\|\phi'(t)\|} =$$



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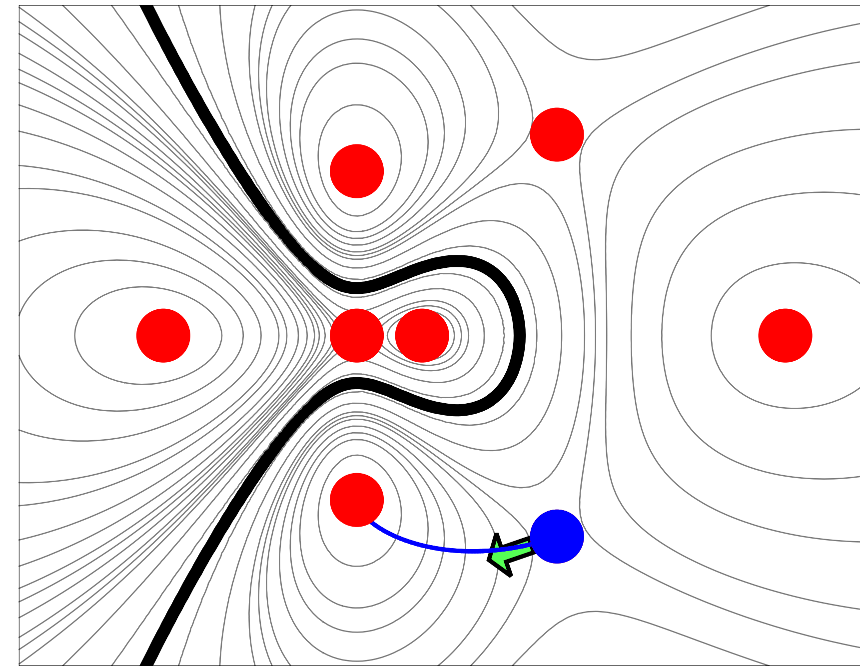

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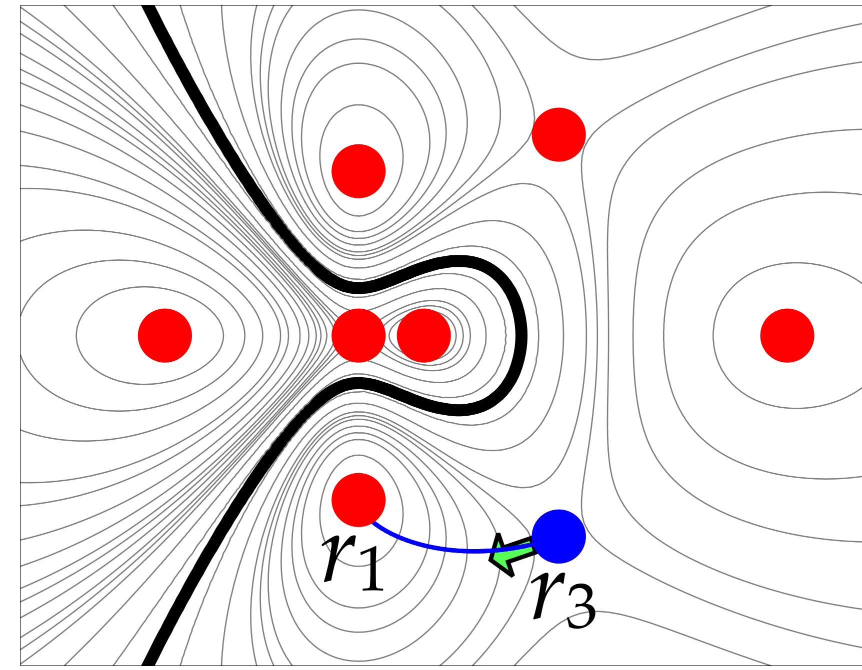

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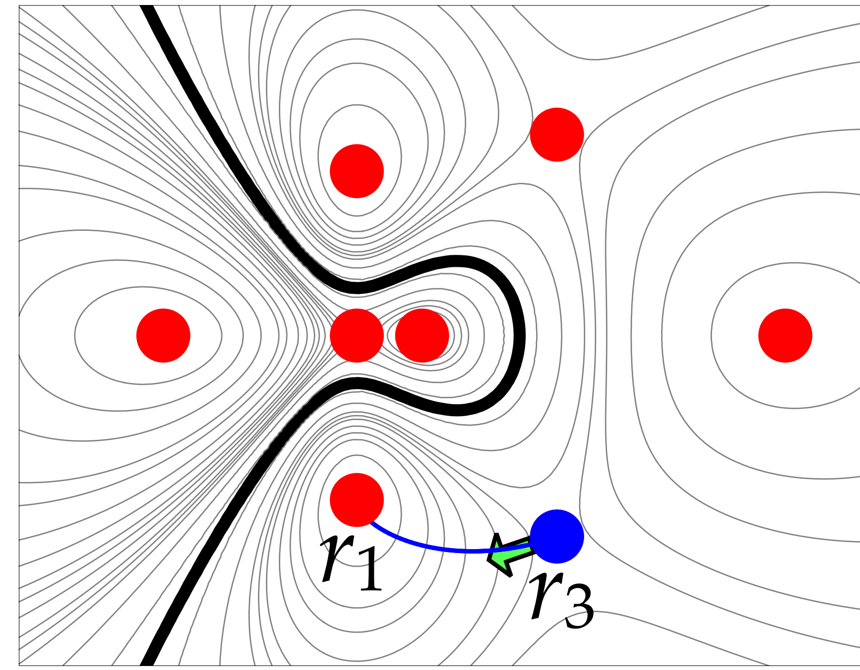

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r_1, r_3 are **connected by steepest ascent paths using outgoing eigenvectors**

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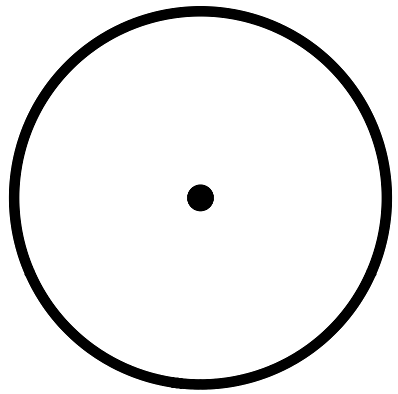
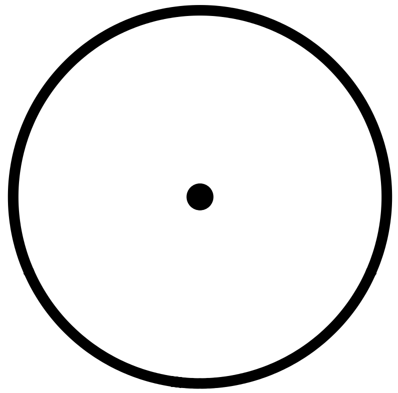


$$f = \left(x_1^2 + x_2^2 - 2\right)\left(x_1^2 + x_2^2\right)$$

1. Correctness: What if g is not a routing function?

$$g = \frac{f^2}{\left(x_1^2 + x_2^2 + 1\right)^{\deg(f)+1}}$$



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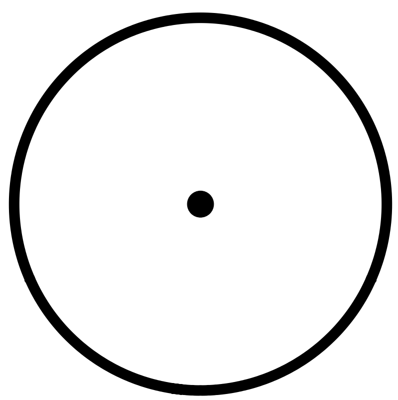

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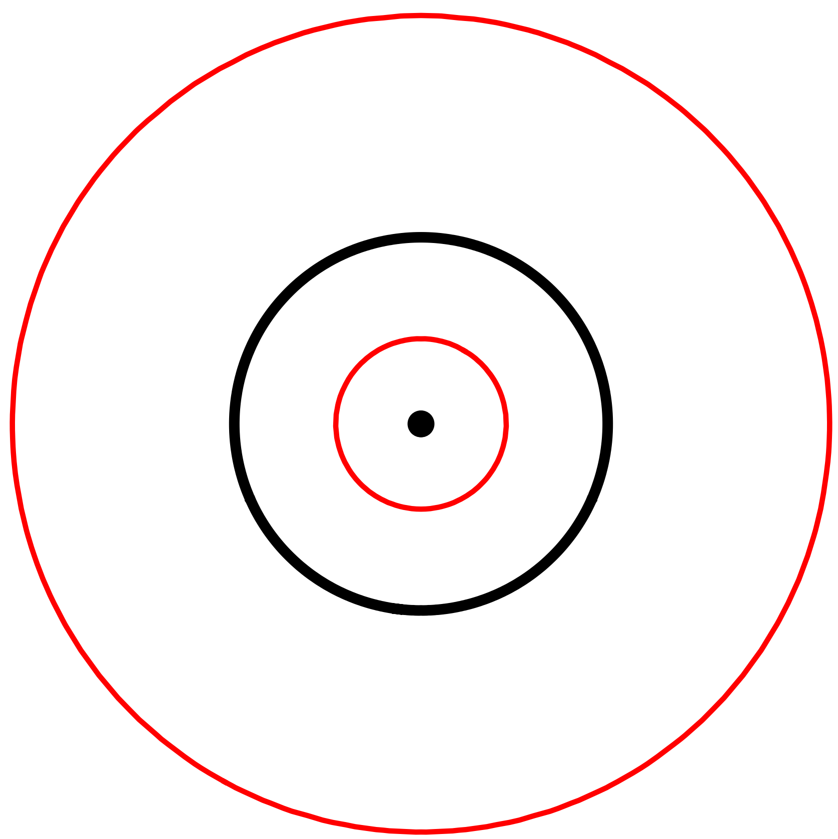

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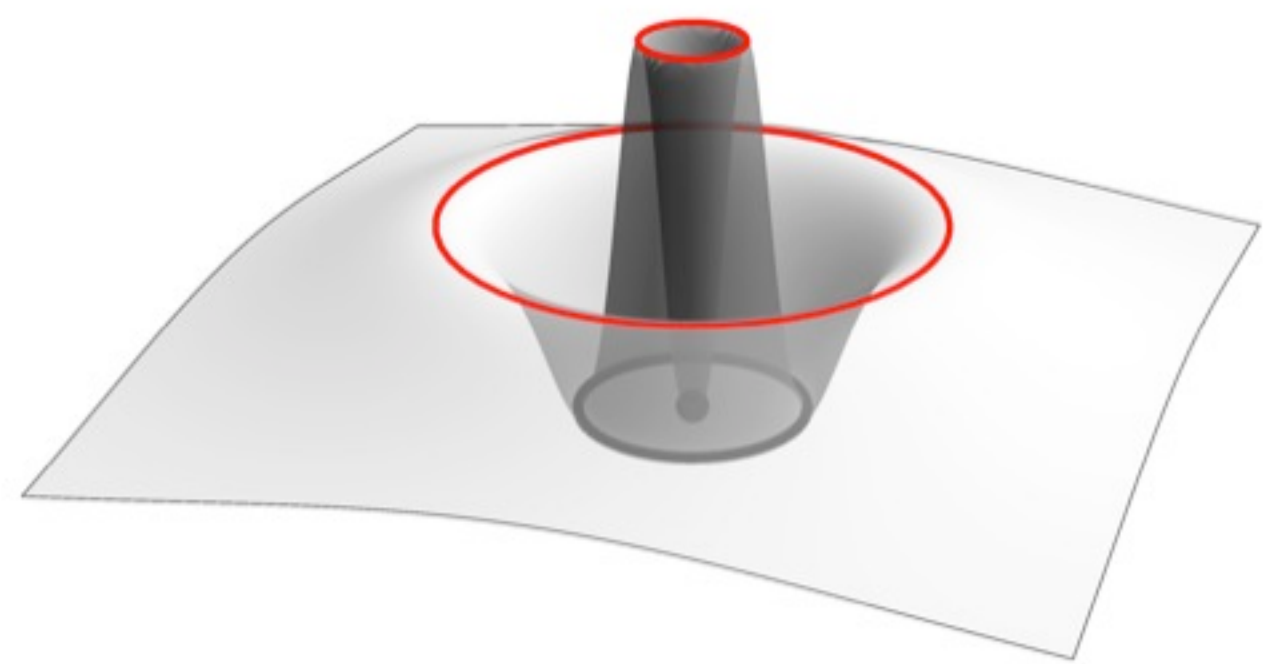

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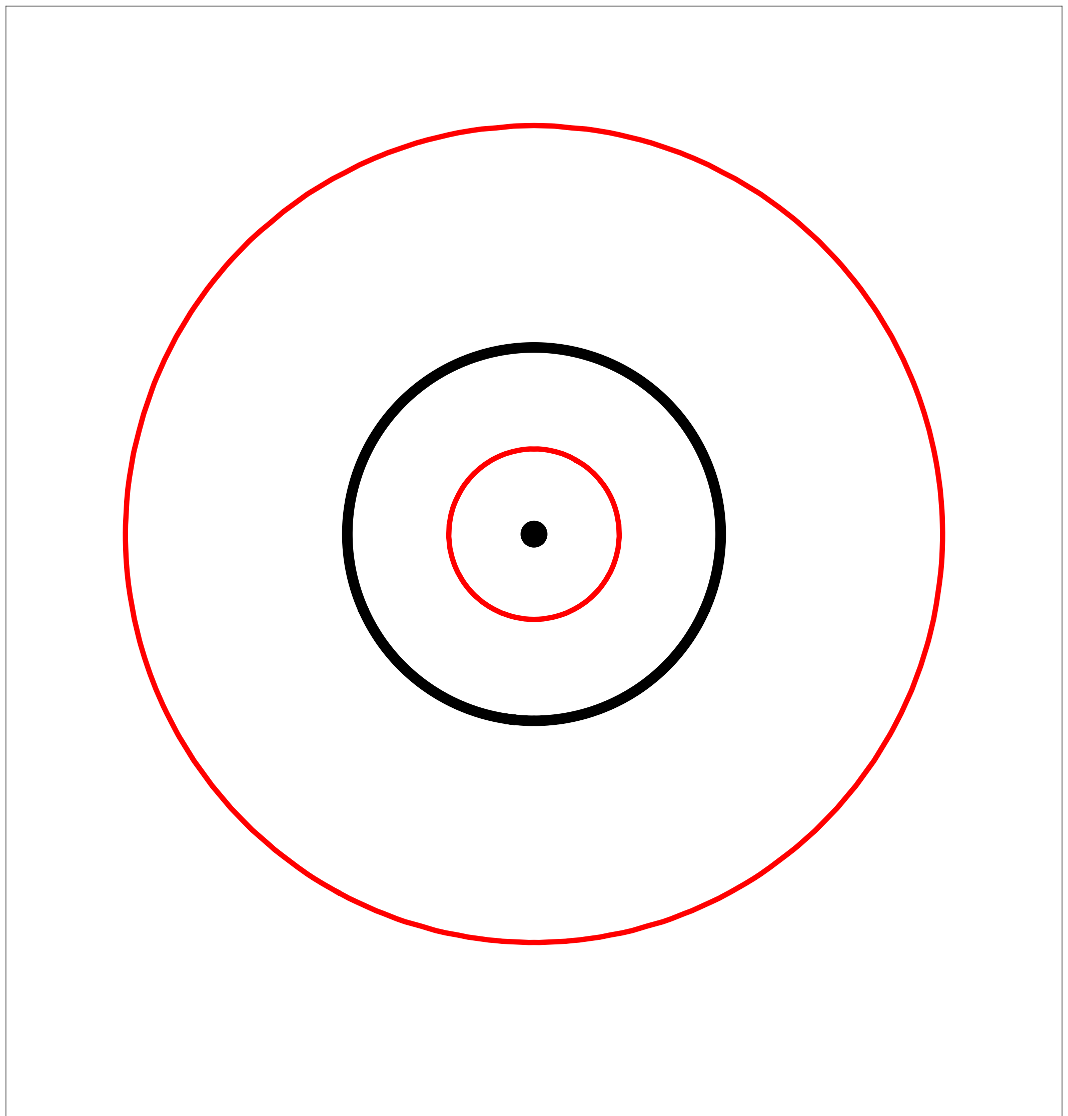

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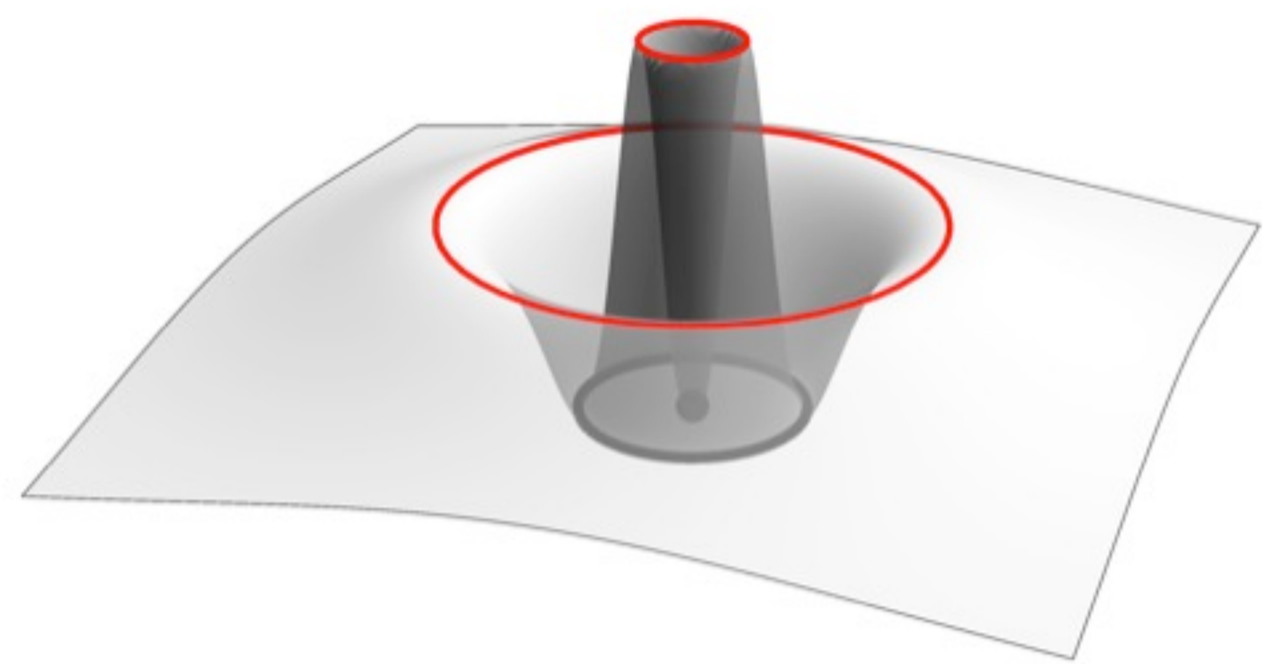

g is not Morse

1. Correctness: Fixing by Perturbation

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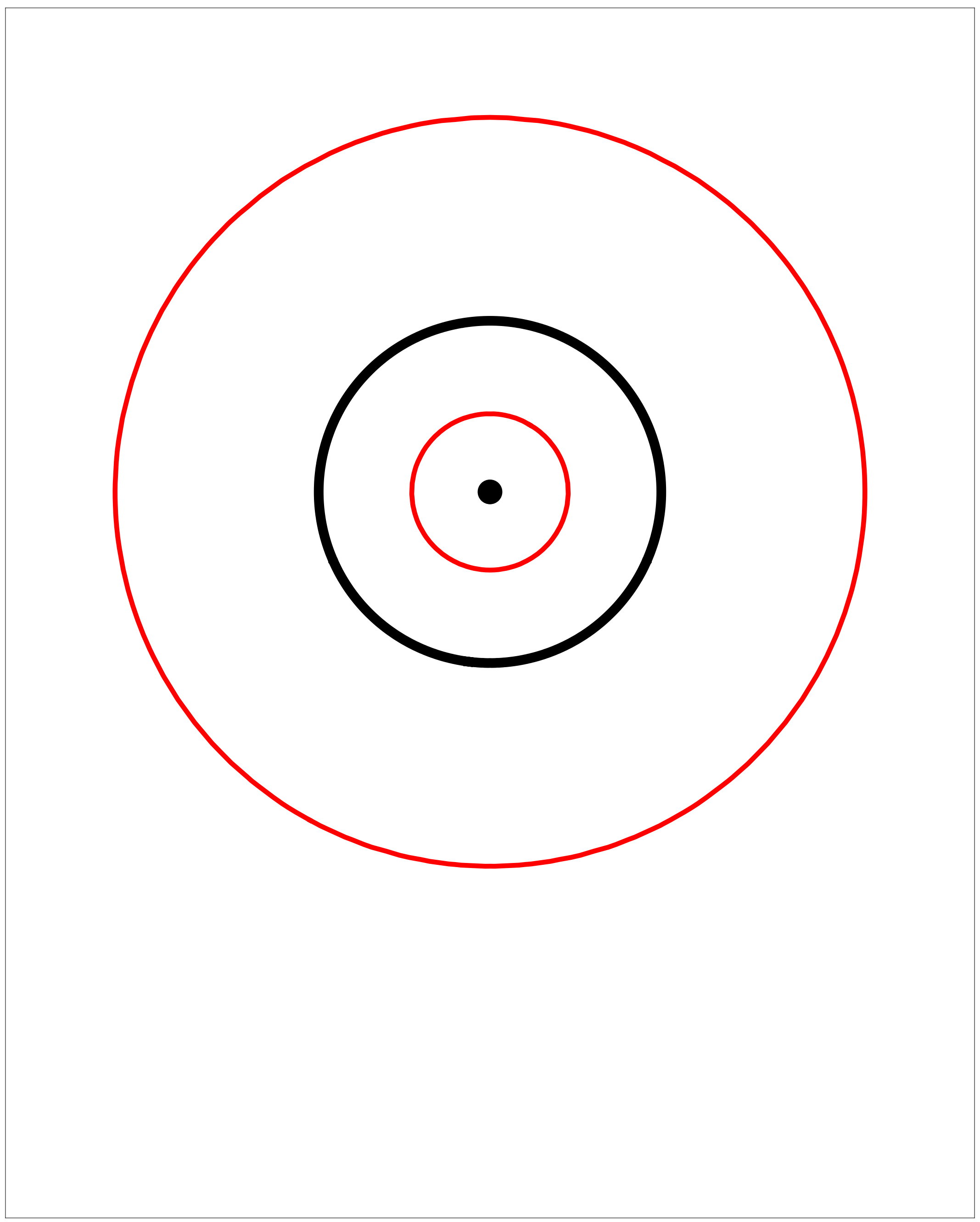
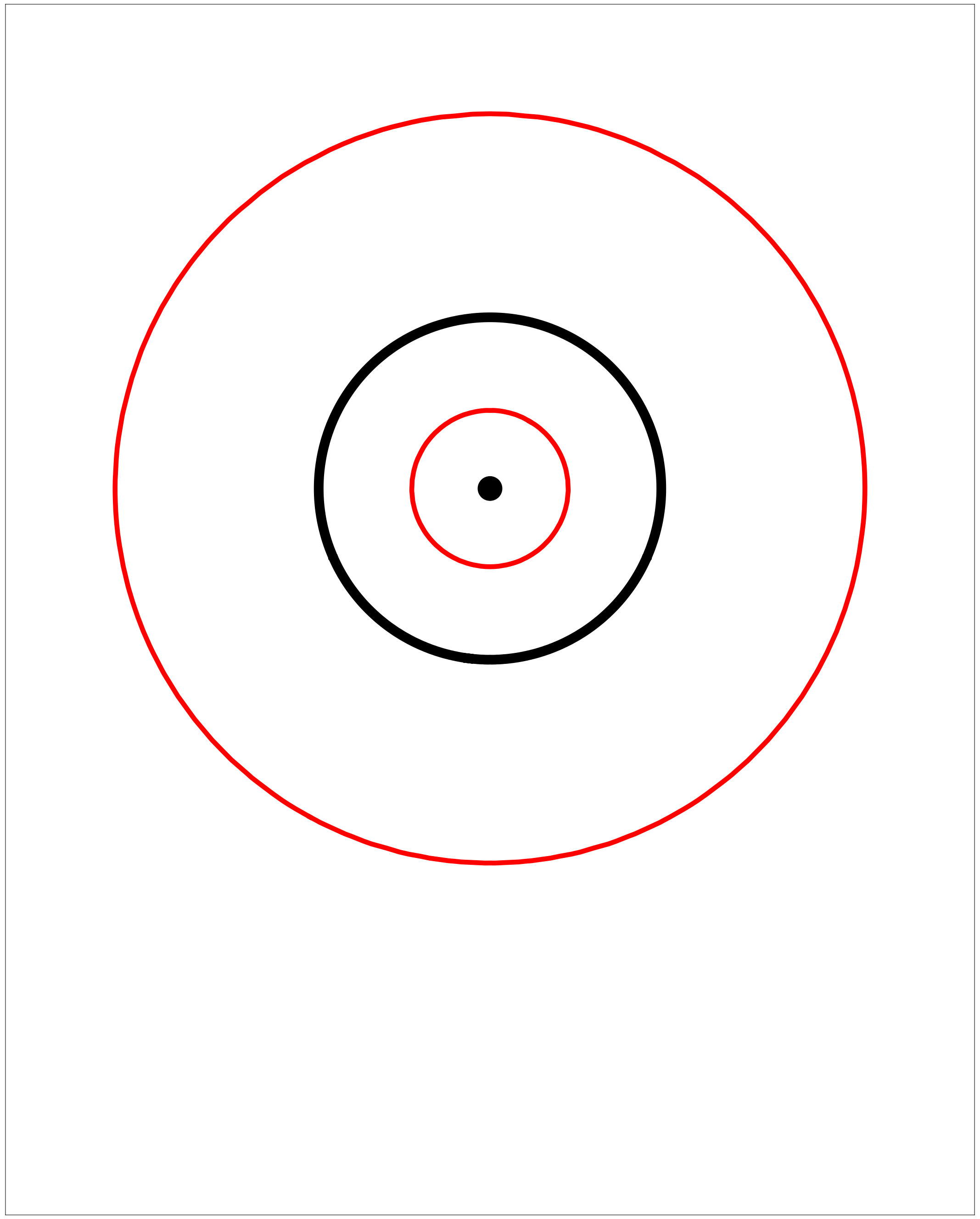


1. Correctness: Fixing by Perturbation

$$g = \frac{f^2}{\left(x_1^2 + x_2^2 + 1\right)^{\deg(f)+1}} \xrightarrow{\text{perturb}} g = \frac{f^2}{\left((x_1 - 0)^2 + (x_2 - 1)^2 + 1\right)^{\deg(f)+1}}$$



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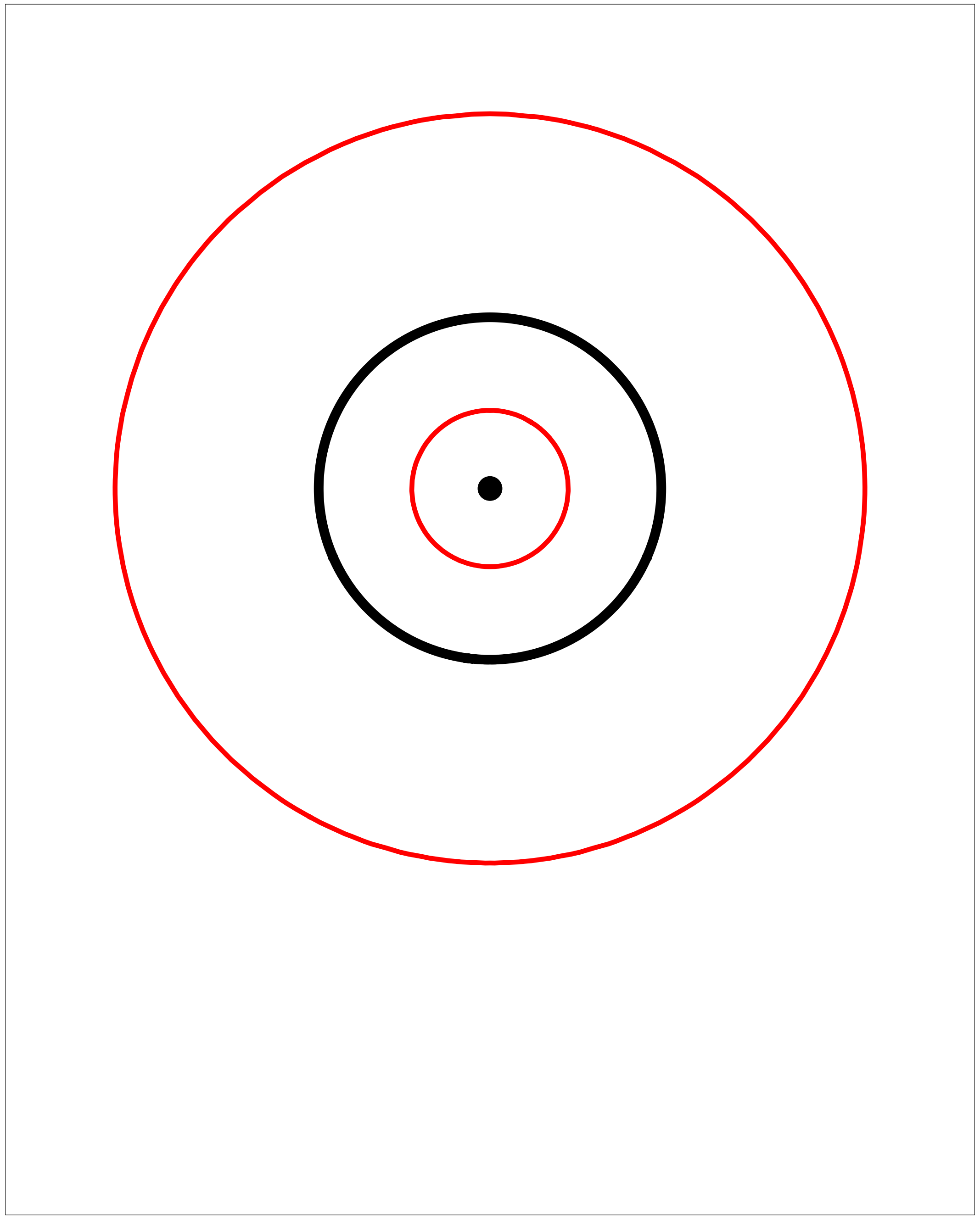

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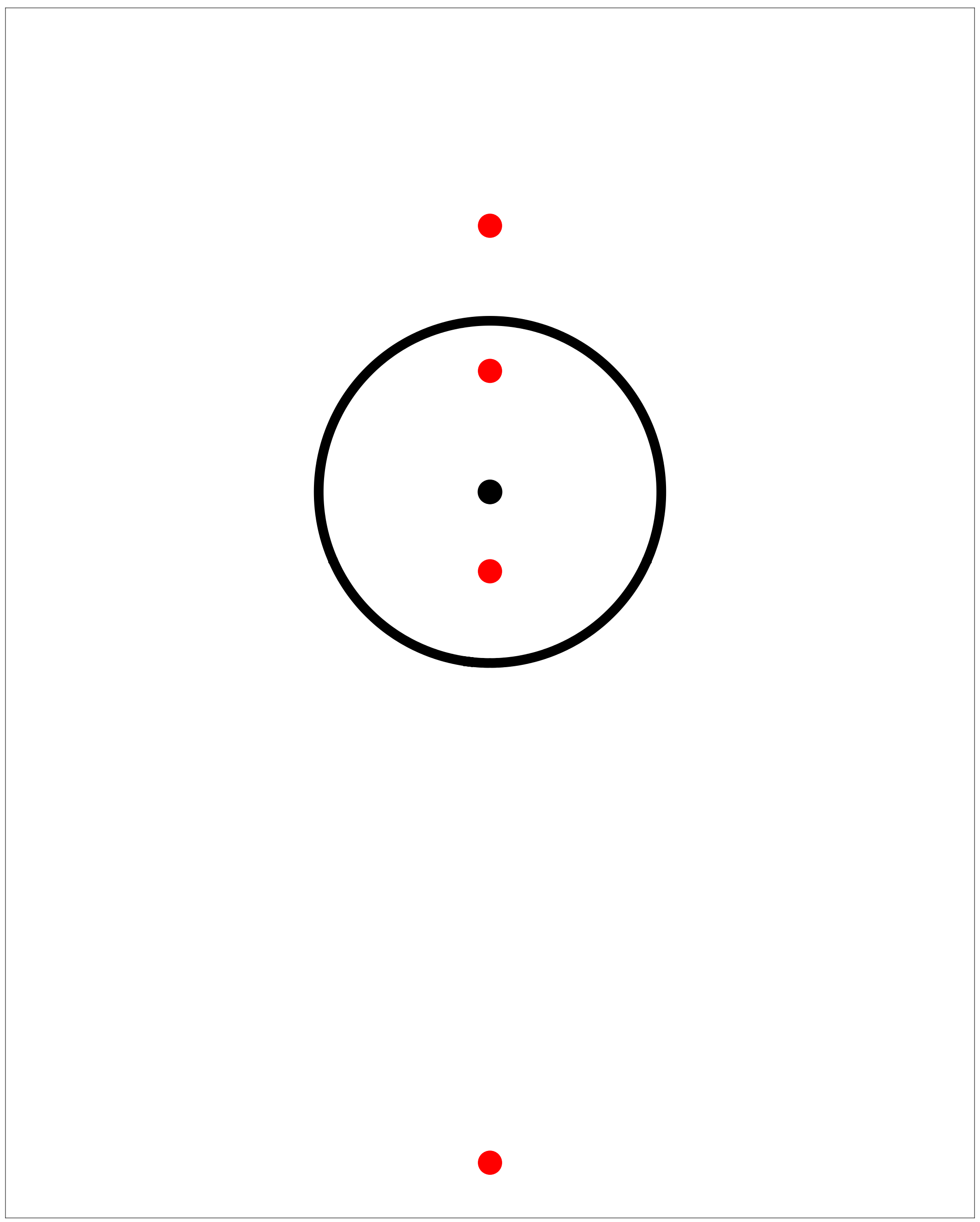

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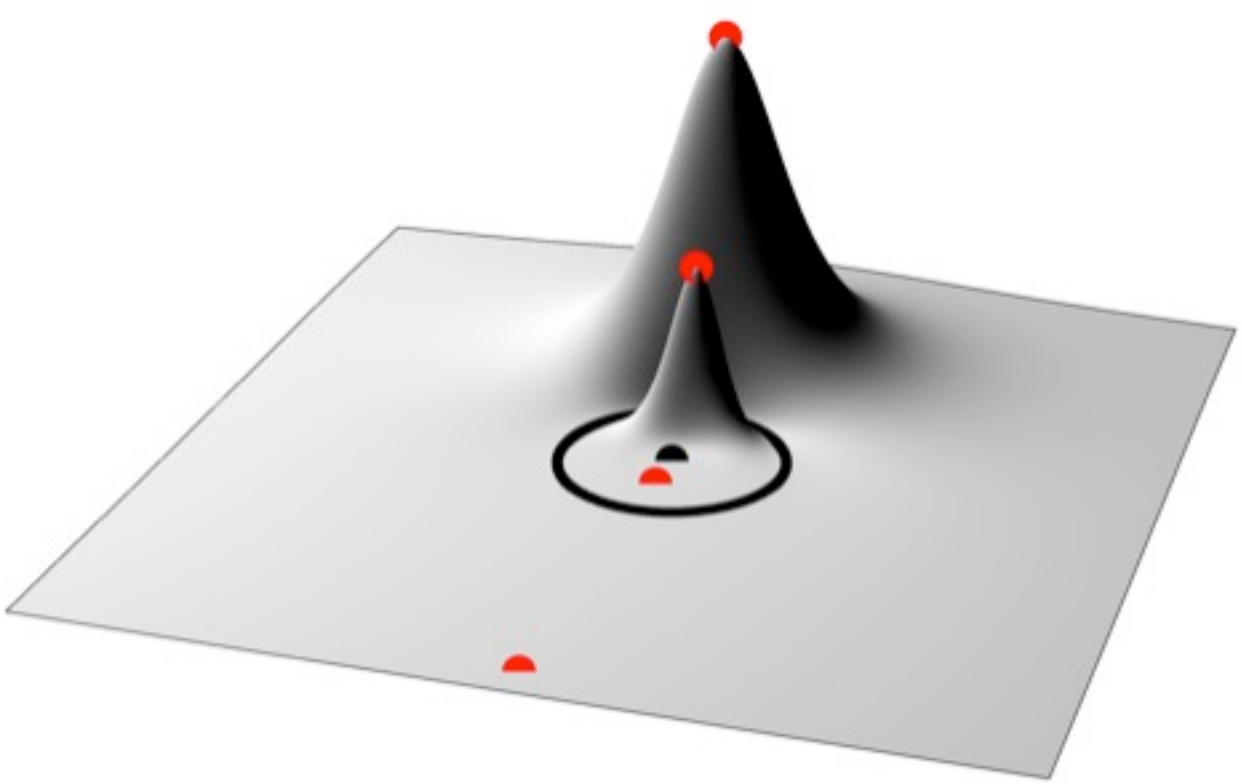

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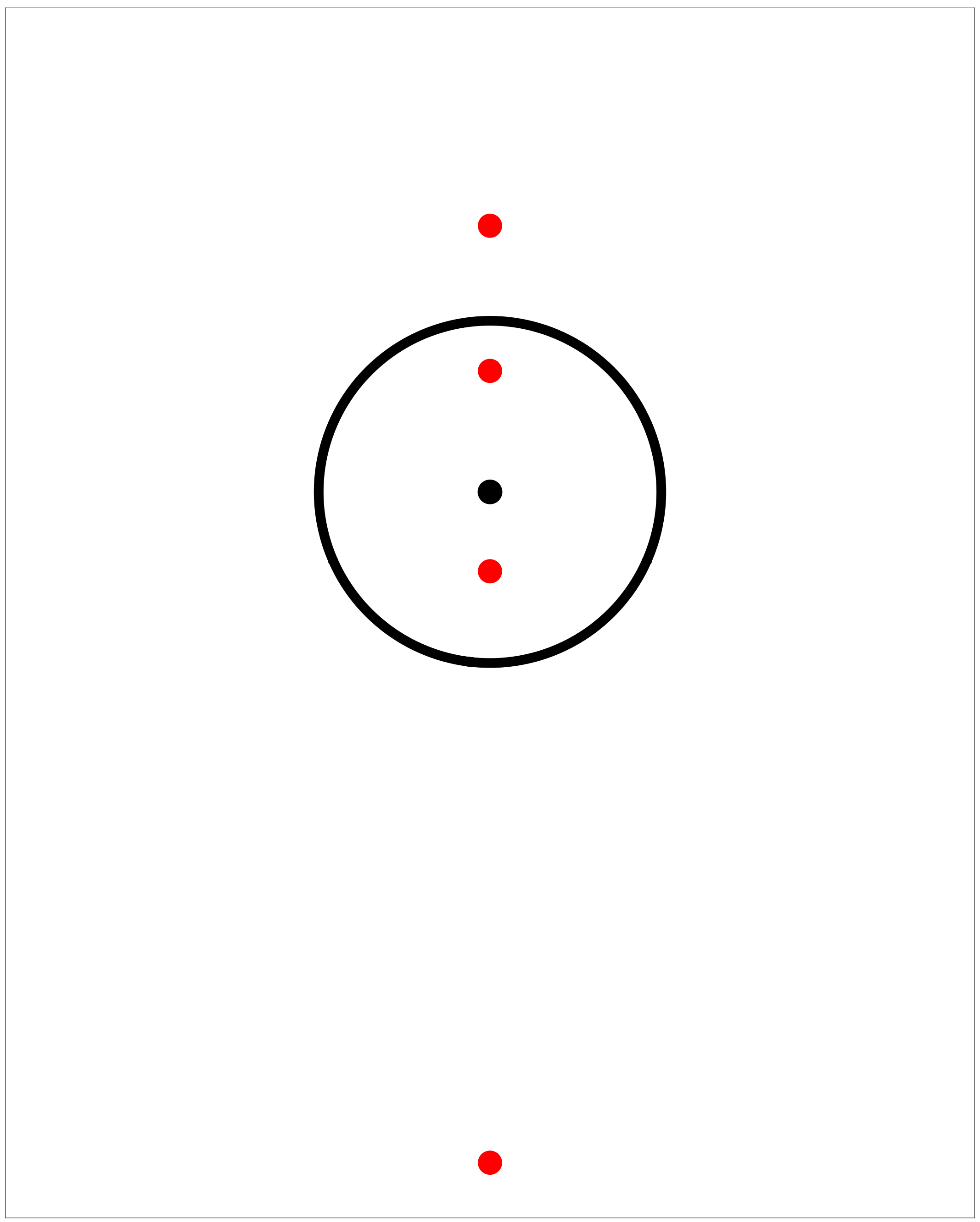

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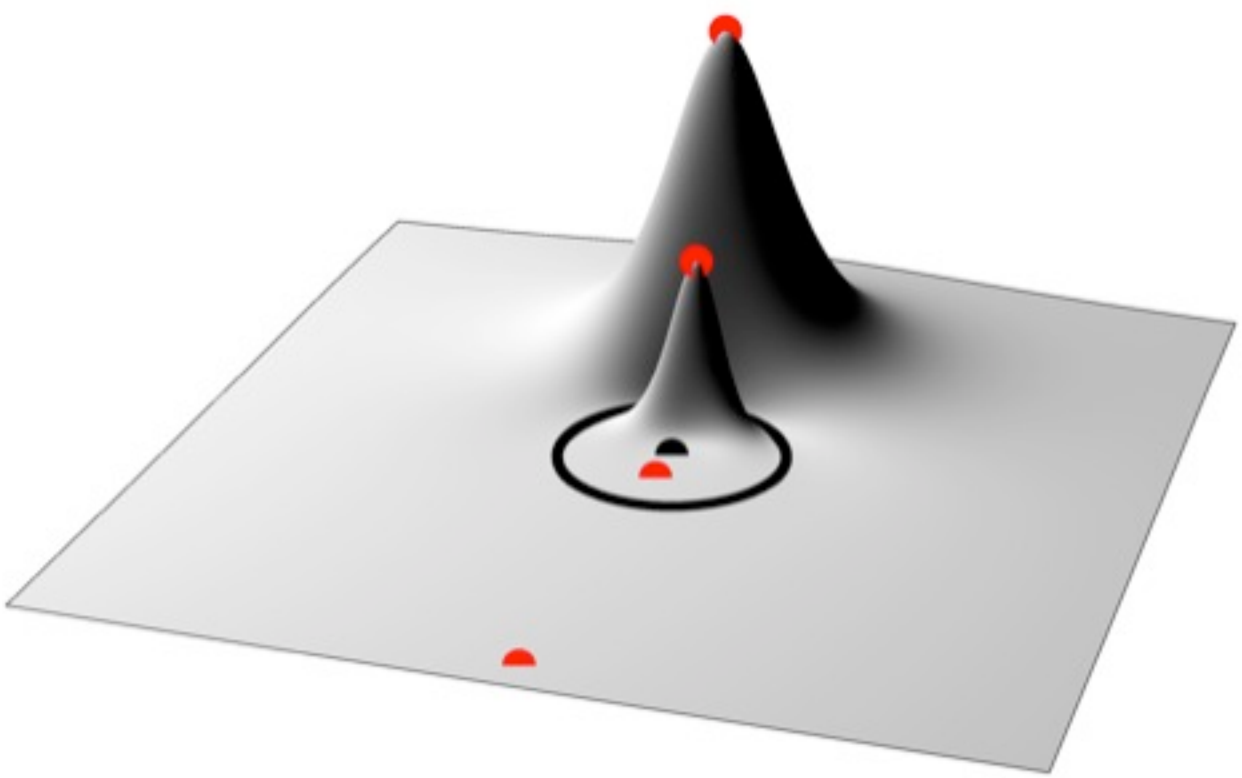

g is Morse

2. Termination: Theorem

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Theorem

$\forall f \in \mathbb{Z}[x_1, \ldots, x_n]$

$\exists$ semialgebraic set $S \subset \mathbb{R}^n$

$\dim(\mathbb{R}^n \setminus S) < n$

$\forall (c_1, \ldots, c_n) \in S$

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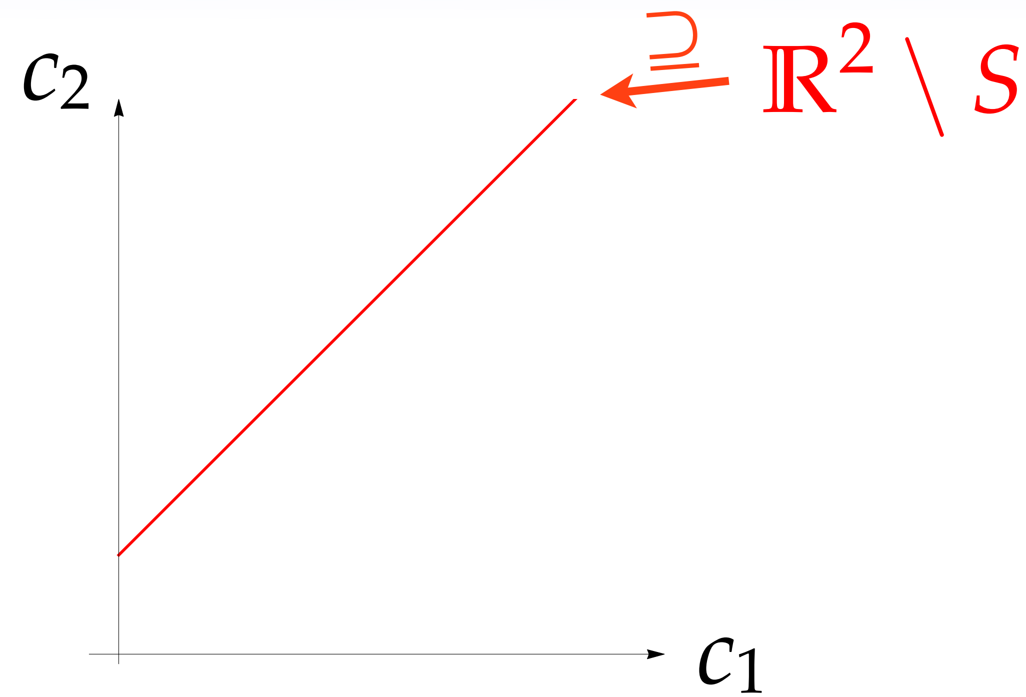

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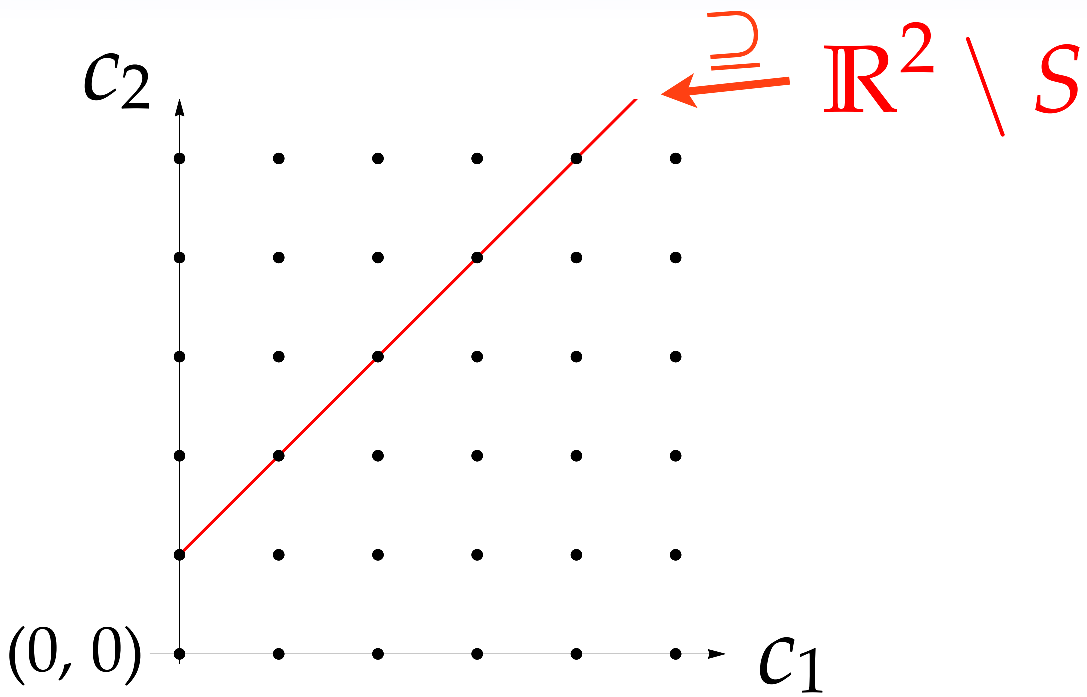

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c_2

$\supseteq$ $\mathbb{R}^2 \setminus S$

$\forall f \in \mathbb{Z}[x_1, \ldots, x_n]$

$\exists$ semialgebraic set $S \subset \mathbb{R}^n$

perturb using graded lex. order

$\dim(\mathbb{R}^n \setminus S) < n$

(0, 0)

c_1

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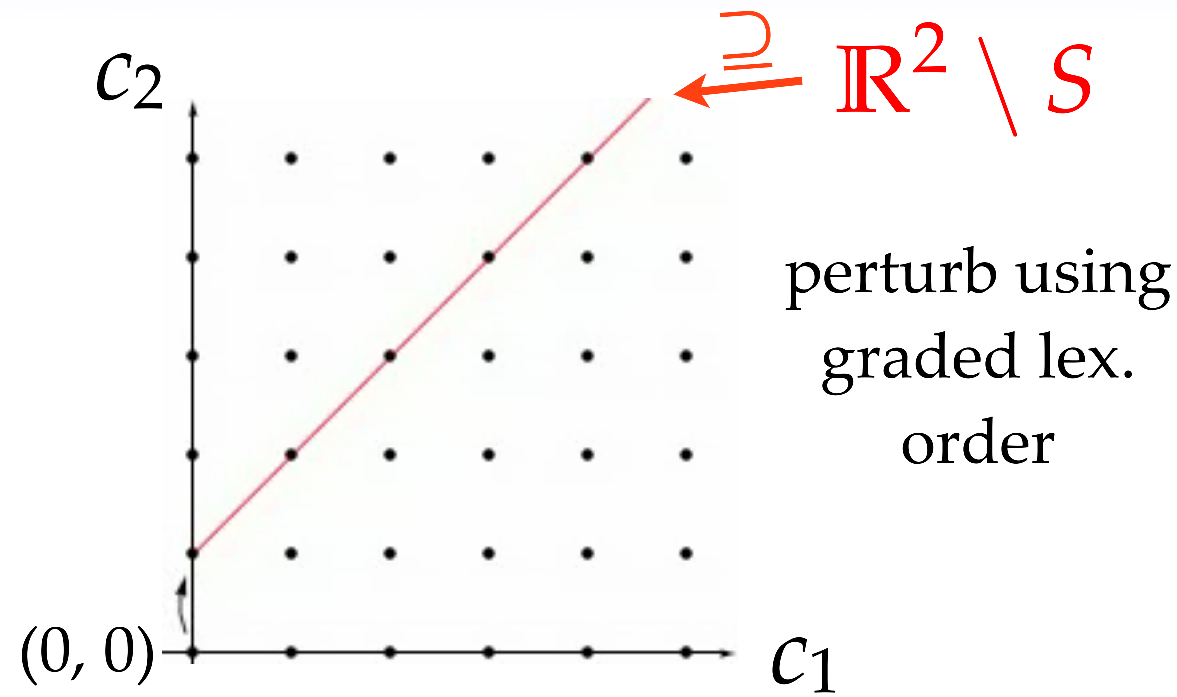

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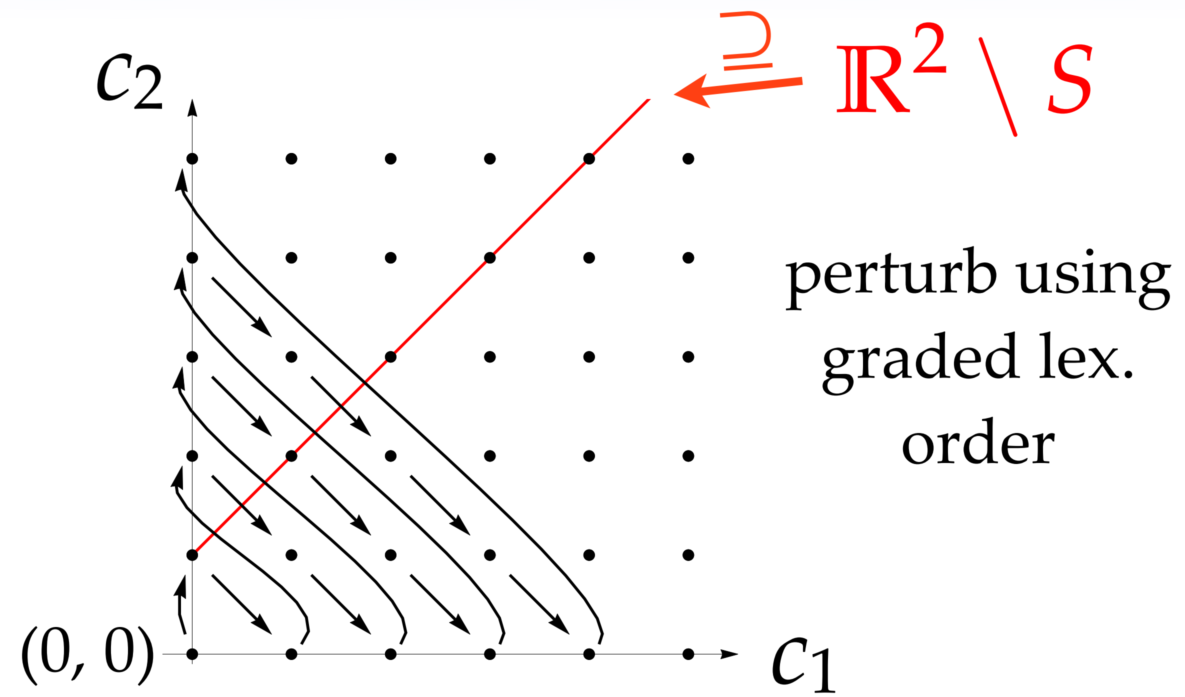

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Proof Idea: Clever application of Sard's Theorem twice

3. Length Bound: Problem

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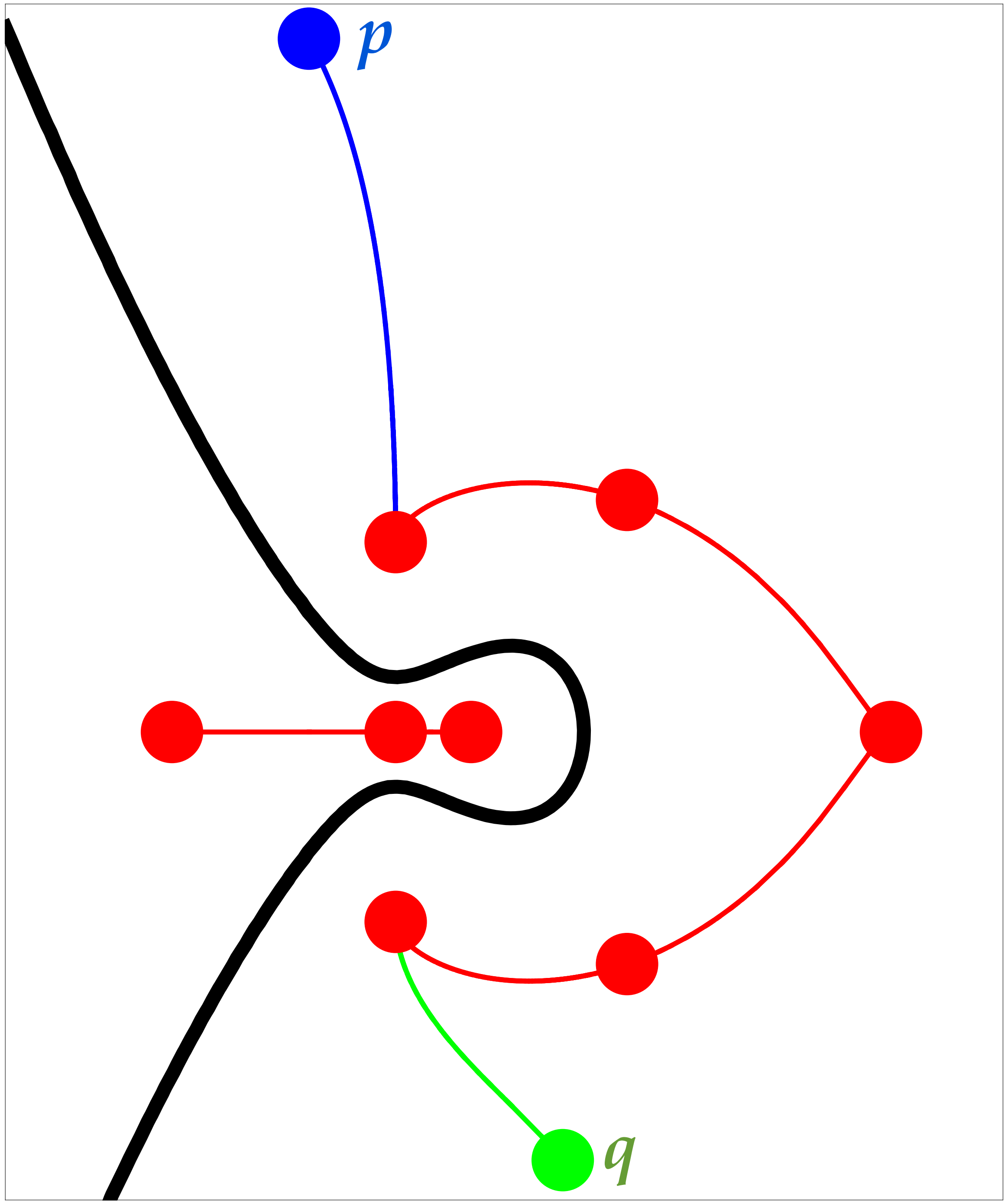

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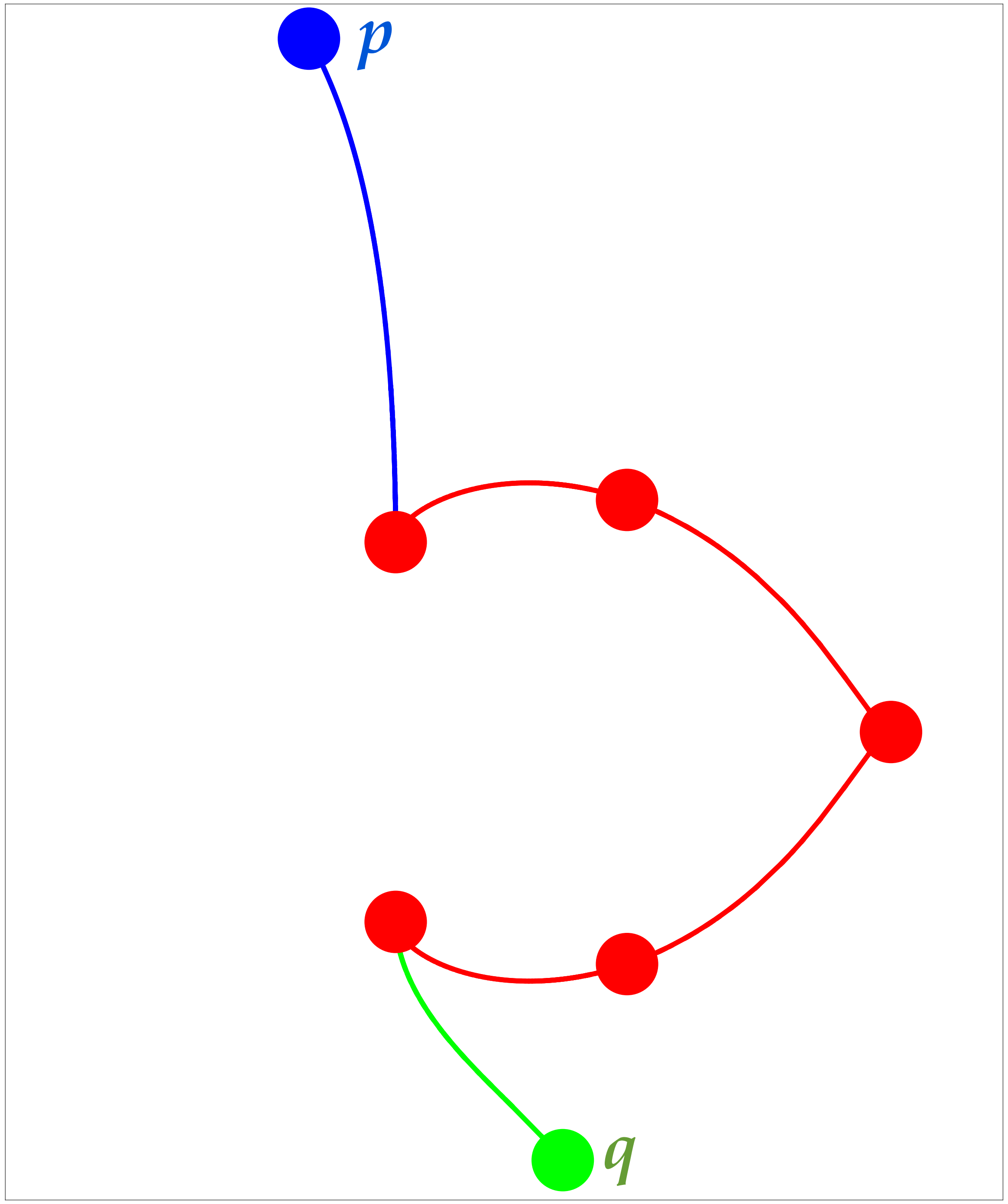

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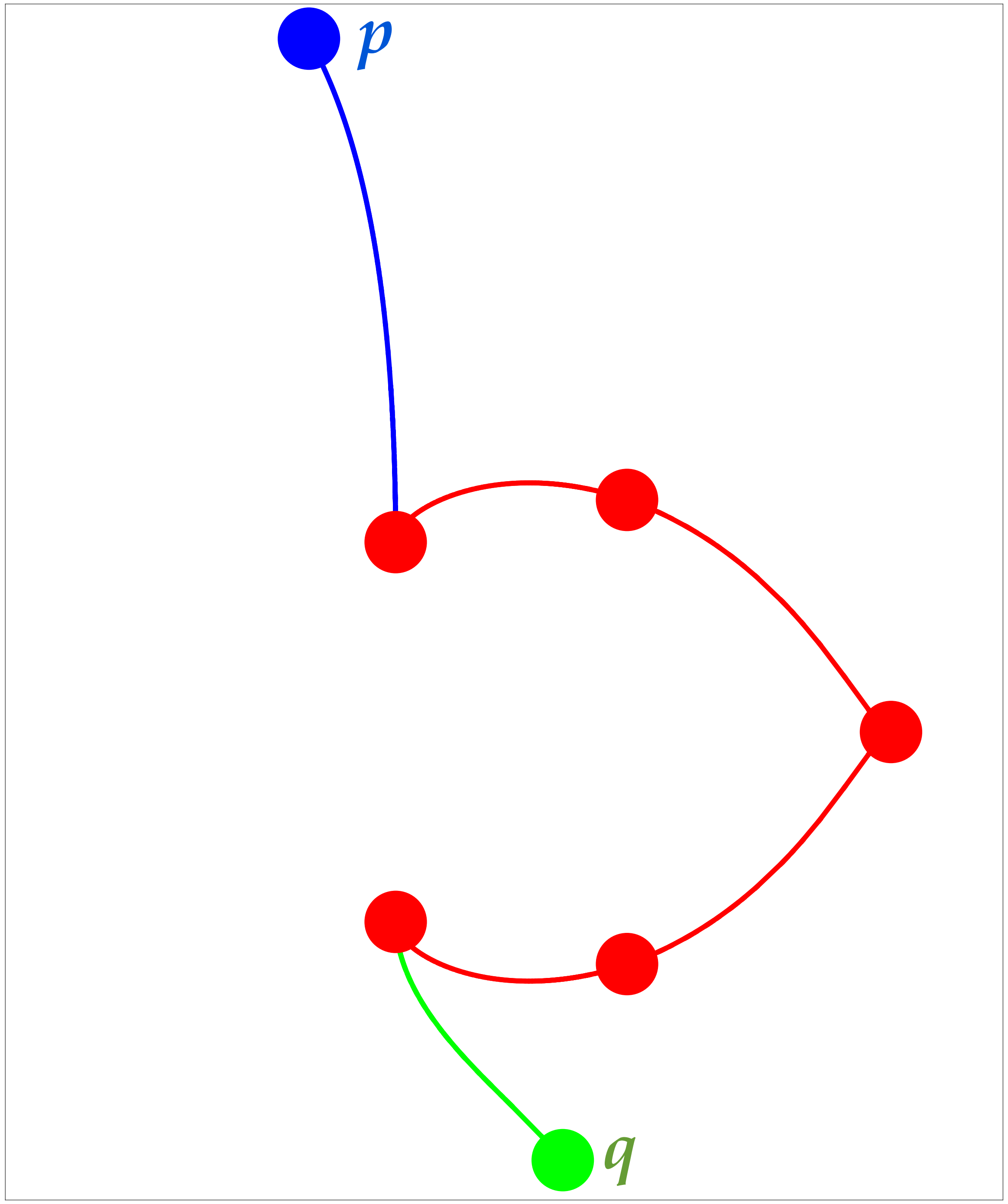

Length ≈

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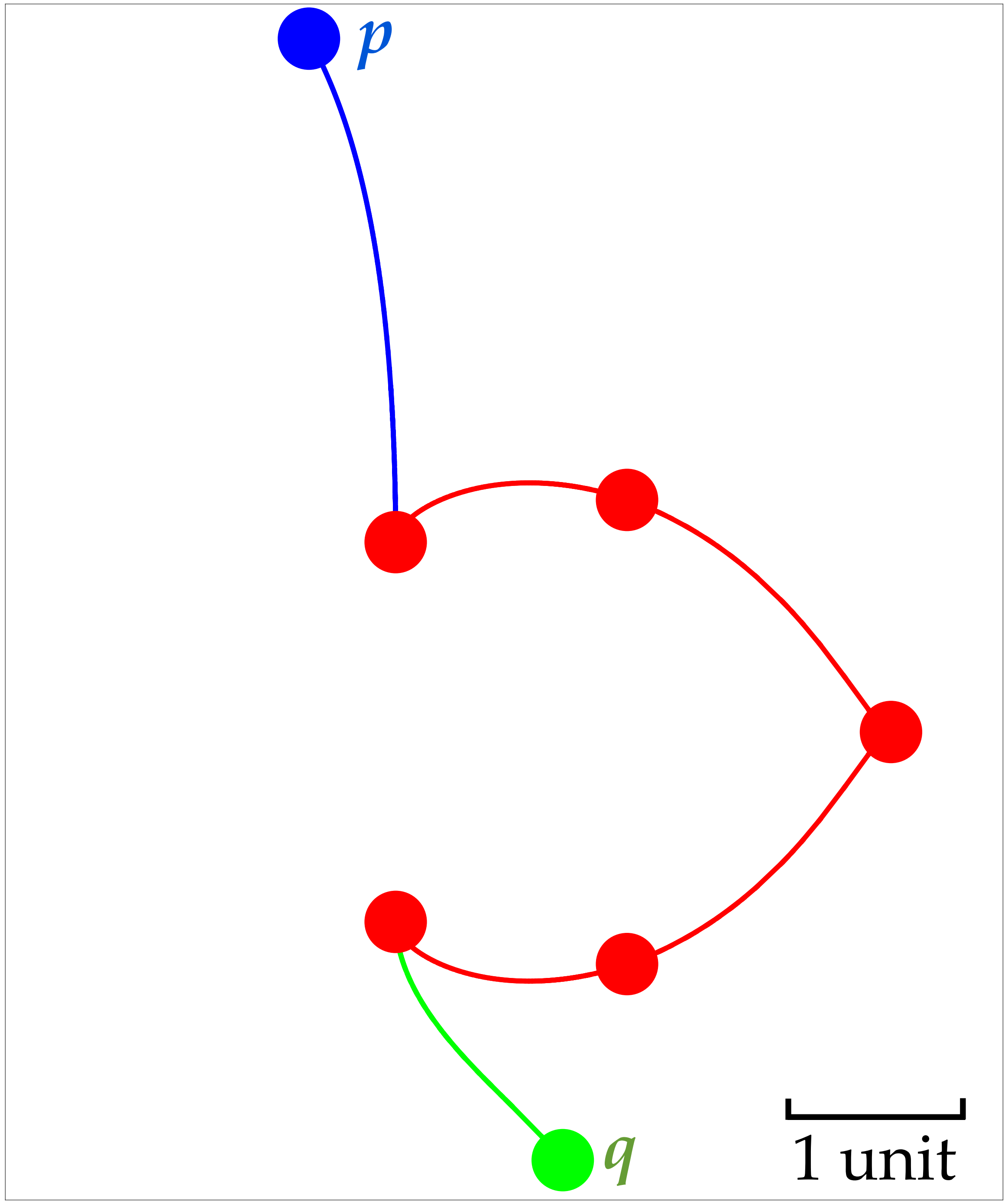

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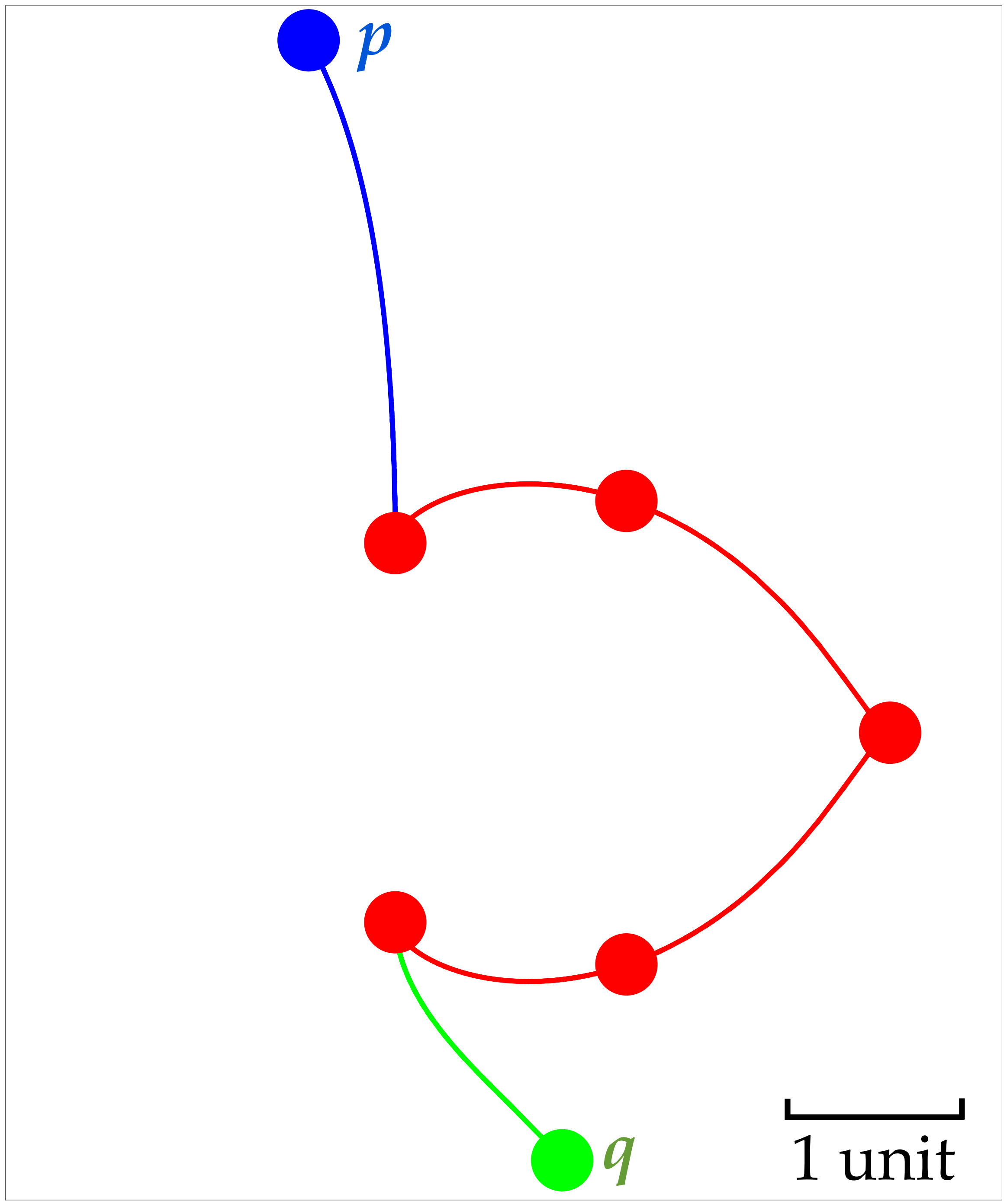
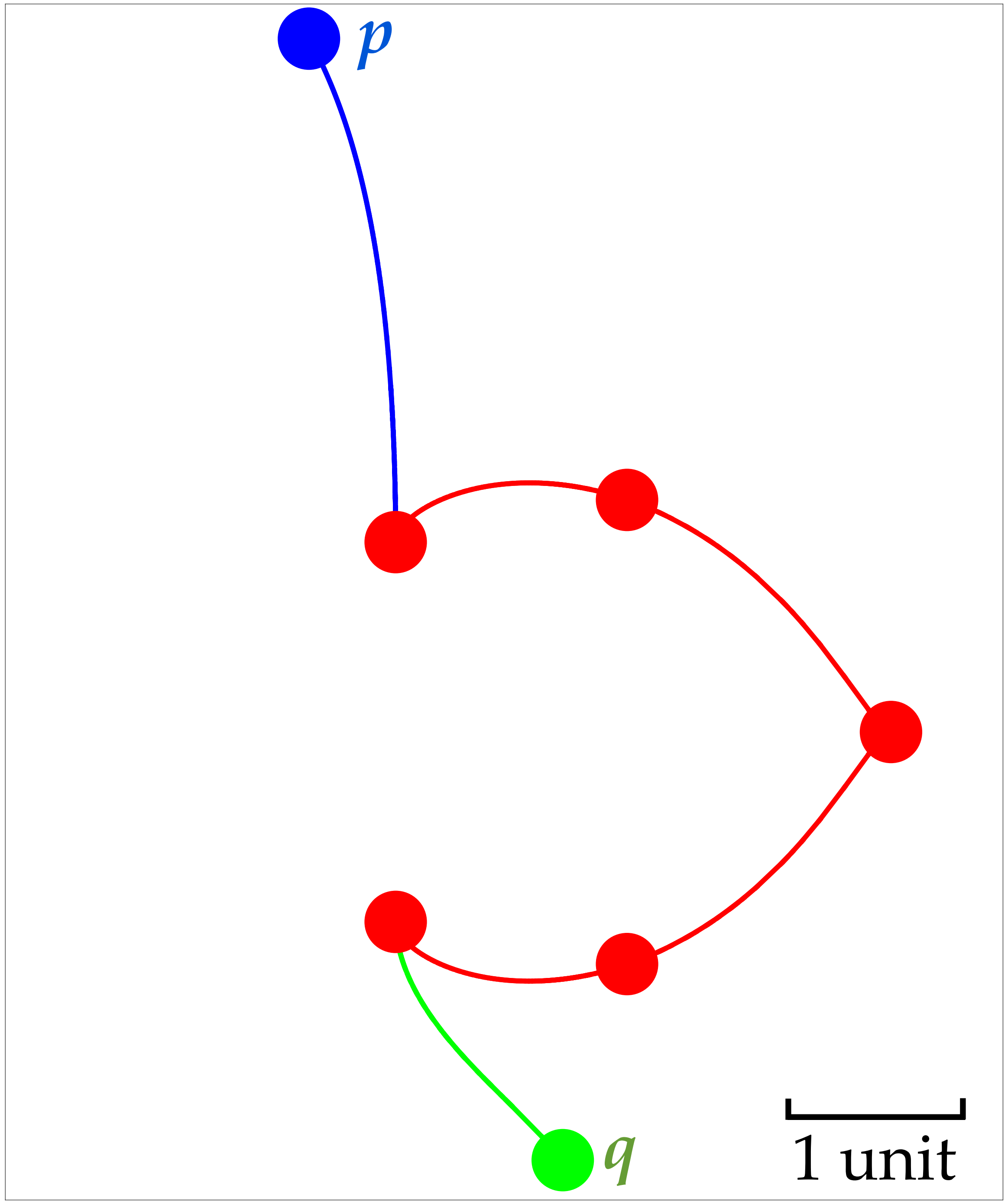


Length ≈ 11.3572

3. Length Bound: Problem



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Given $f \in \mathbb{Z}[x_1, \ldots, x_n]$ $\quad d = \deg(f) \geq 2$

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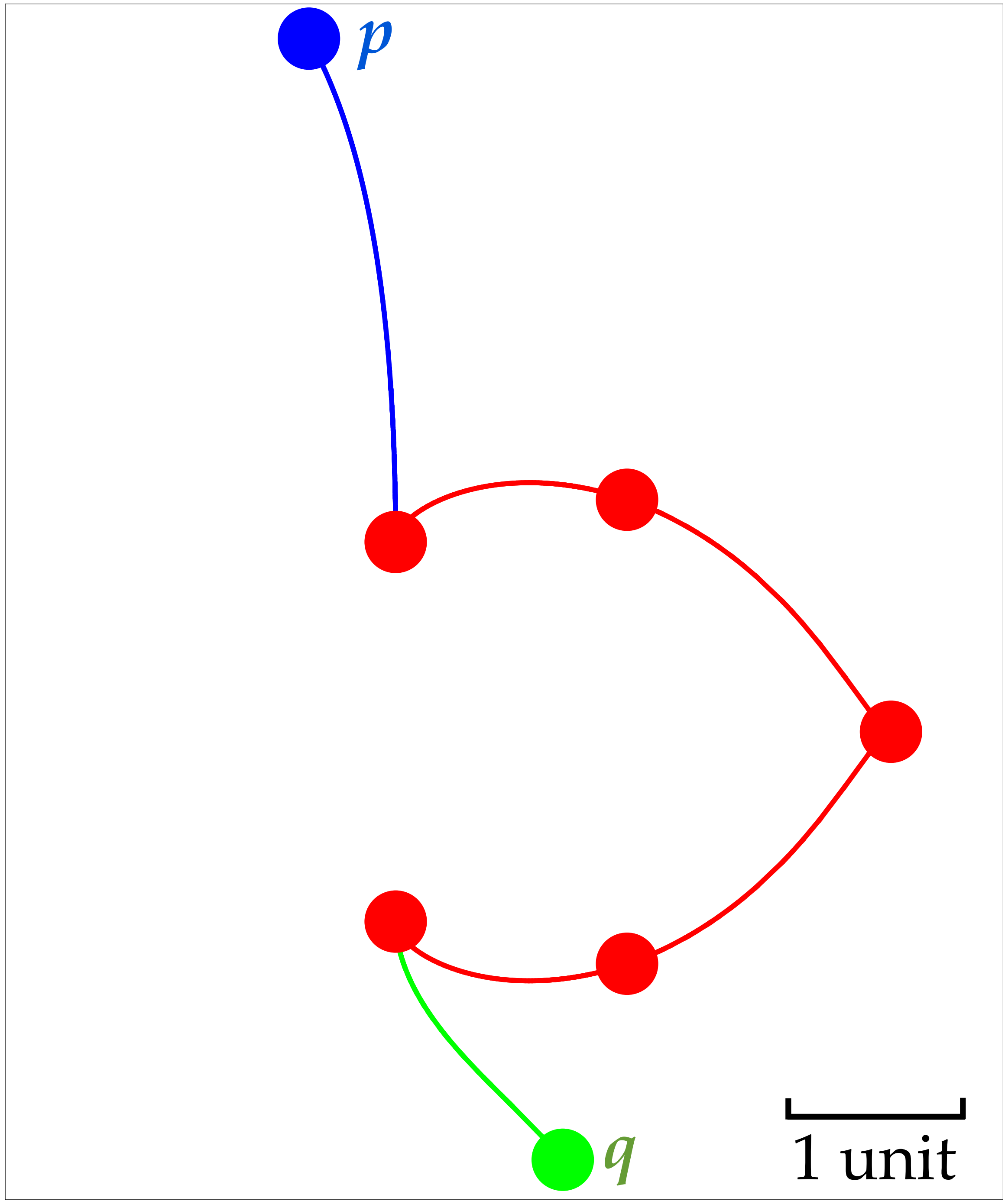

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Find A such that

Length $\leq A(n, d, H, c_1, \ldots, c_n, p, q)$

$H = \max \left| \text{coefficients of } f \right|$

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1. Radius Bound

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2. Trajectory Bound

3. Length Bound: Proof Steps

1. Radius Bound
2. Trajectory Bound
3. Proof Sketch

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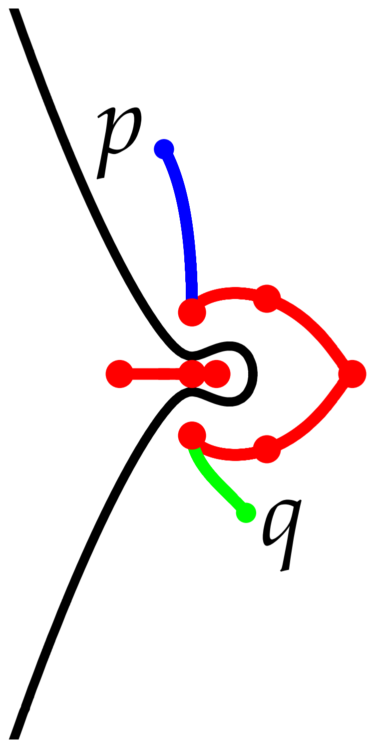

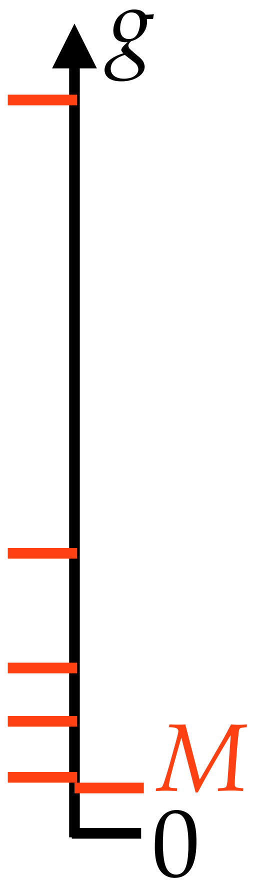

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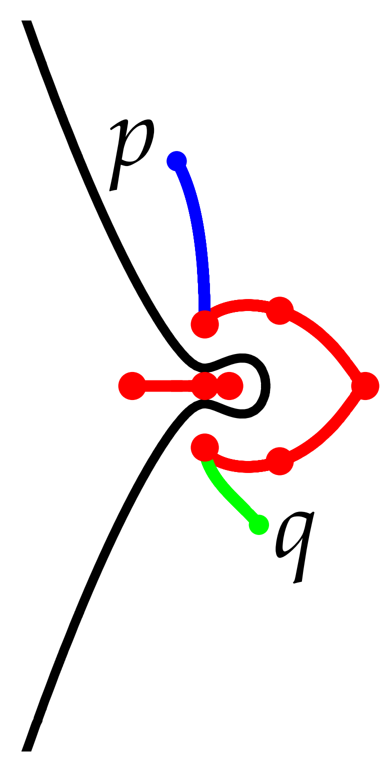

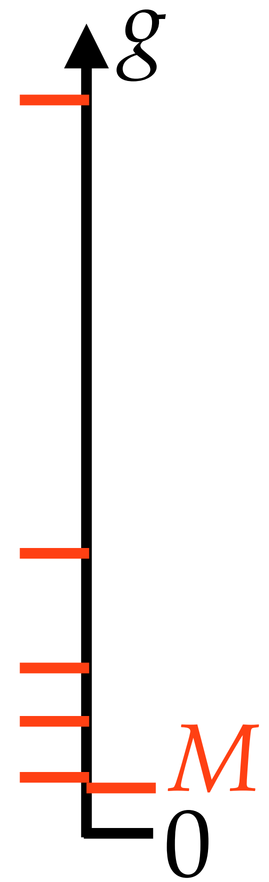

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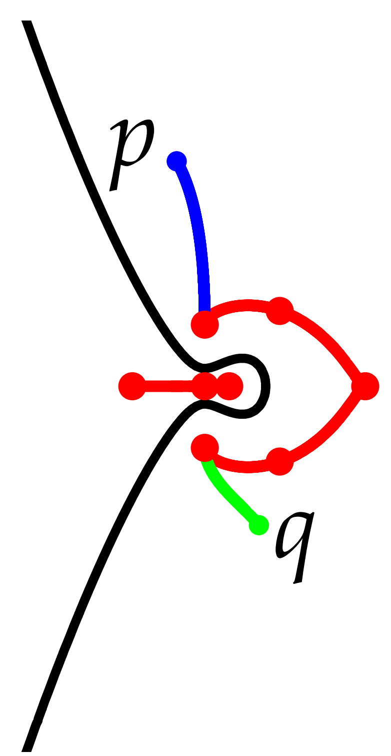

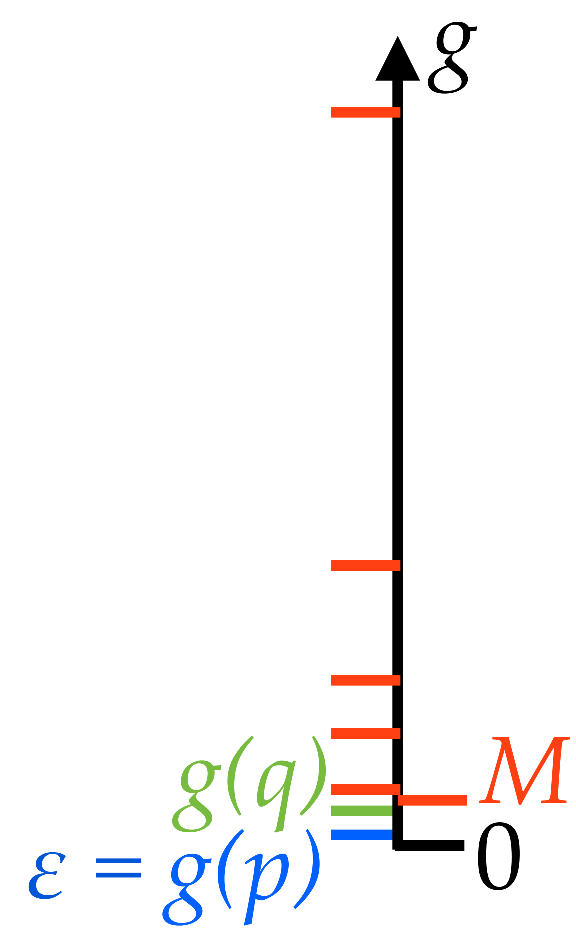

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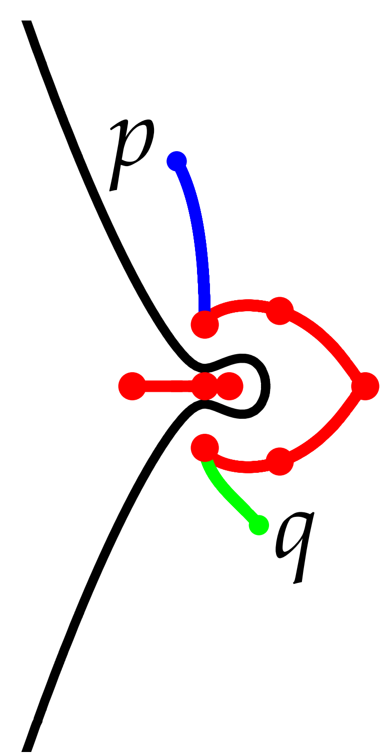

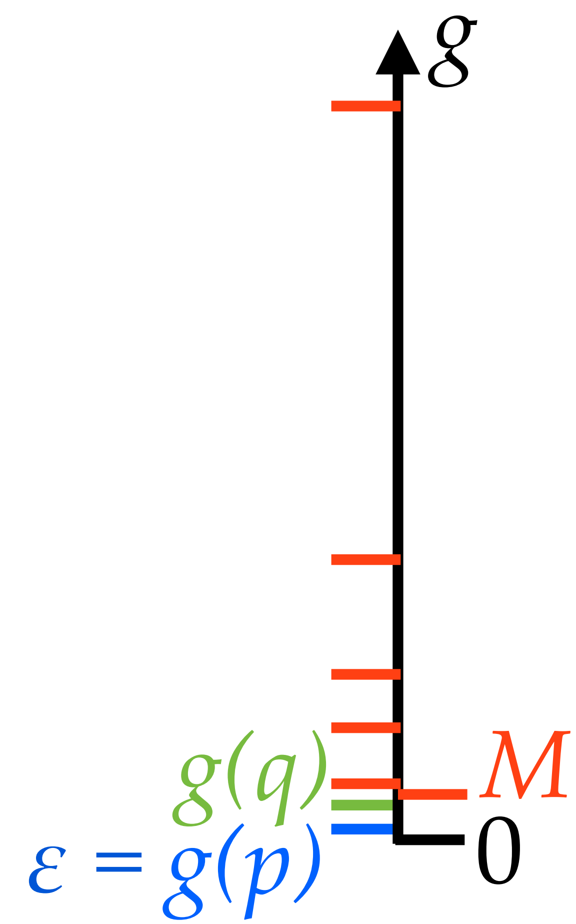

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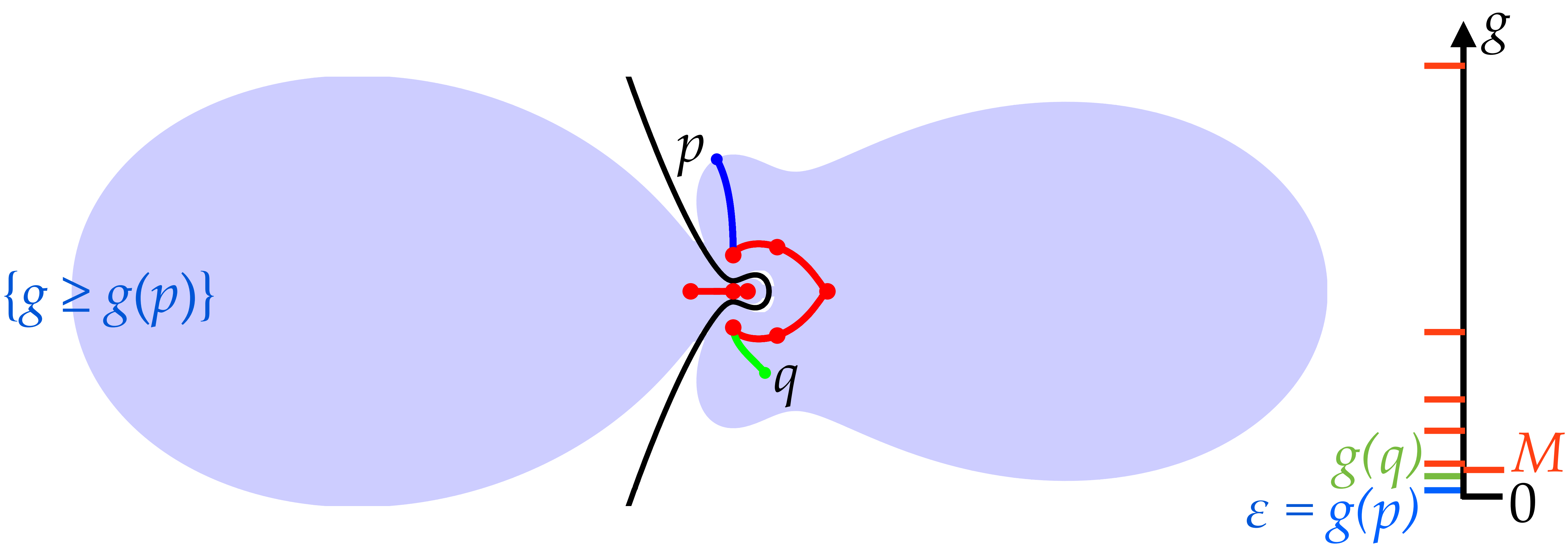

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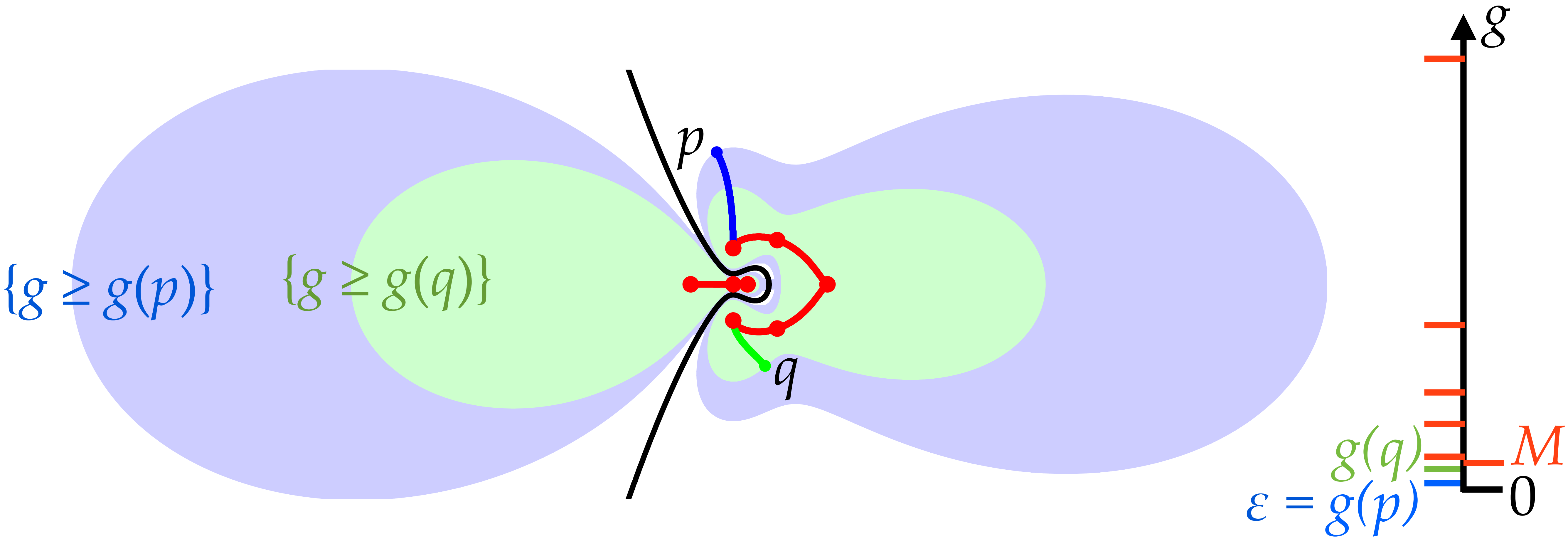

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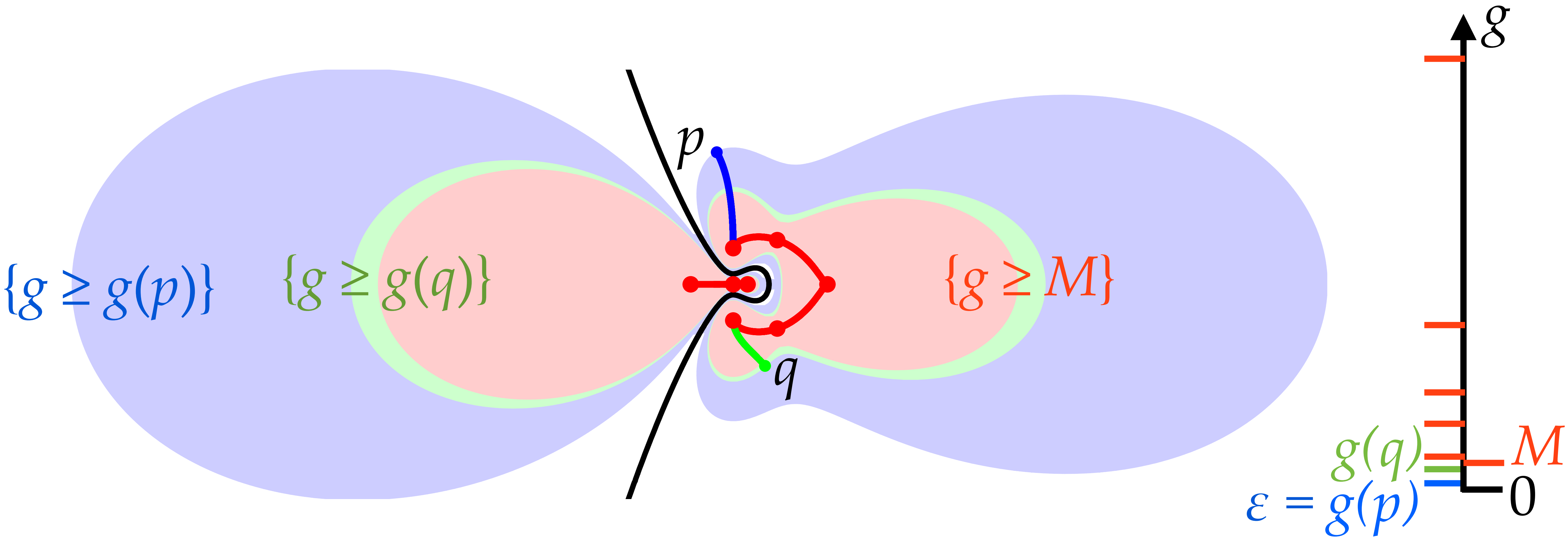

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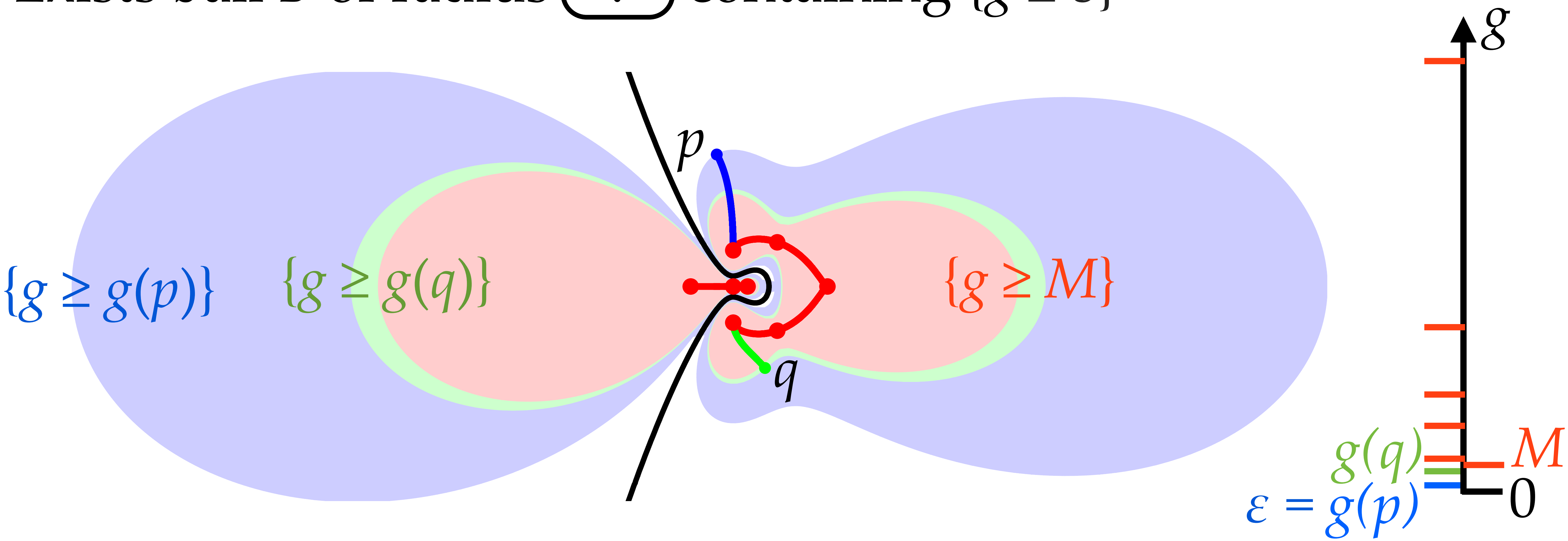

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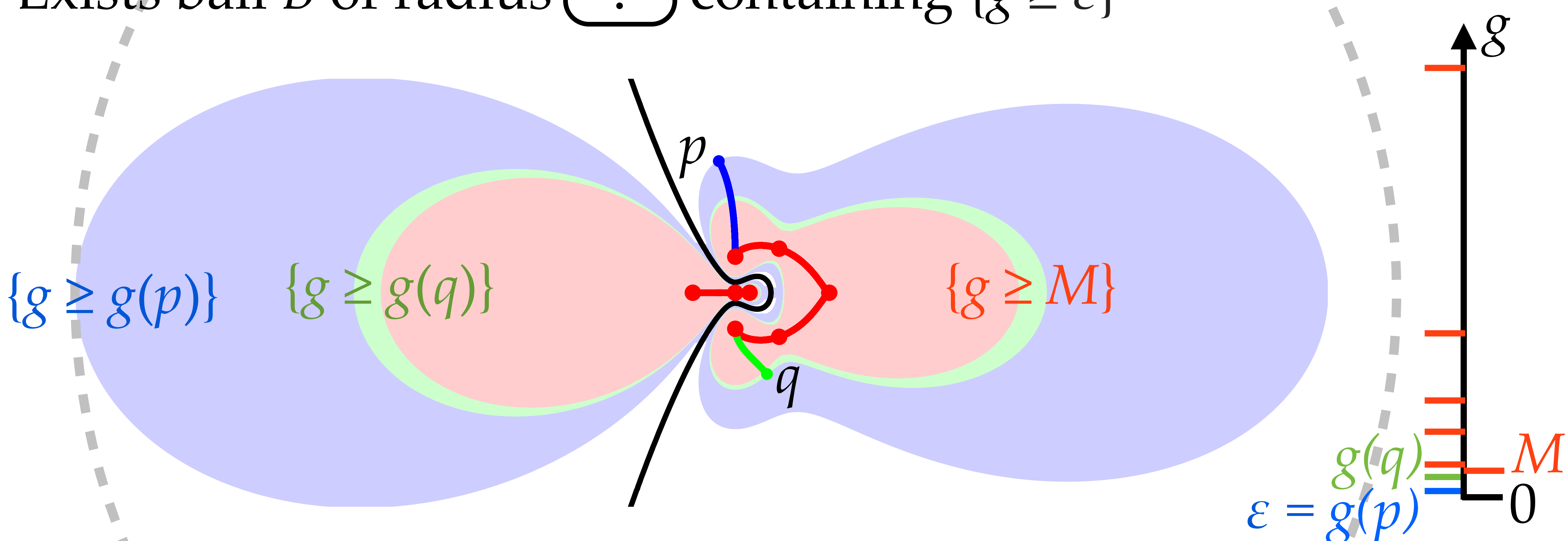

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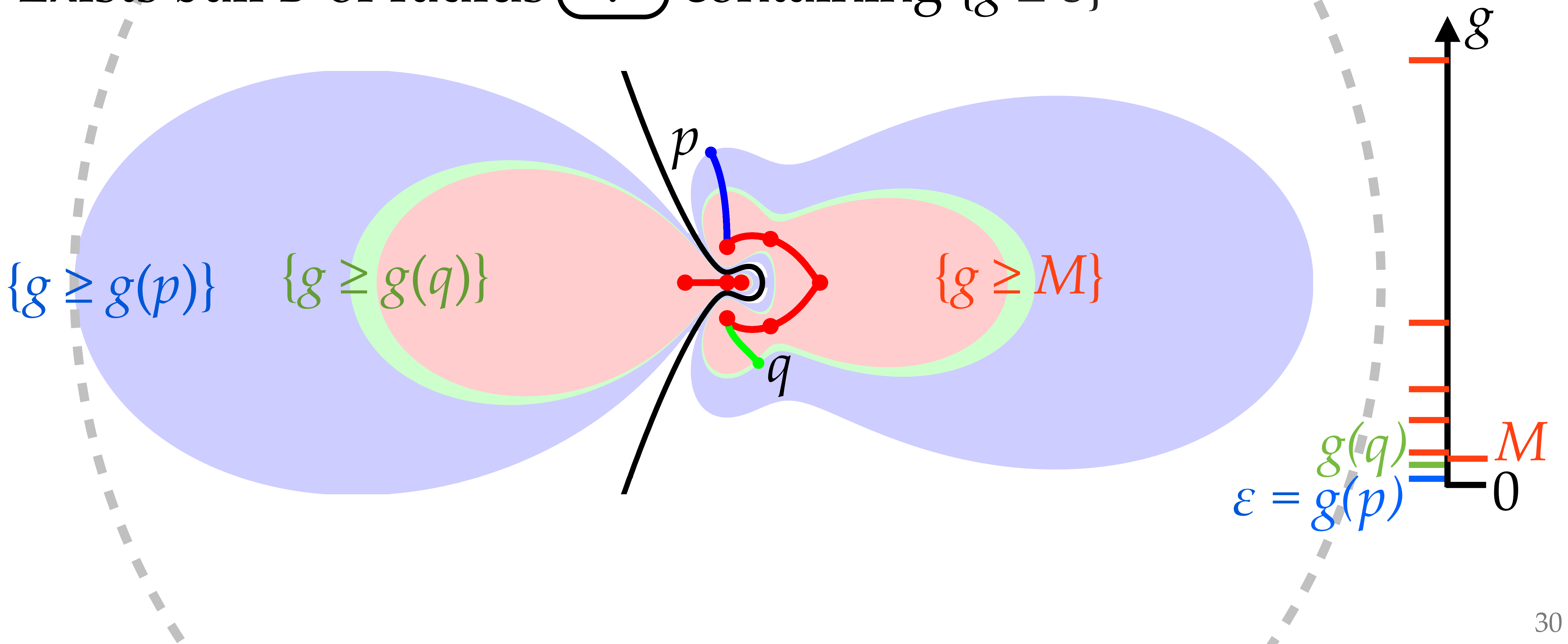

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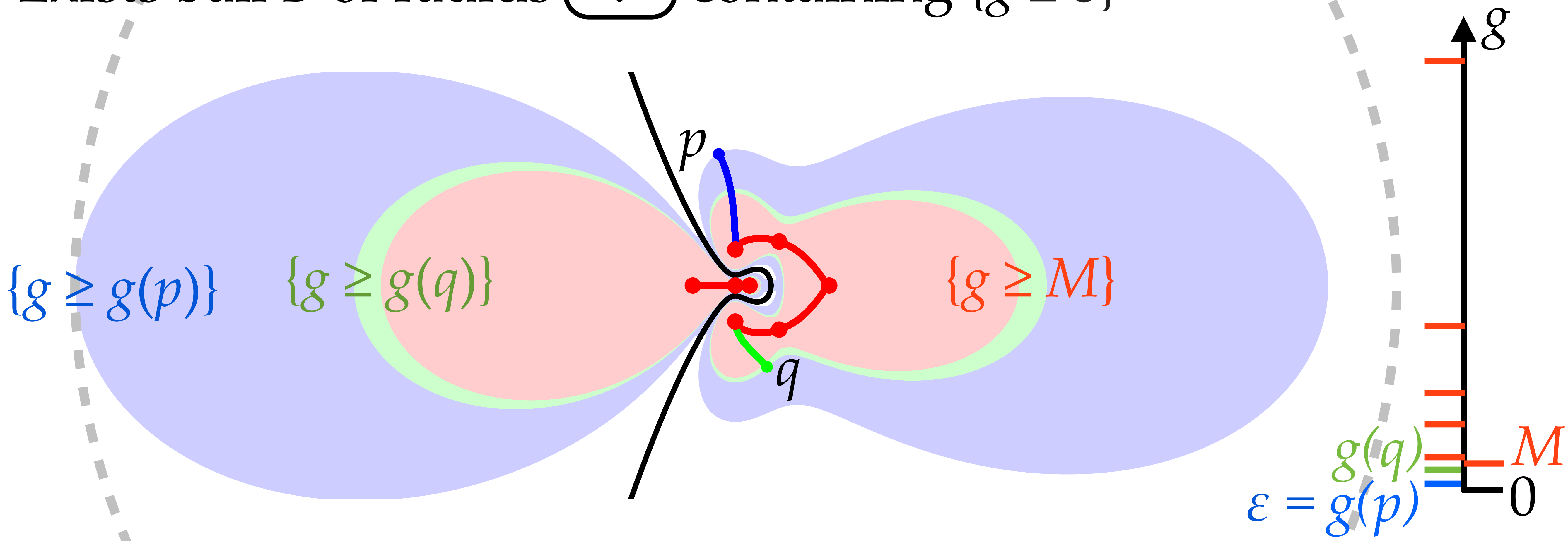

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M bound

Application of DMM bound in [Emiris, et al. 2010]

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Application of bound
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radius bound

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We can contain $\left\{g \geq \dfrac{A_1}{A_2}\right\}$ in a ball of radius r.

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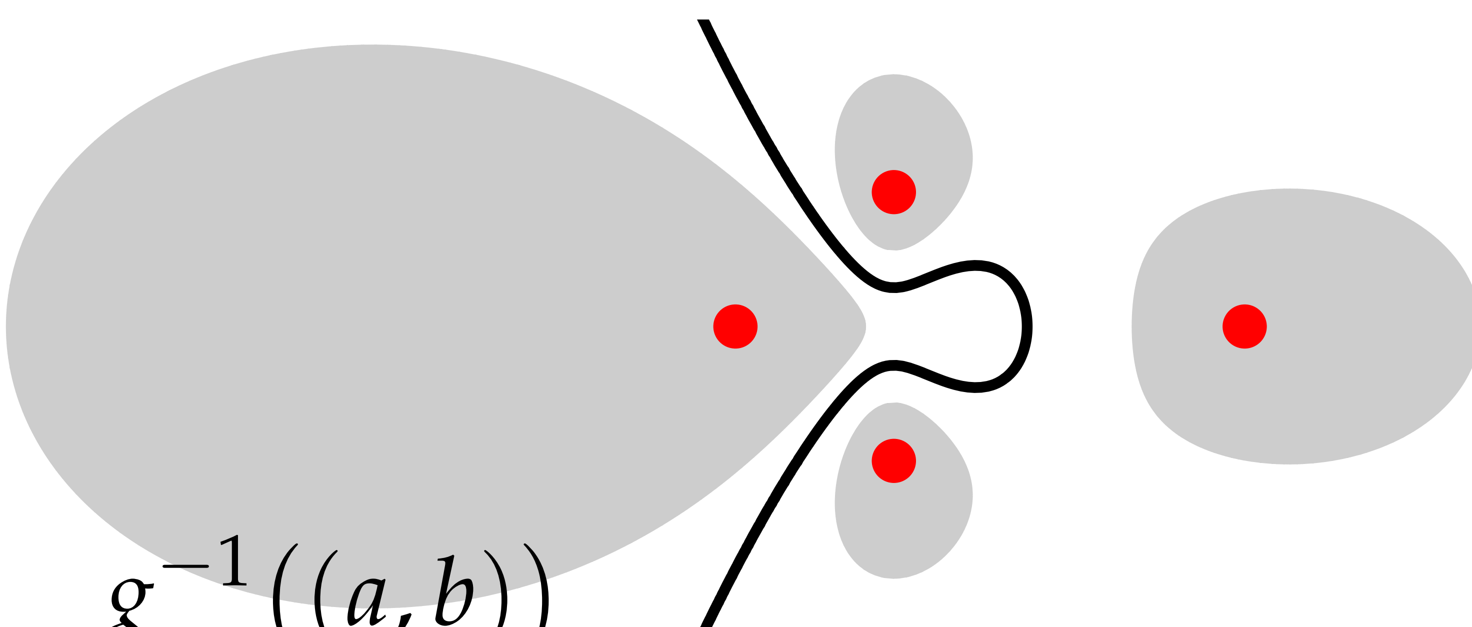

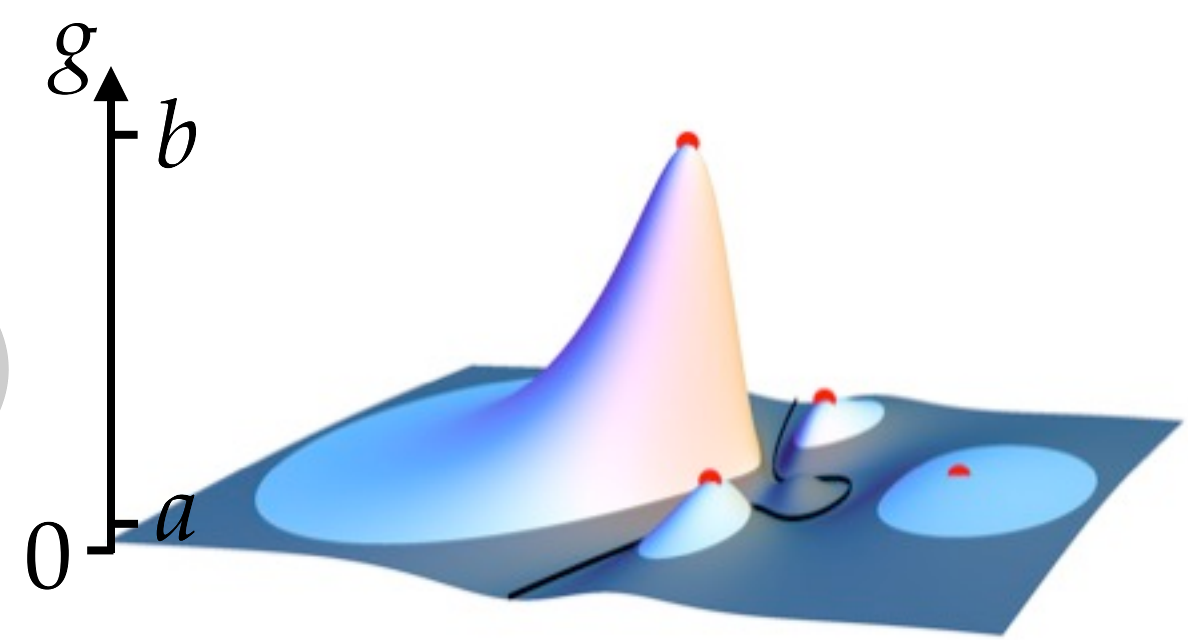

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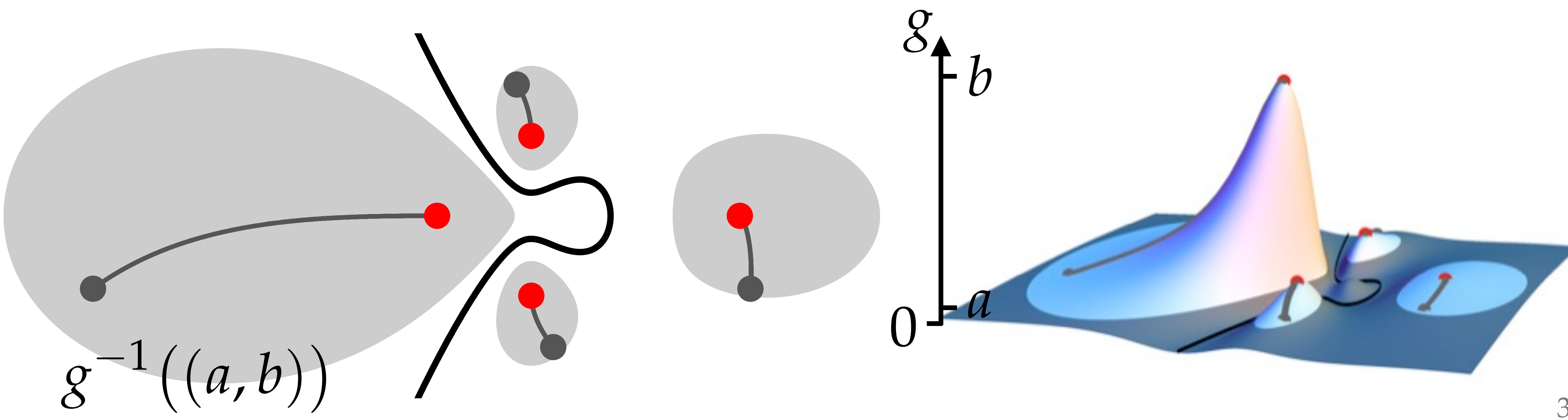

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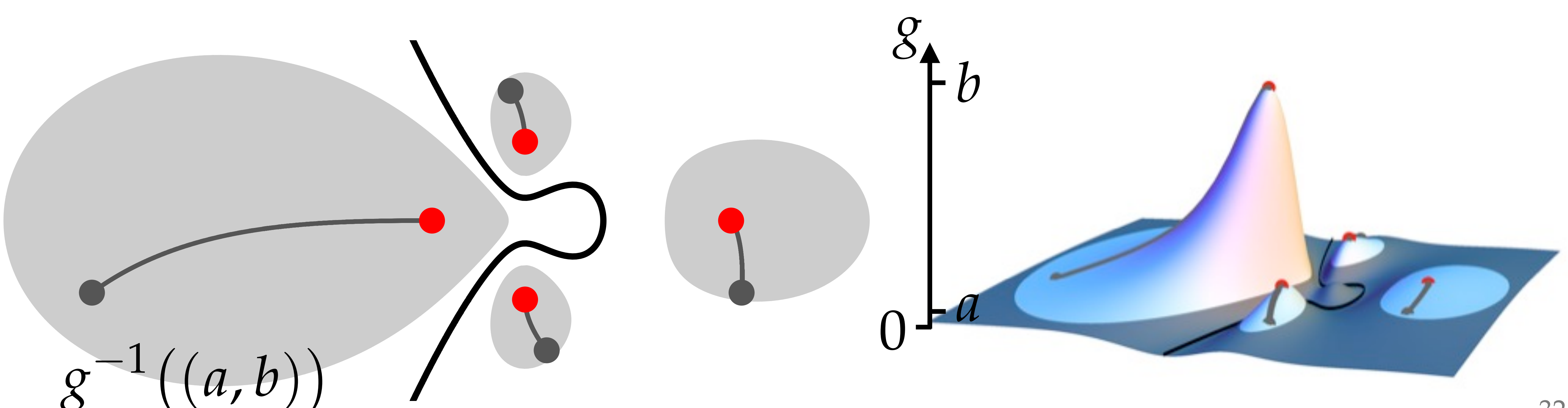
Proof Idea: Motivated by [D'Acunto, Kurdyka. 2004]



3. Length Bound: Proof Sketch

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p and q in a same component of $\{f \neq 0\}$ can be connected by a connectivity path of length bounded by

$$4nr(6d+4)^{n-1}$$

where

$$r = n\left(120 A_1 A_2 H d\left(c_1^2 + \cdots + c_n^2 + 1\right)\right)^{4n^3(6d)^{3n}}$$

$$\frac{A_1}{A_2} = \min\left\{g(p), g(q), \frac{1}{\left(2dH\left(c_1^2 + \cdots + c_n^2 + 2\right)\right)^{104n^3(5d)^{5n}}}\right\}$$

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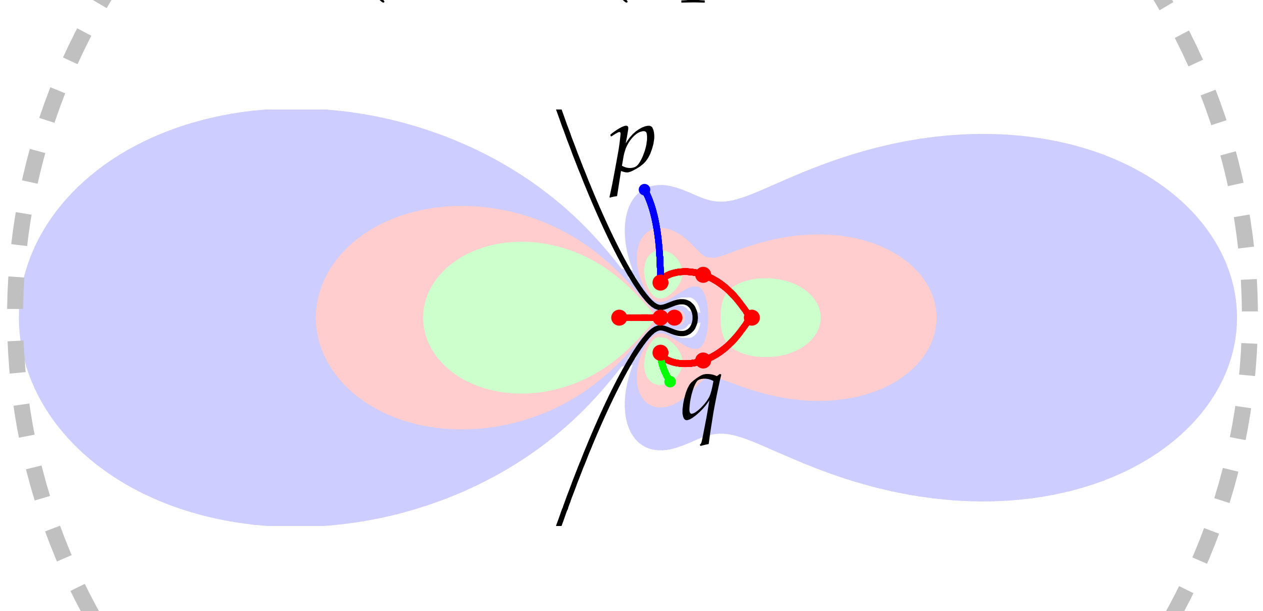

ball of radius r containing $\left\{g \geq \frac{A_1}{A_2}\right\}$

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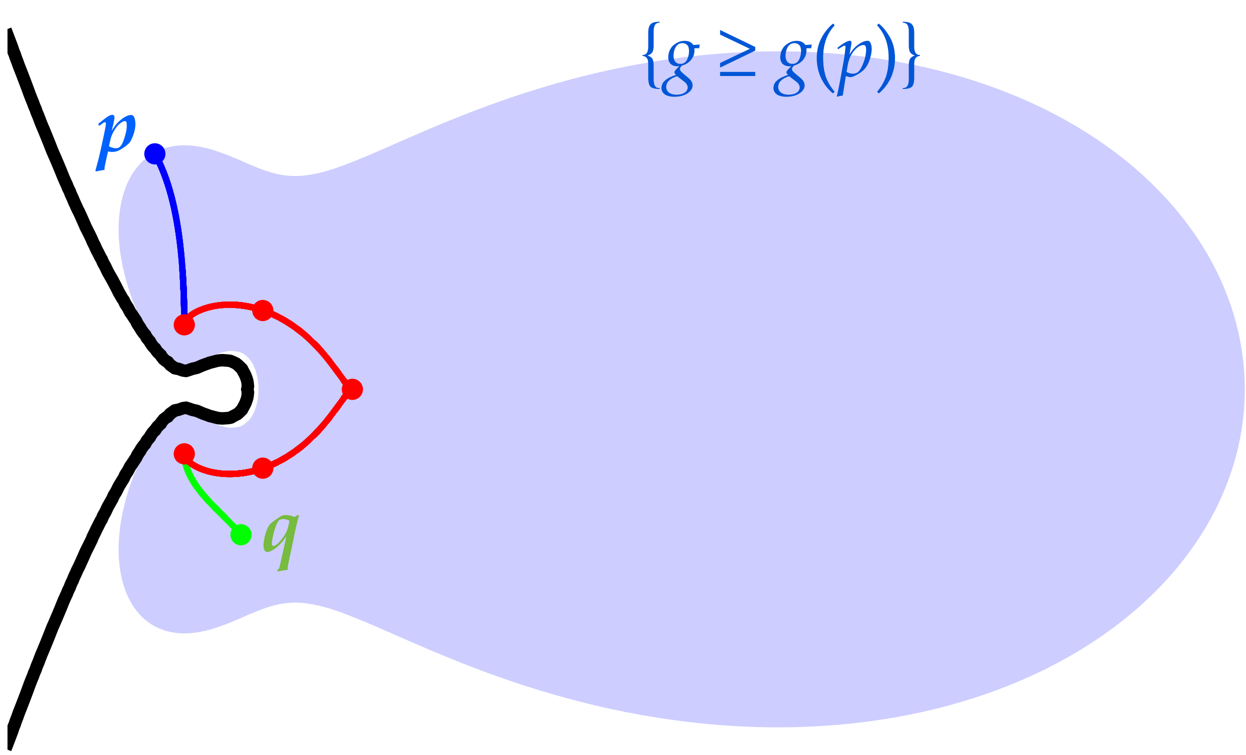

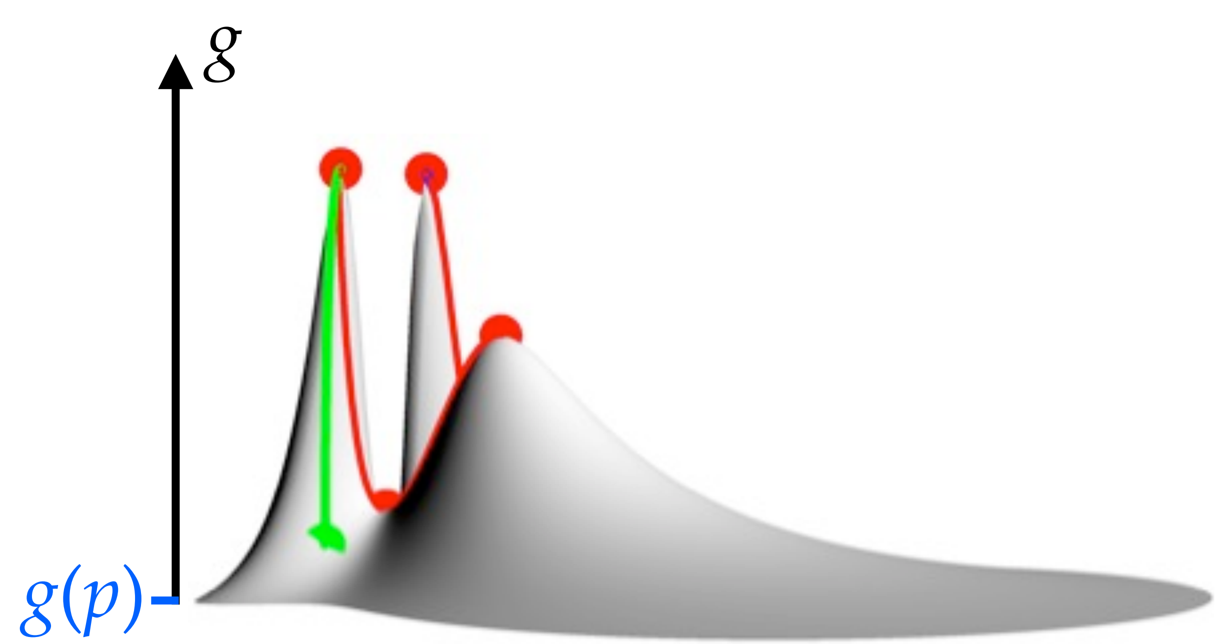

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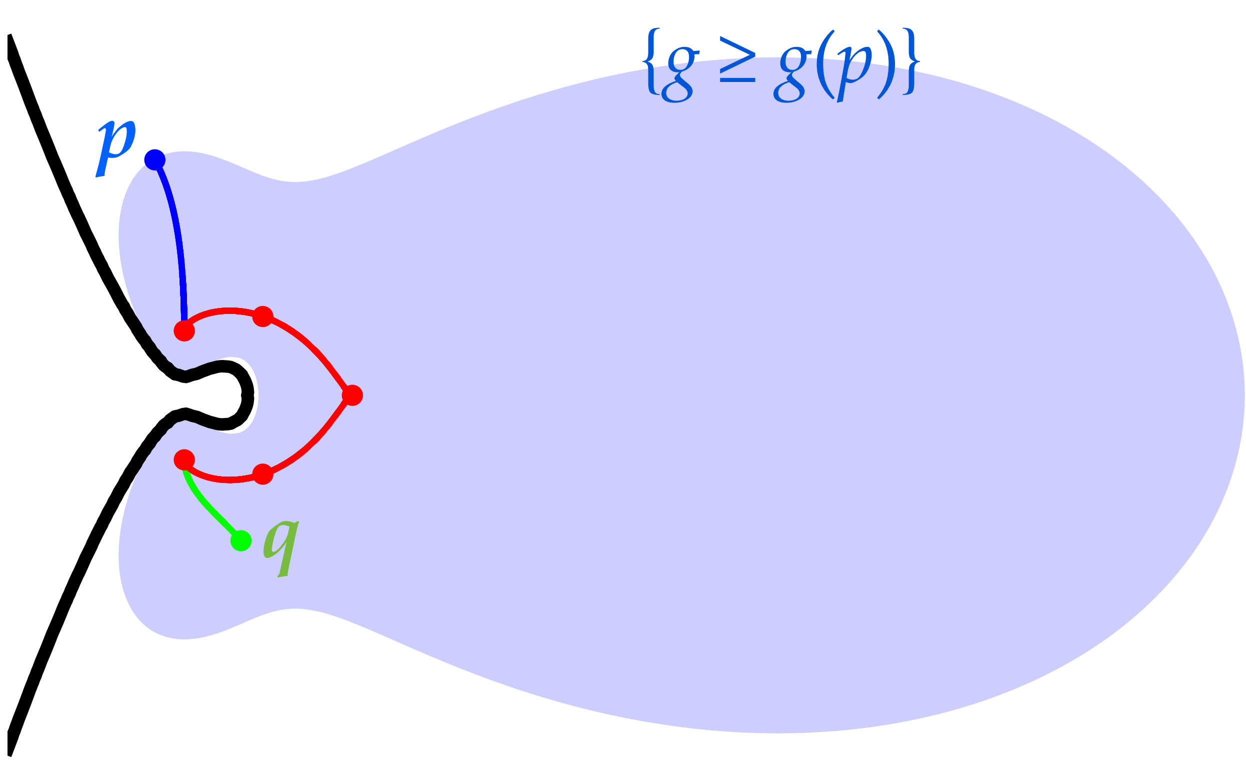

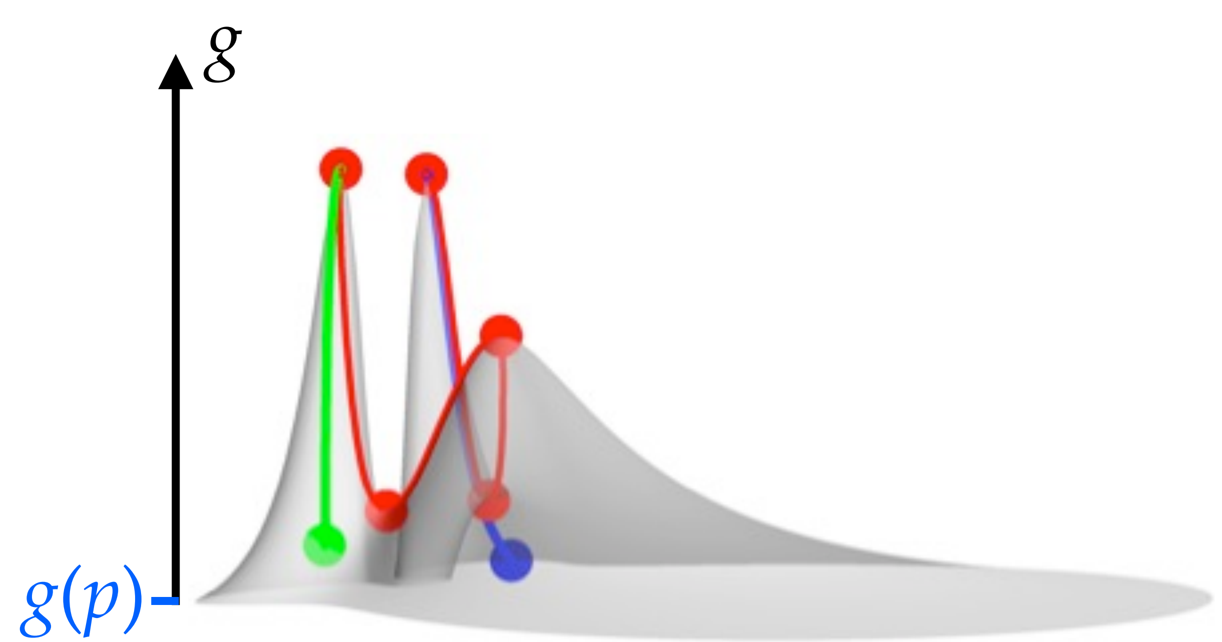

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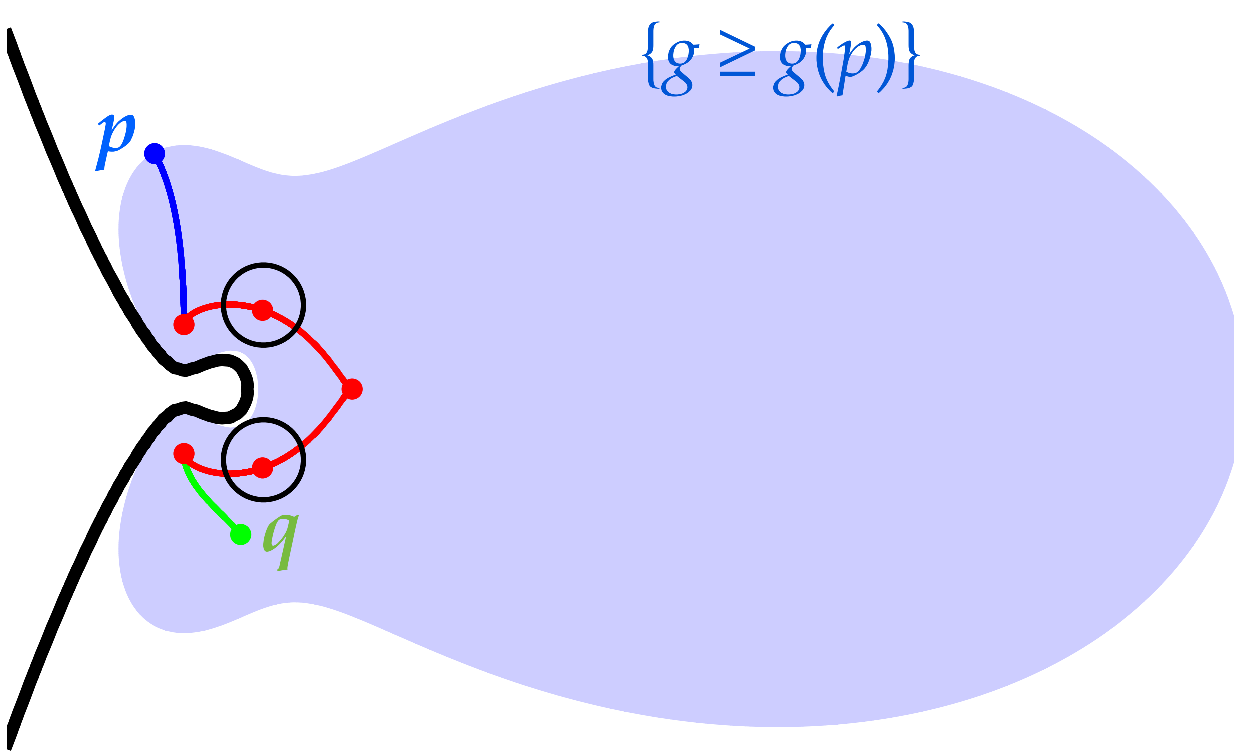

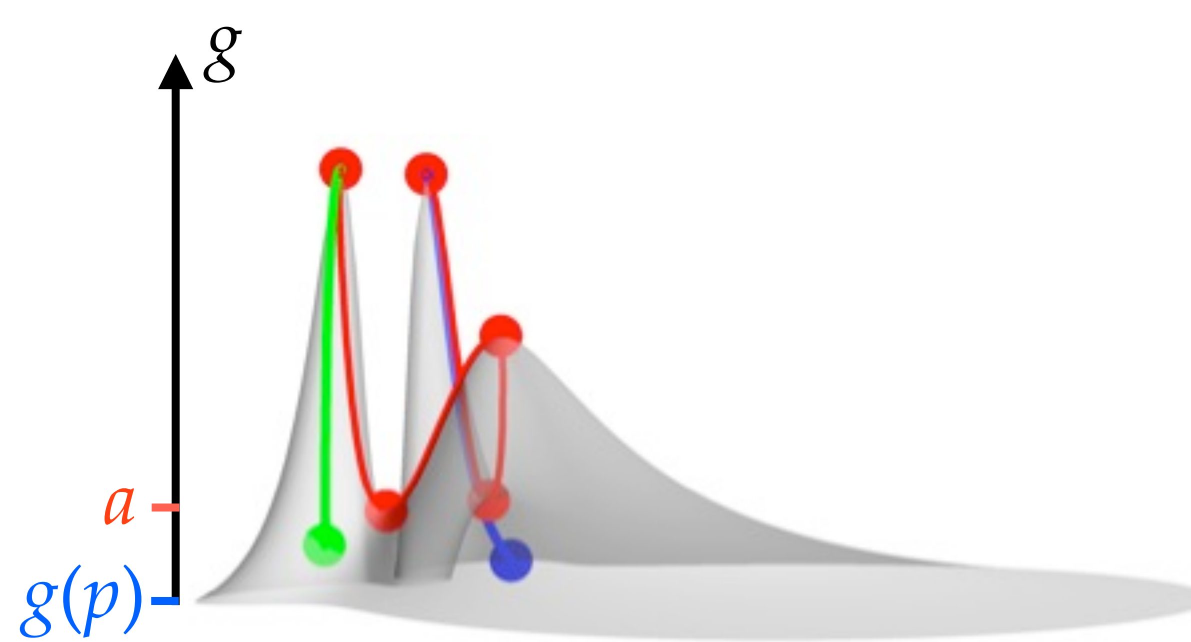

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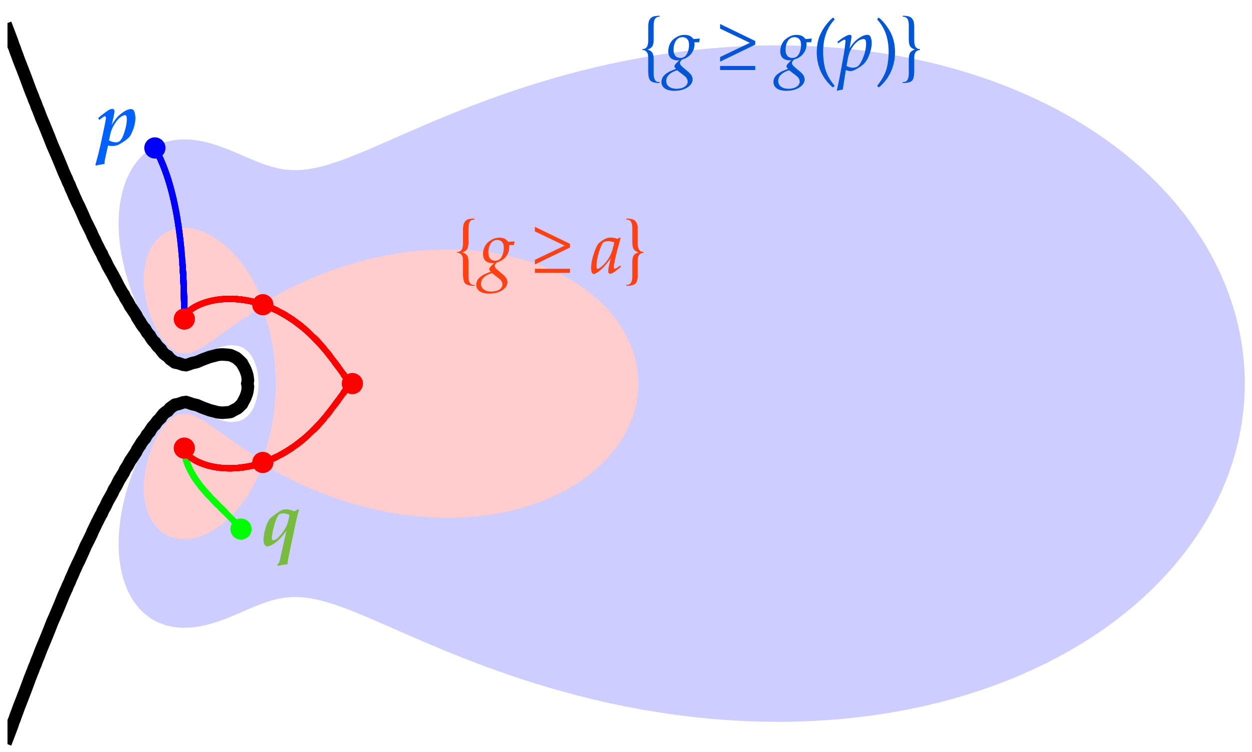

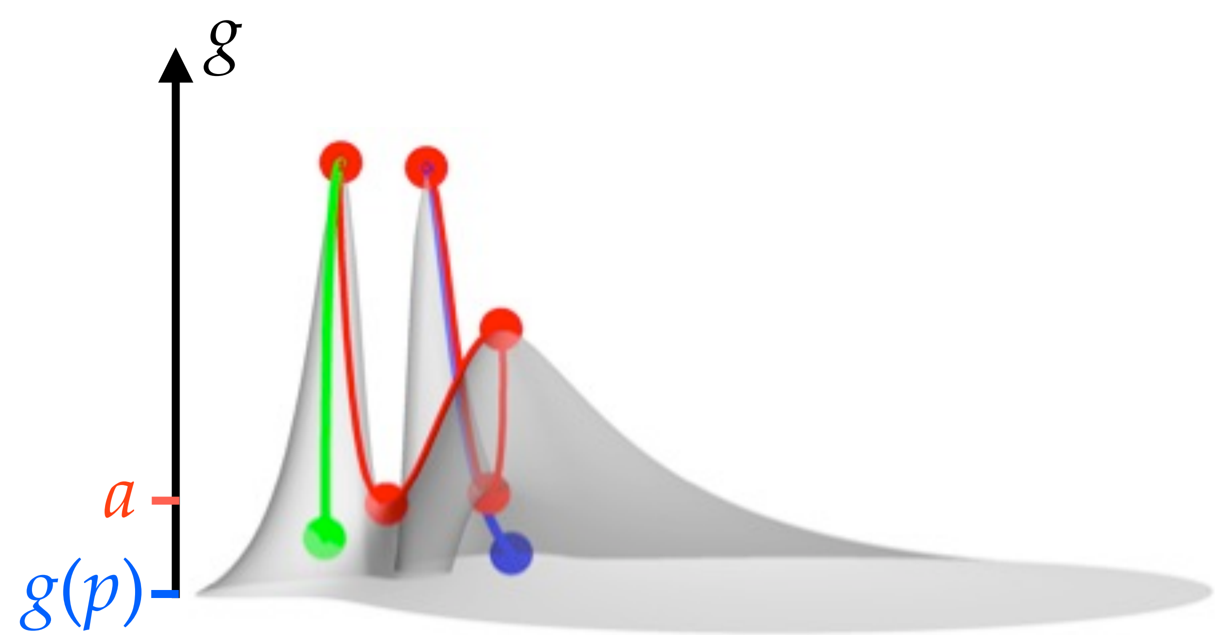

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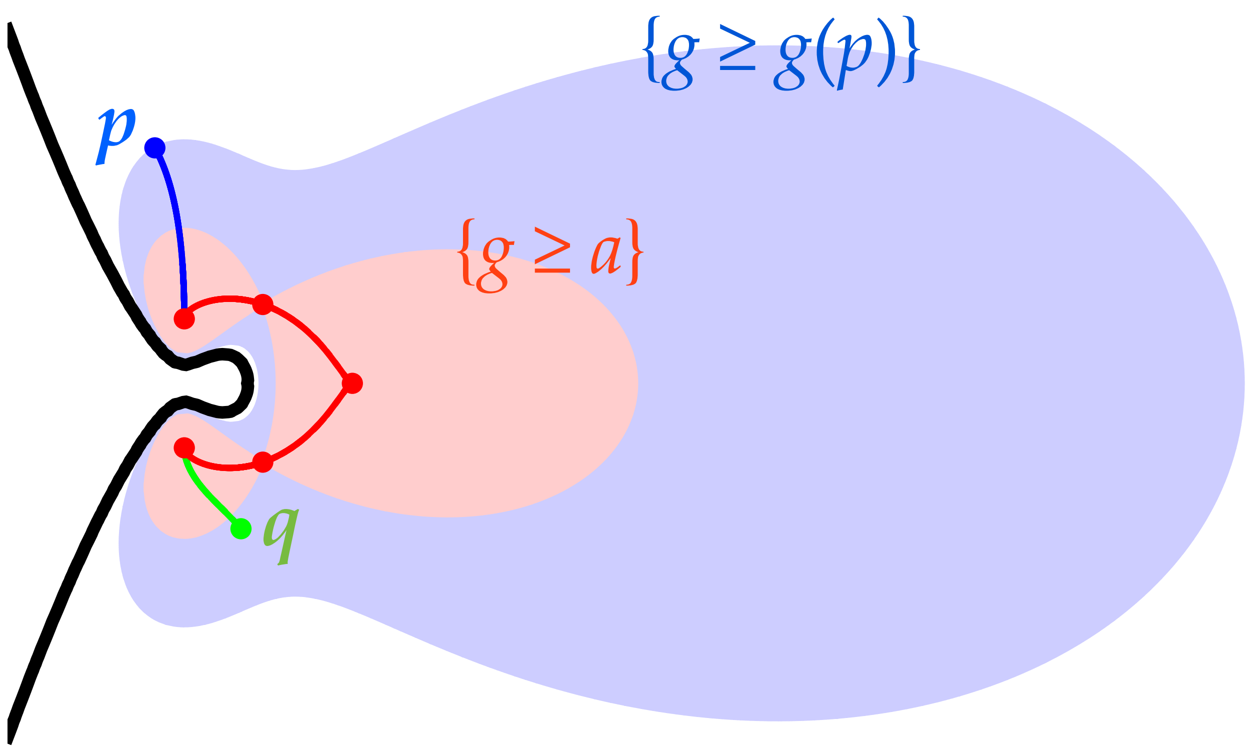

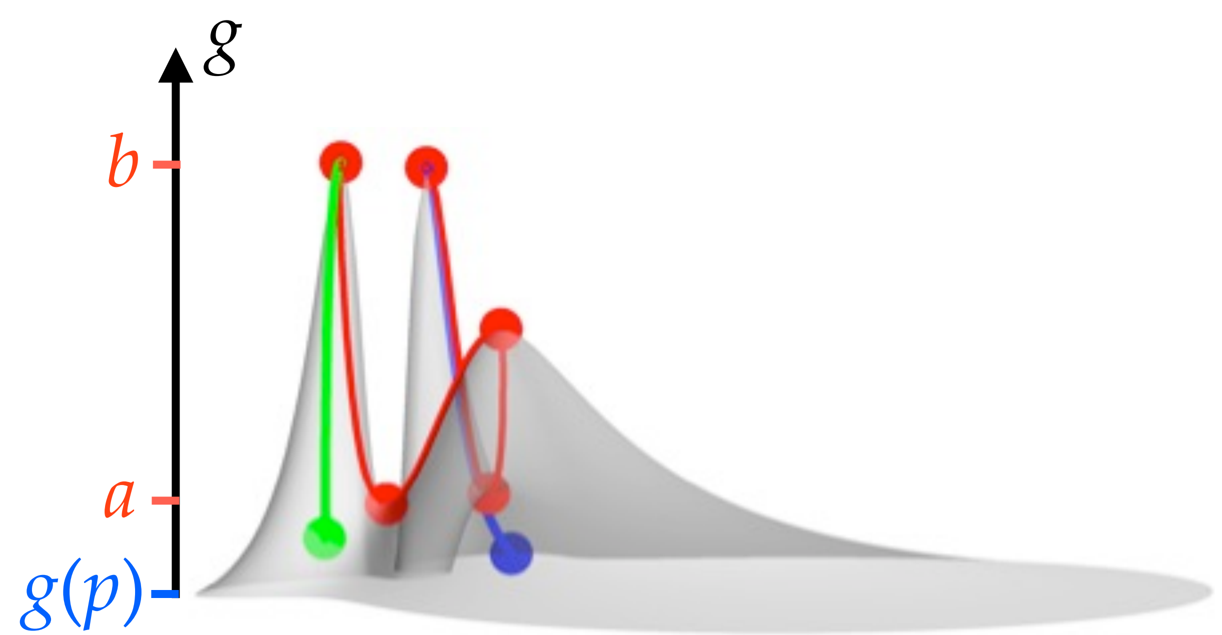

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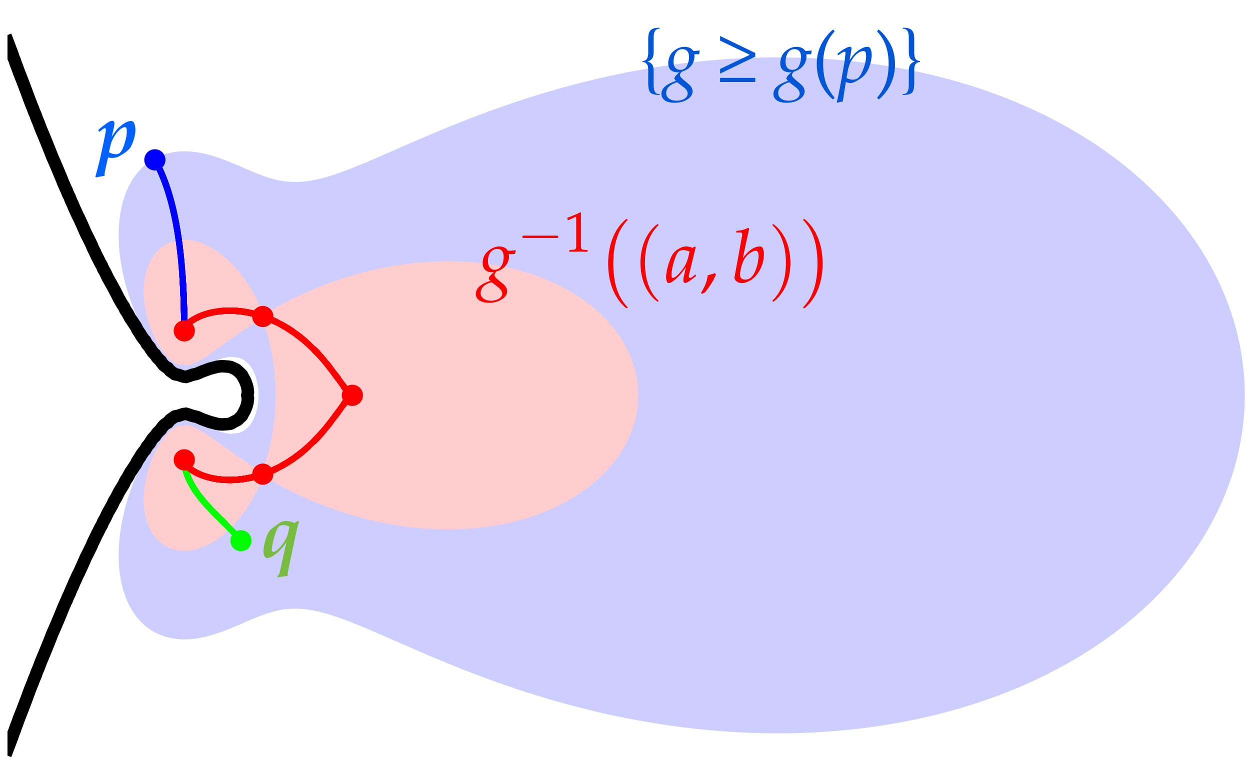

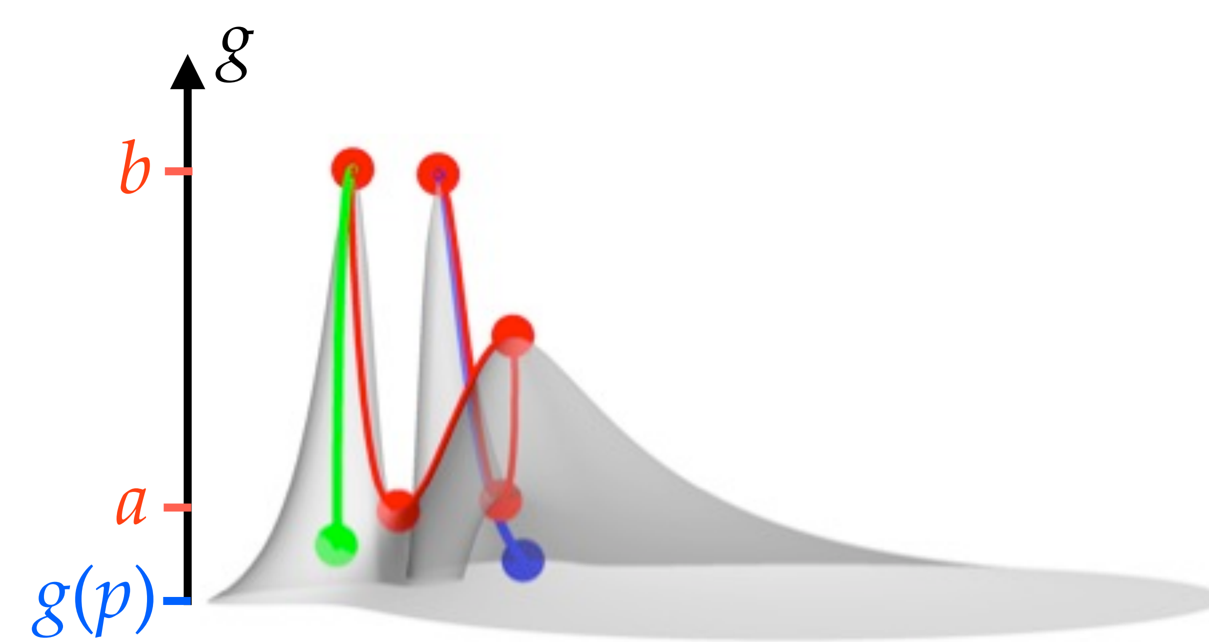

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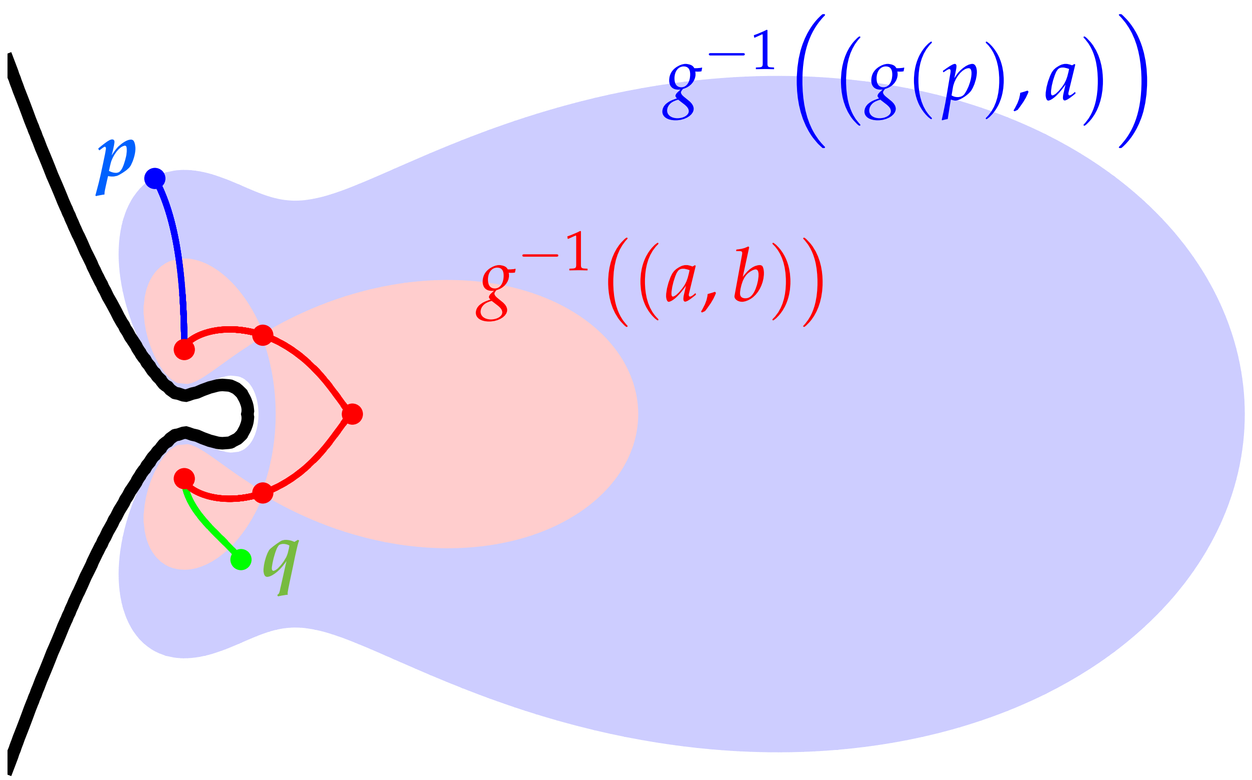

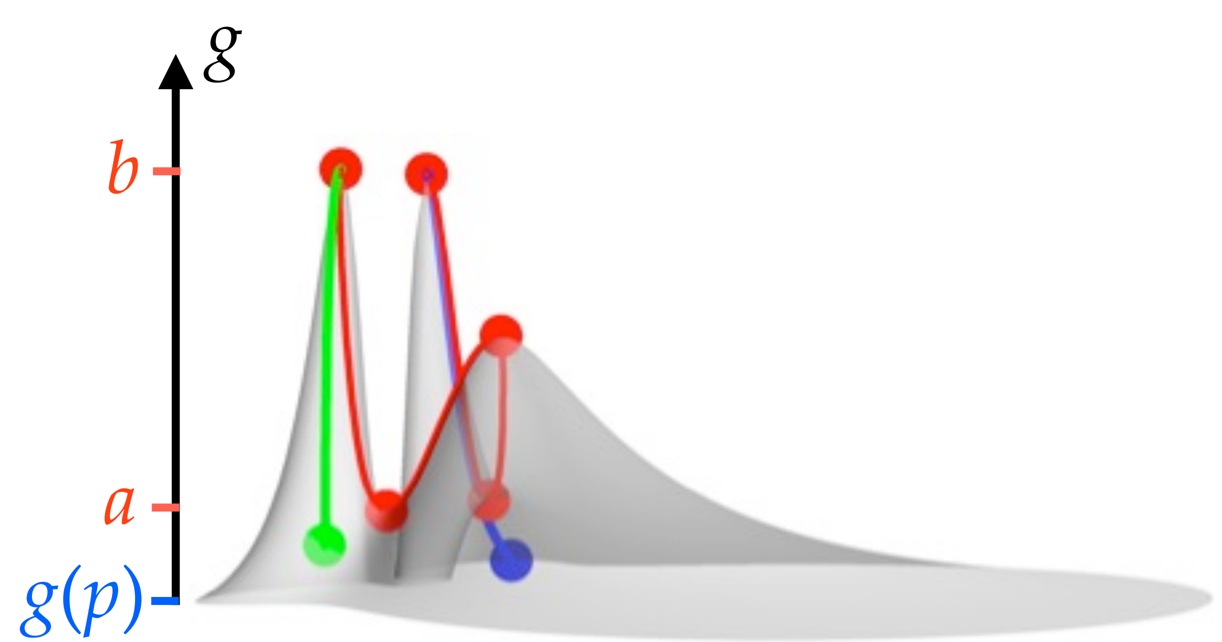

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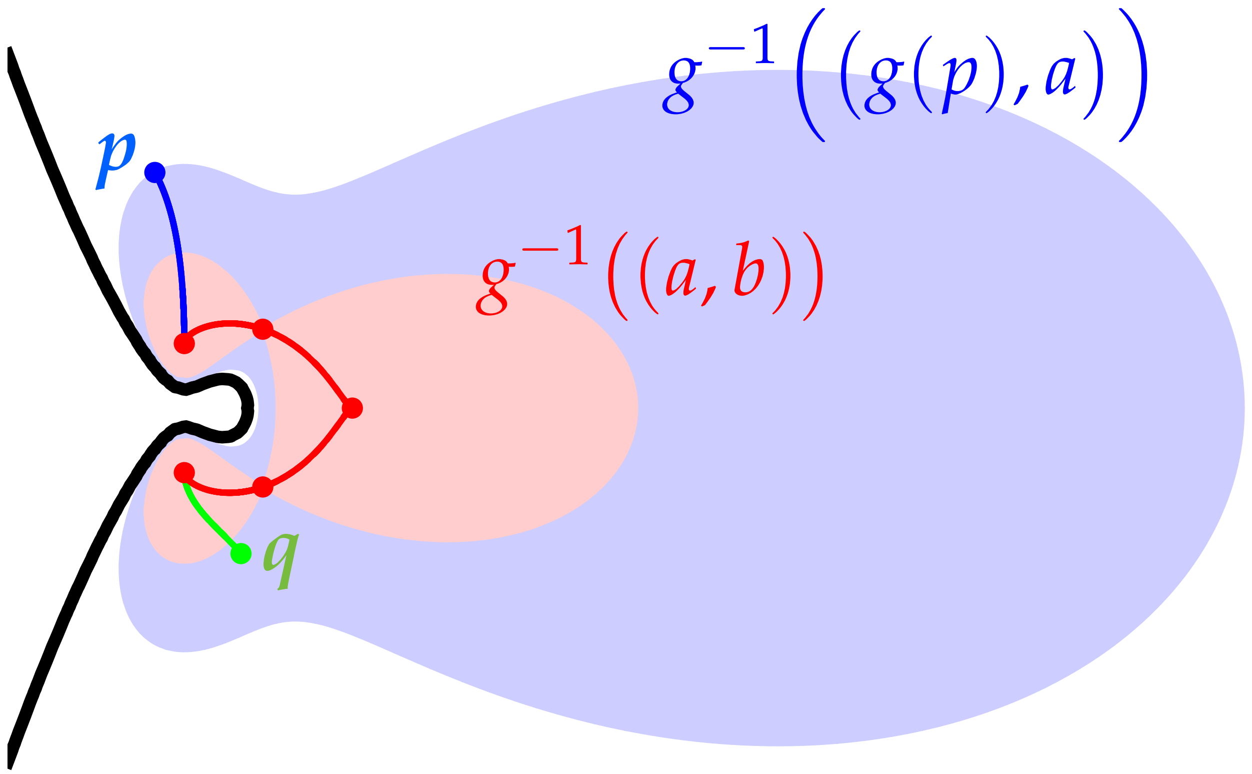

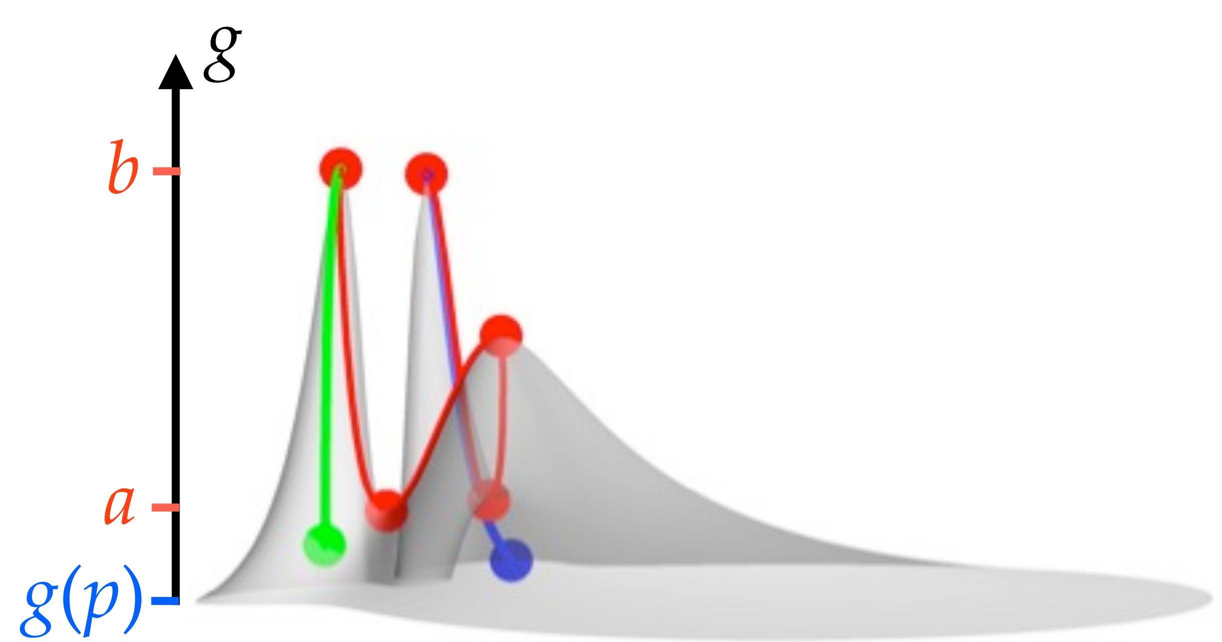

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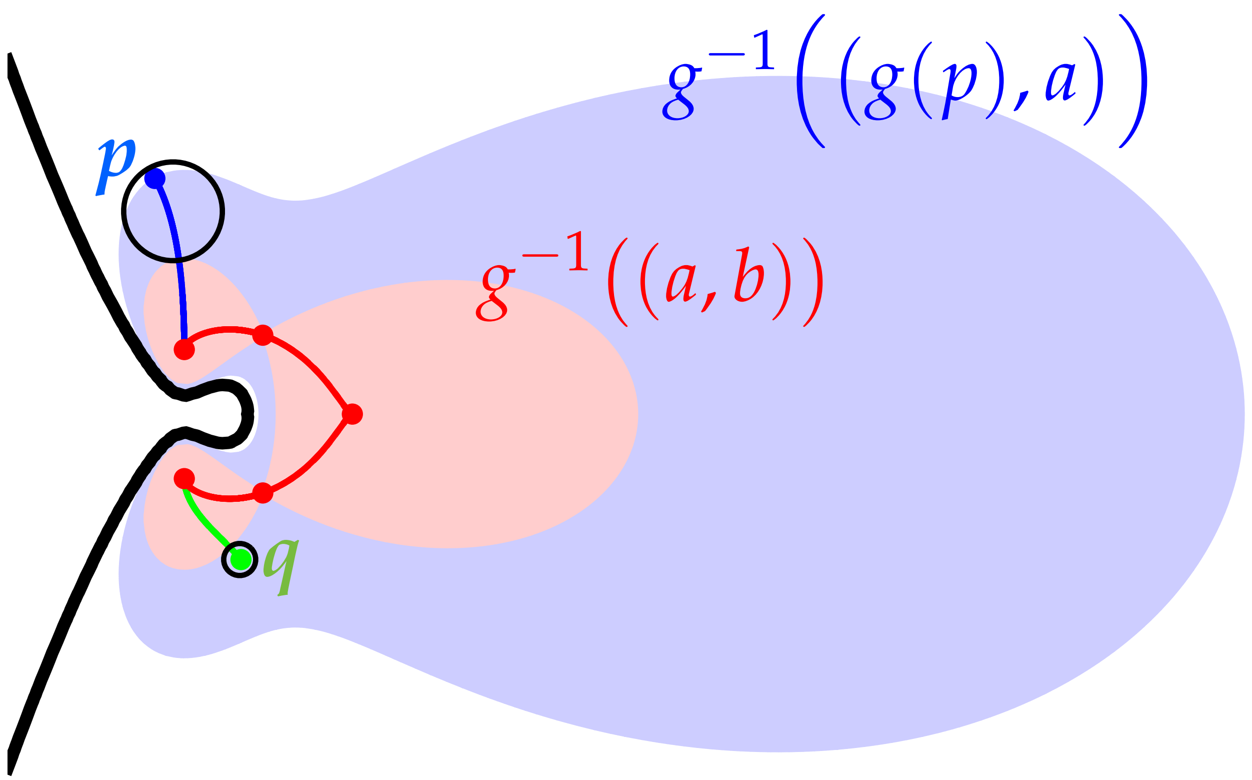

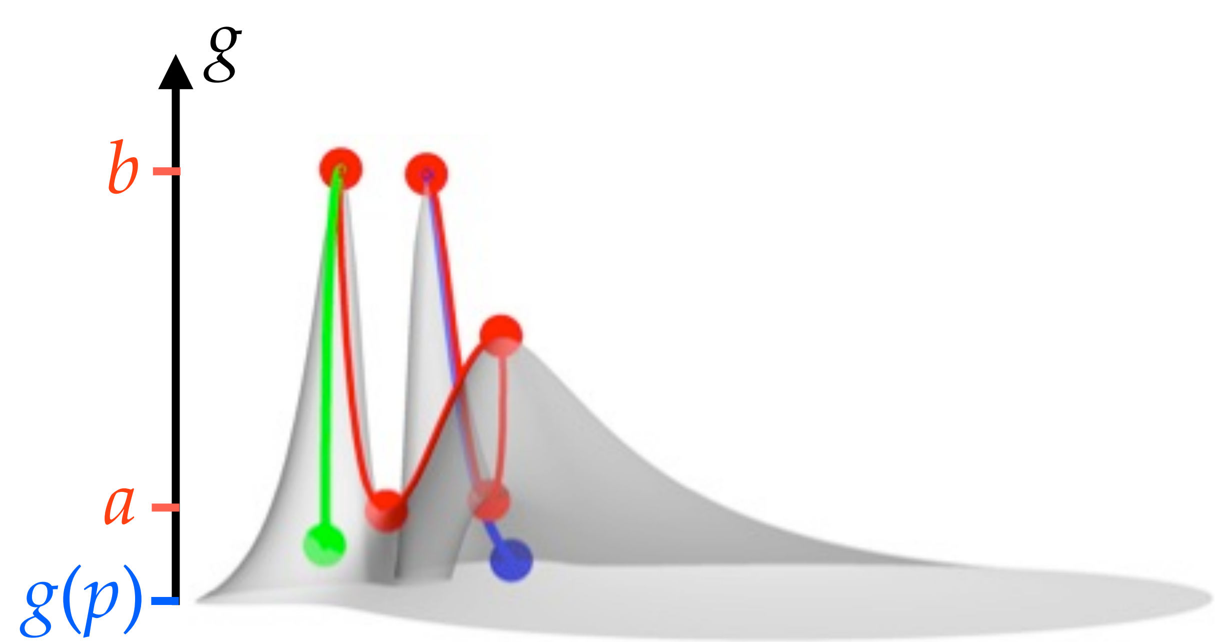

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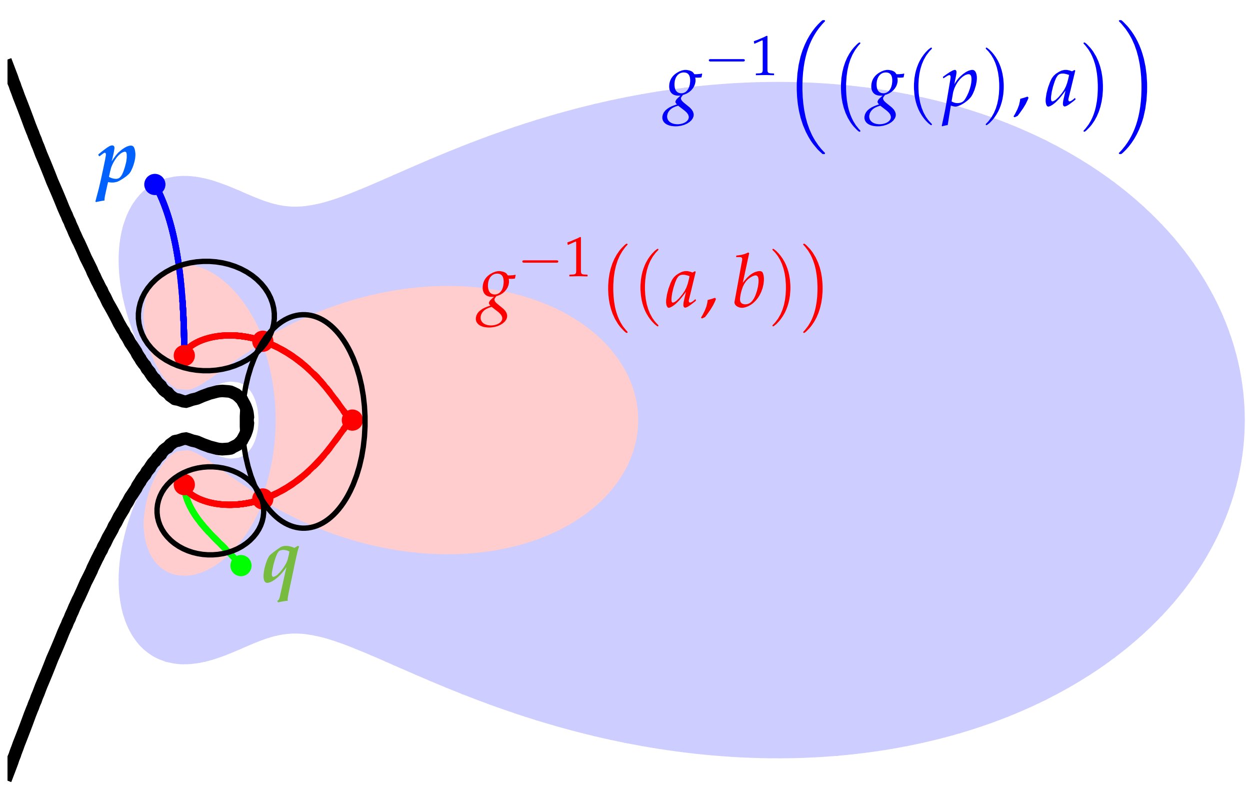

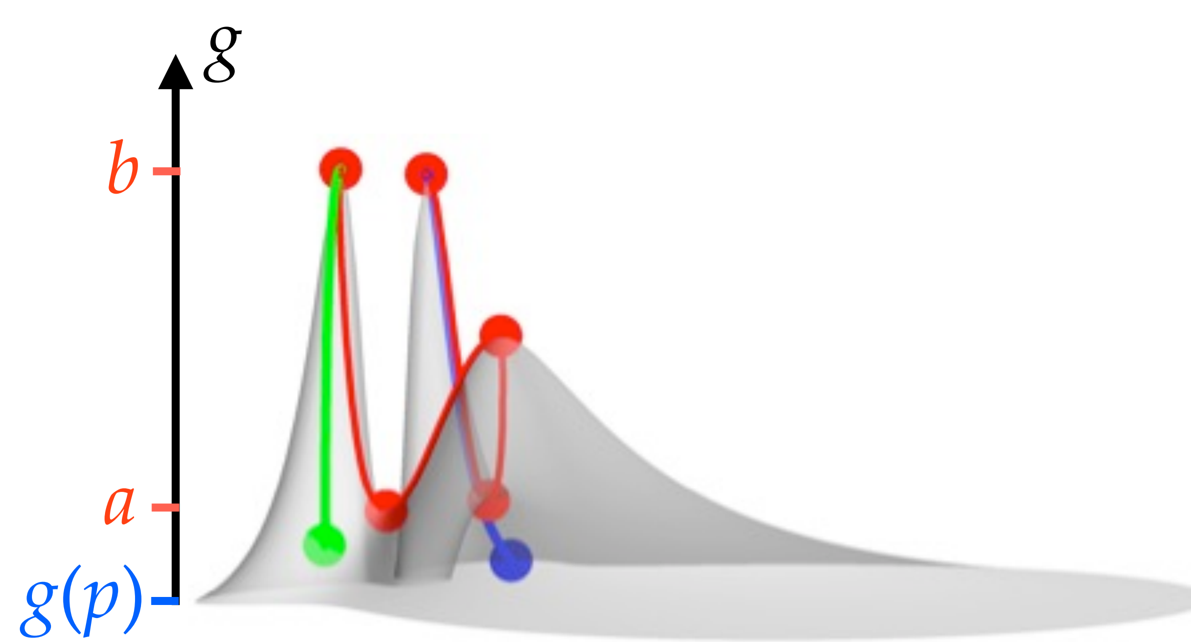

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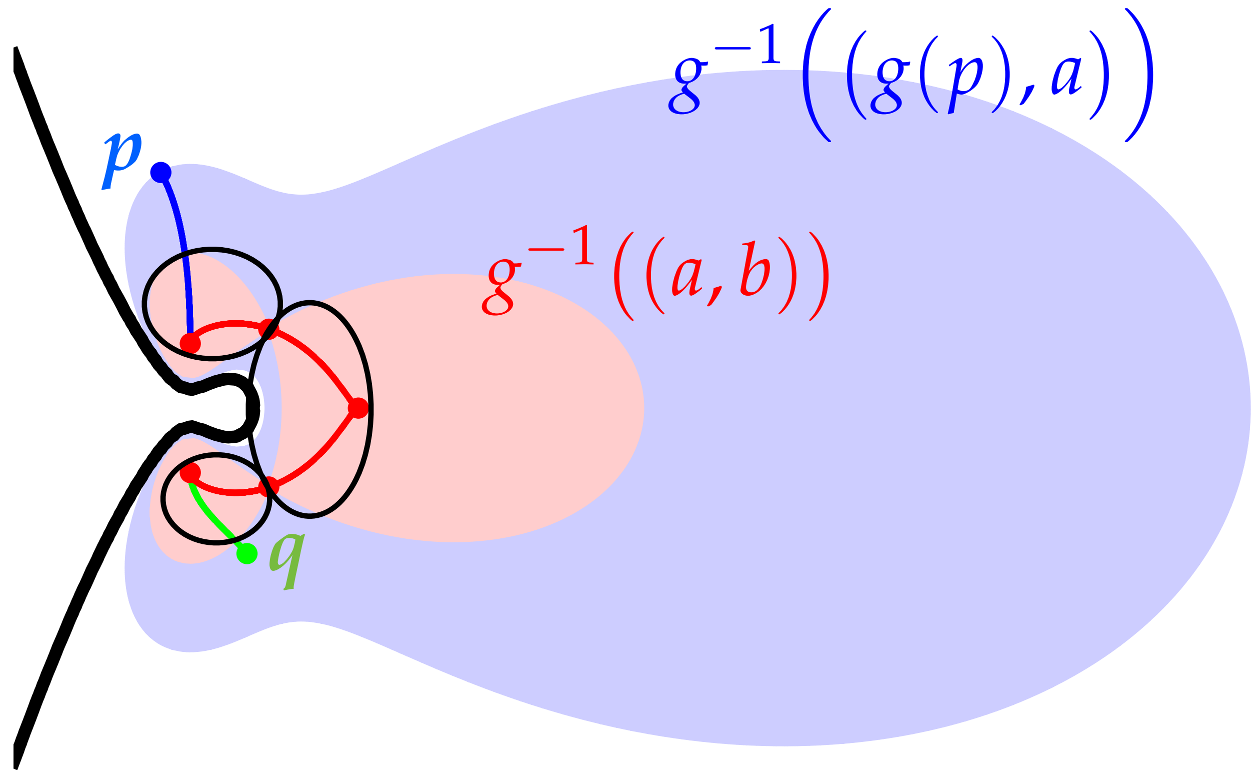

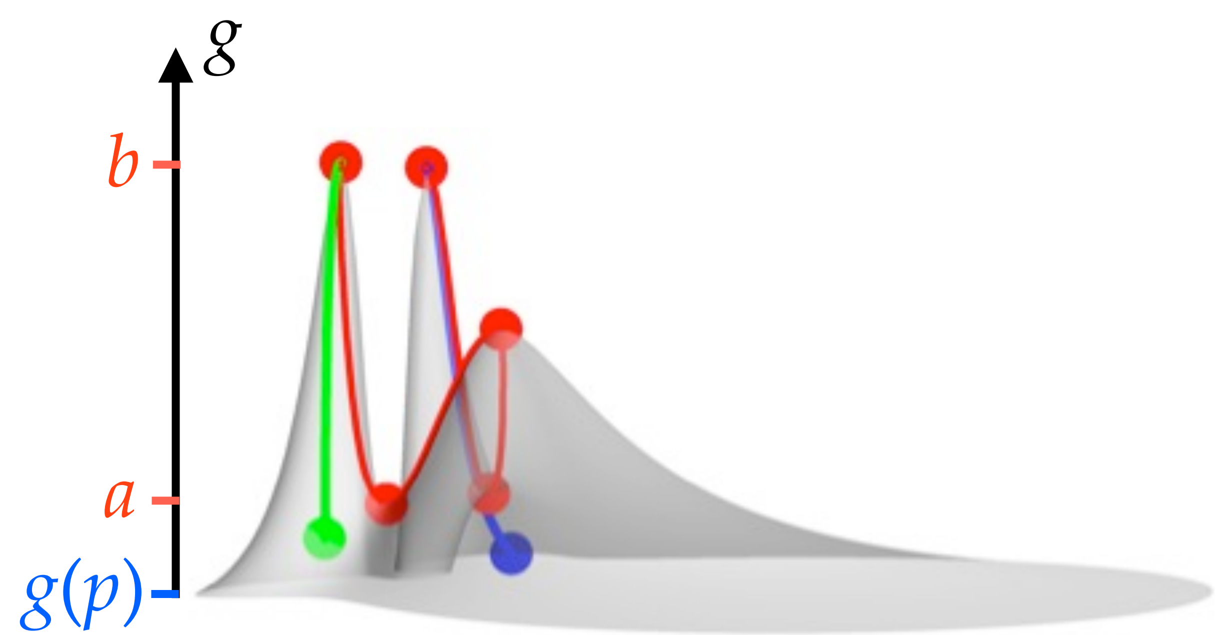

Future Work

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- Rigorously tracing steepest ascent paths

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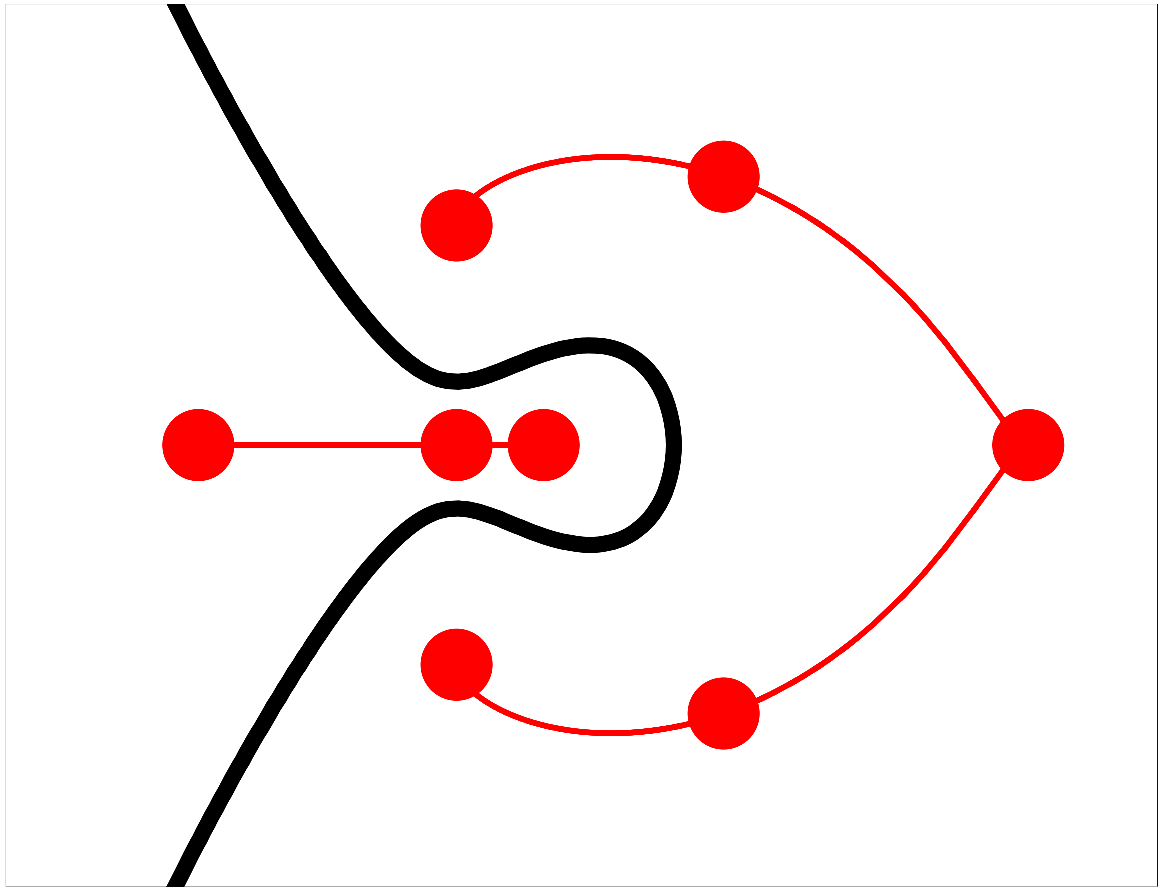

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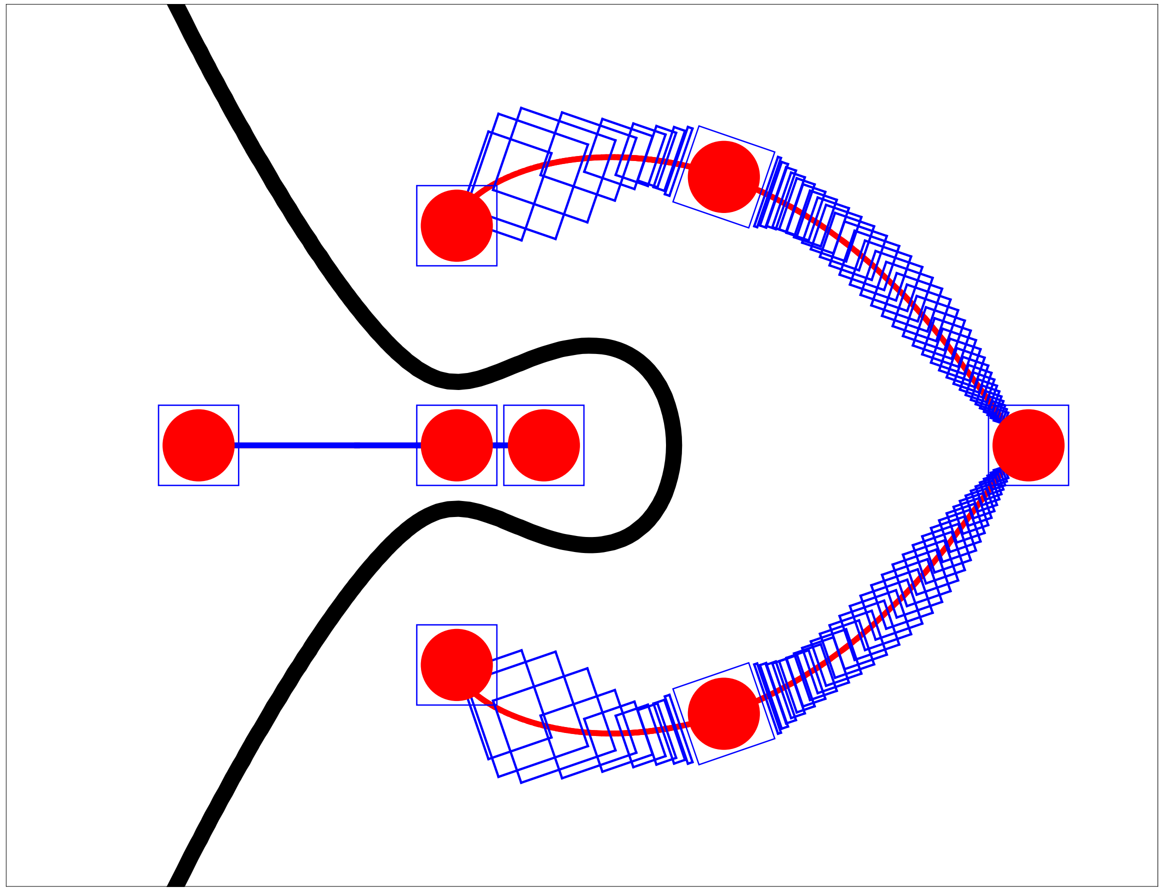

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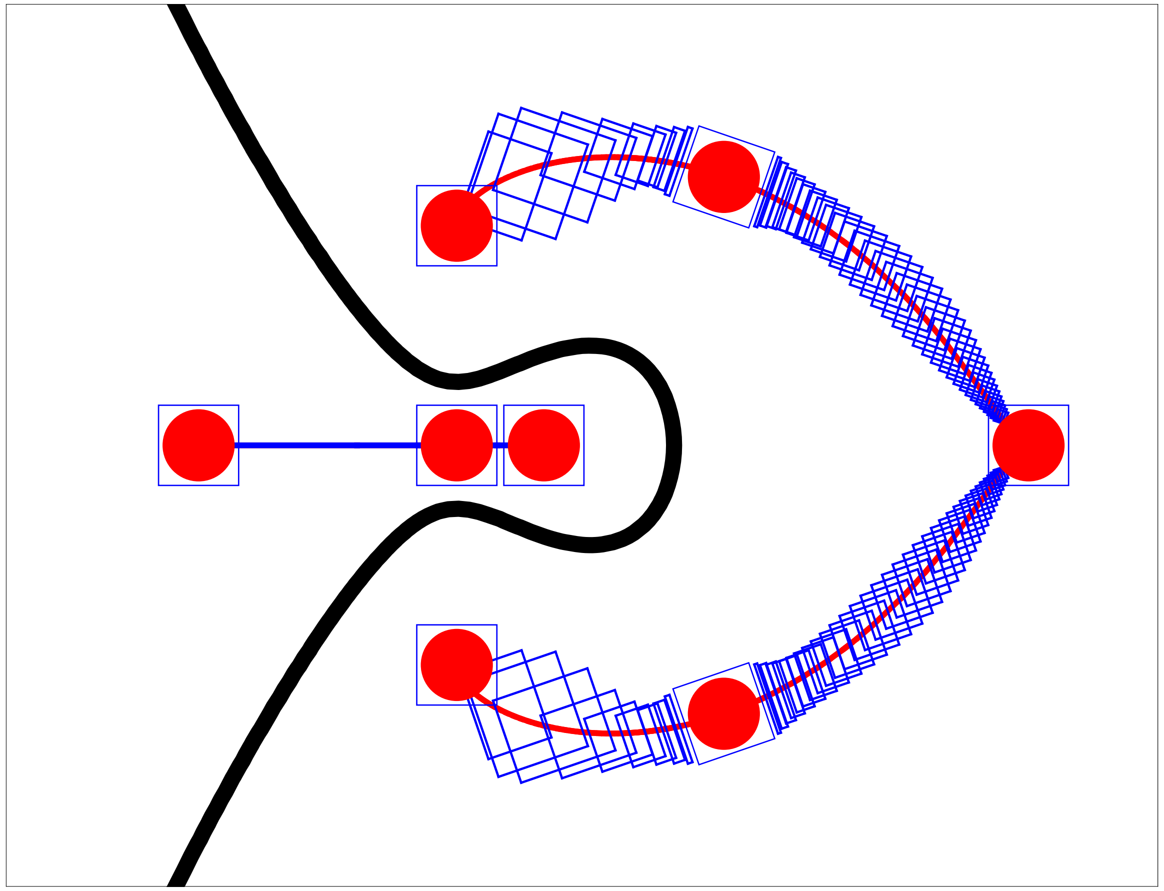

- Improve bounds

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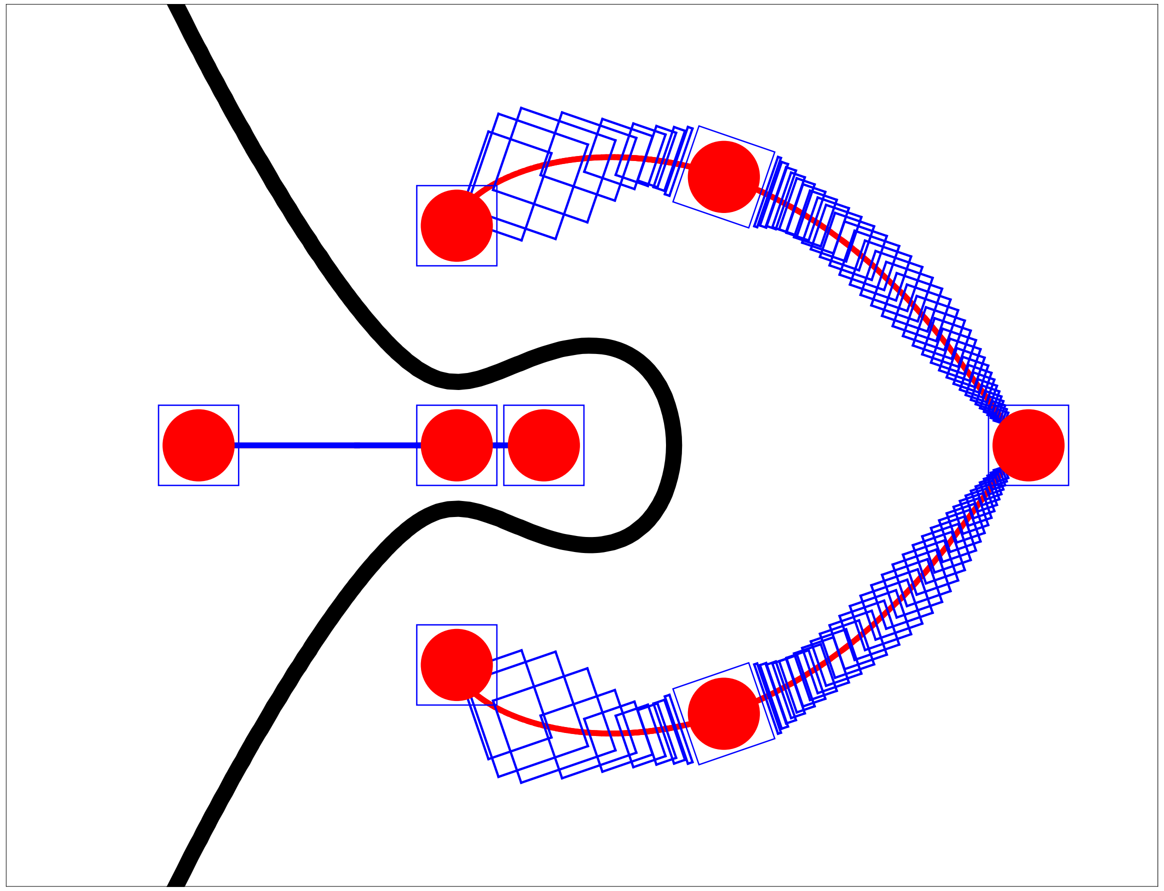

- Improve bounds
- Complexity analysis