Connectivity in Semialgebraic Sets

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NCSU Symbolic Computation Seminar

Problem: Connectivity

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$$f = x_1^3 - x_1^2 + x_2^2$$

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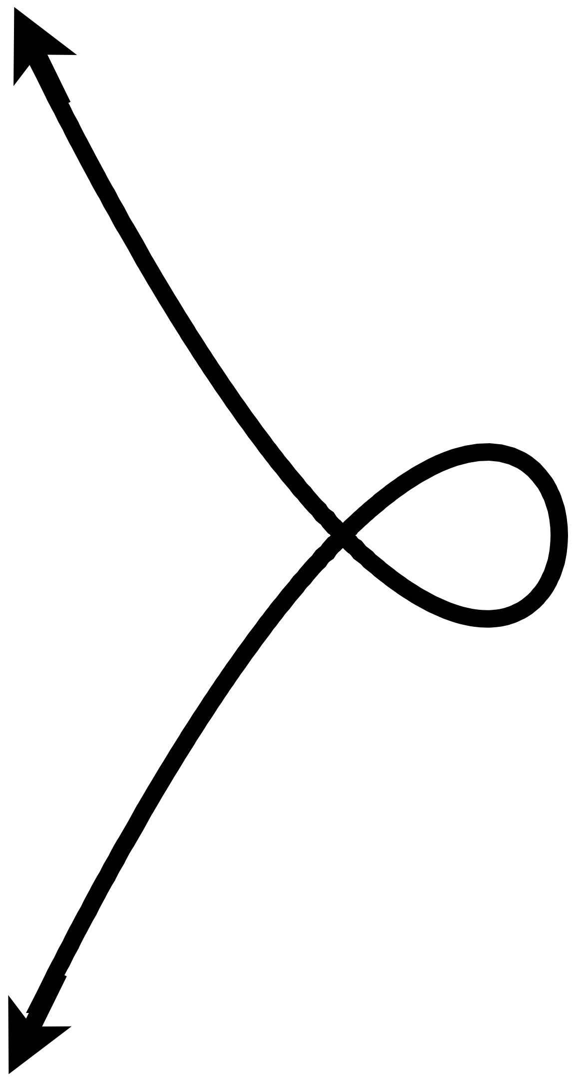

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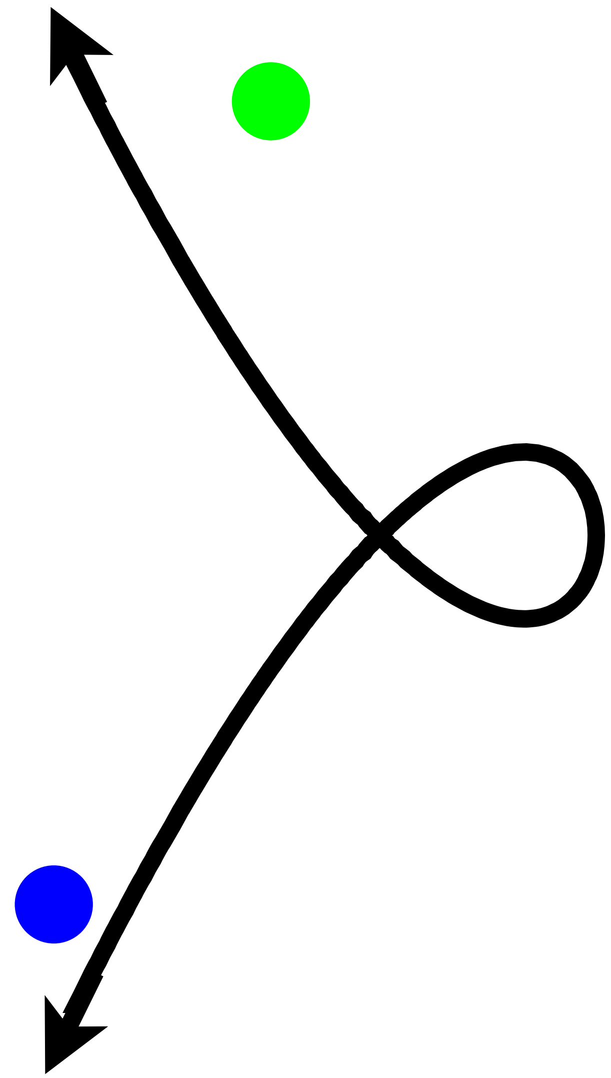

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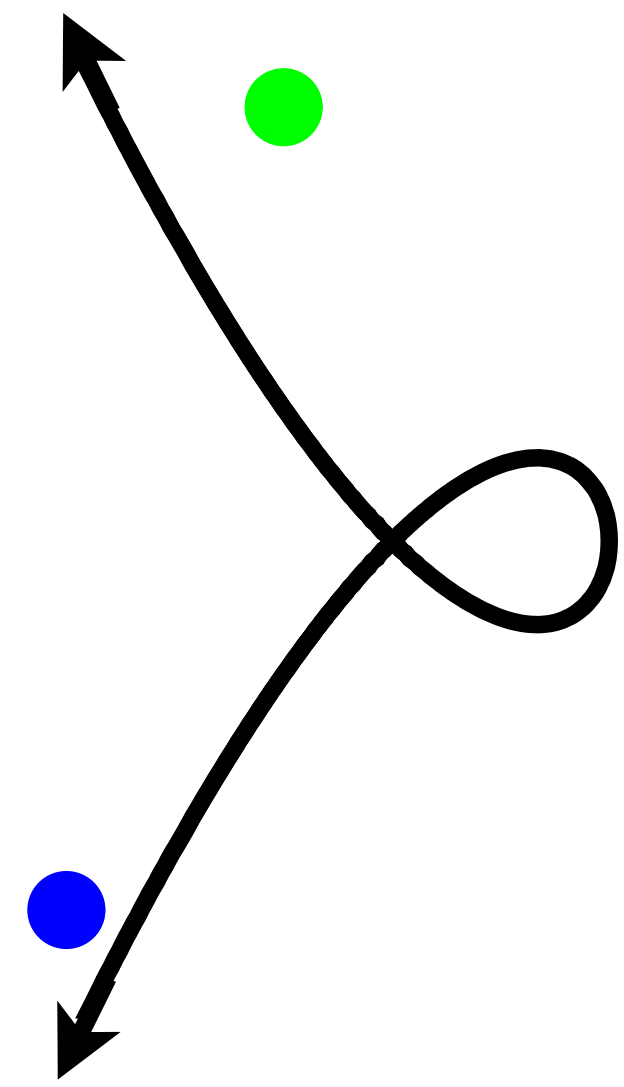

False

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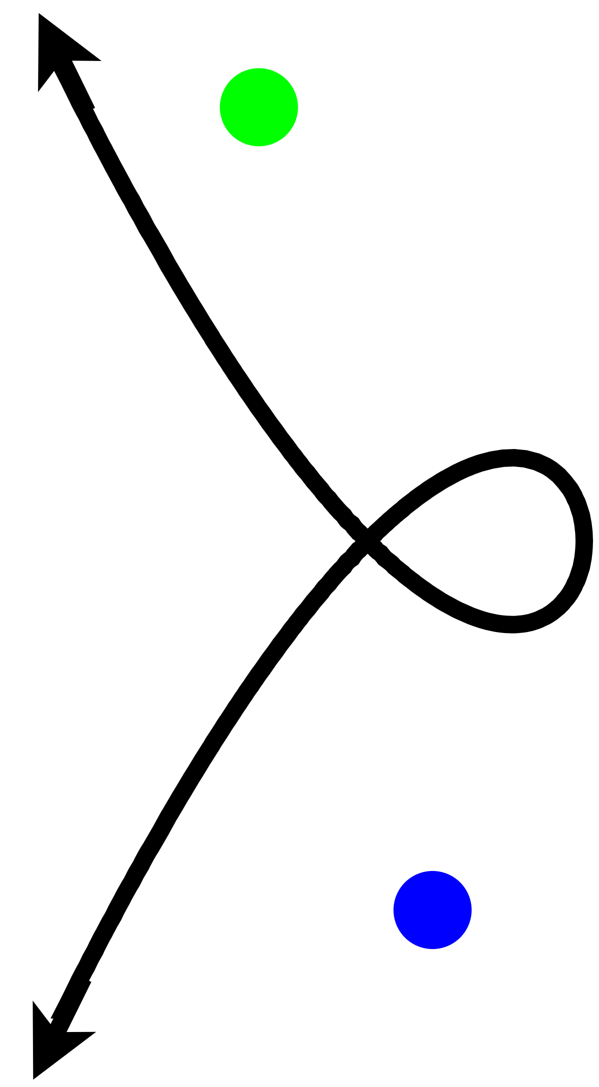

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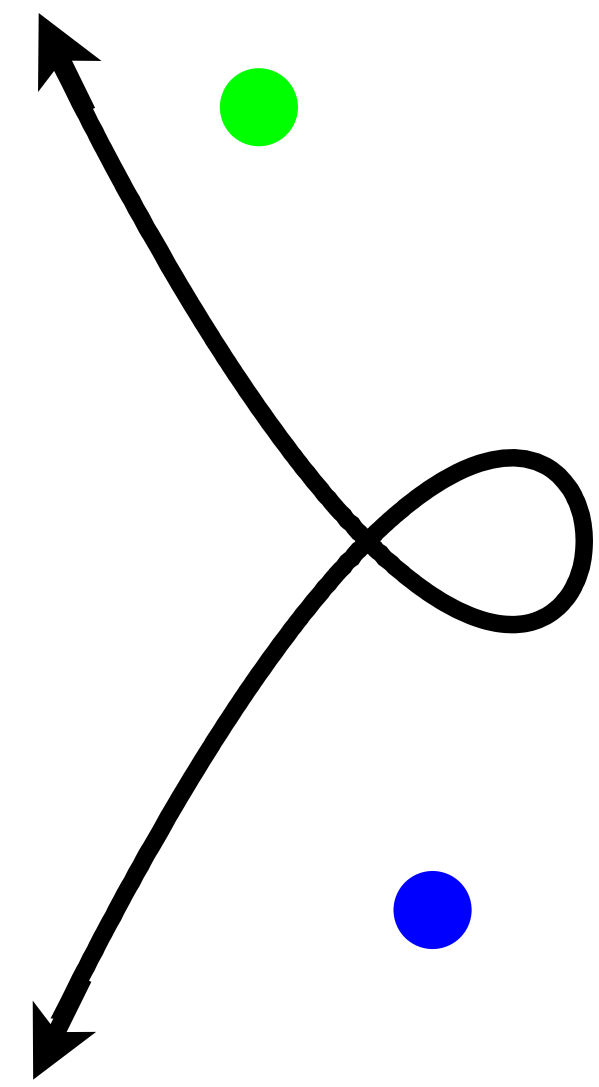


True

Problem: Connectivity

Input

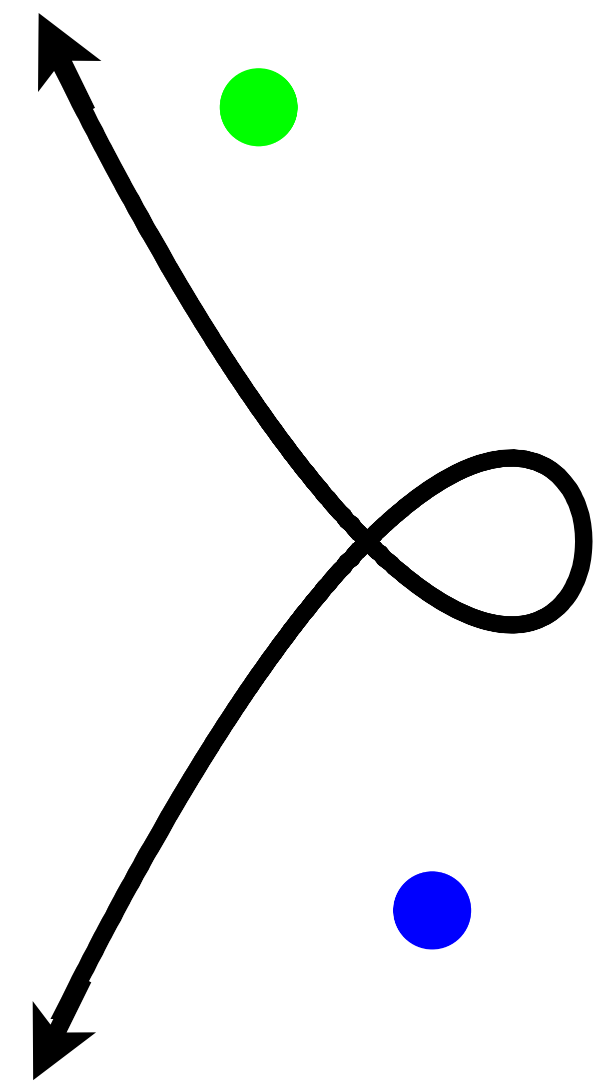
$f \in \mathbb{Z}[\mathsf{x}_1, \ldots, \mathsf{x}_n]$, squarefree,

finitely many singular points,

$n \geq 2$, $\deg(f) \geq 1$

●, ● $\in \mathbb{Q}^n \cap \{f \neq 0\}$

$$f = x_1^3 - x_1^2 + x_2^2$$



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Output

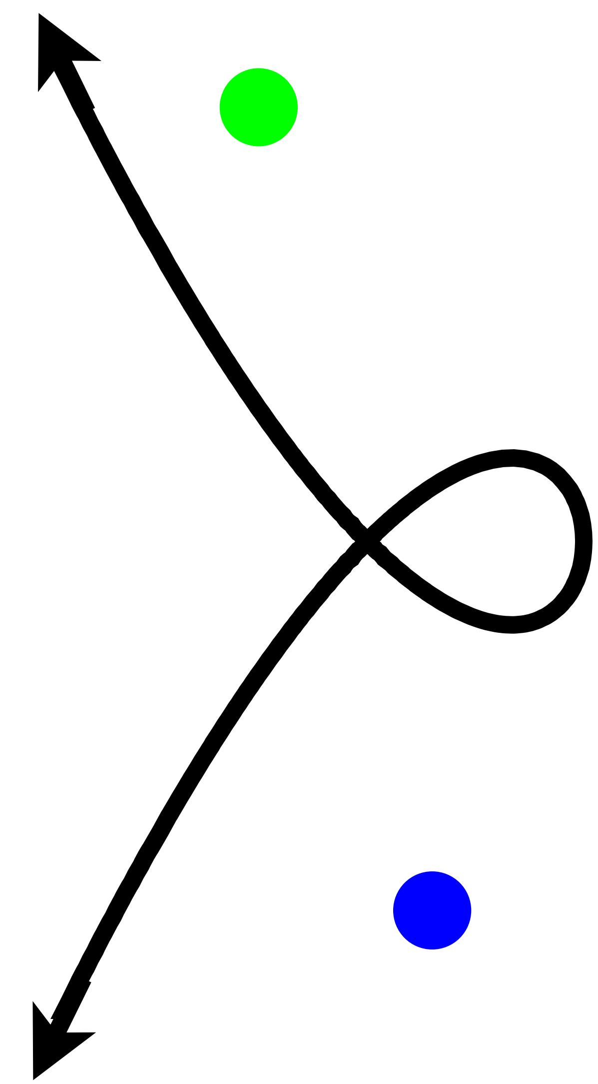
True

if ●, ● are in a same semialgebraically connected component of $\{f \neq 0\}$

False

otherwise

$$f = x_1^3 - x_1^2 + x_2^2$$



True

Motivations and Previous Works

- Fundamental in computational real algebraic geometry.
- Many important applications in science and engineering.
- Previous work:

1975	Collins
1983	Schwartz, Sharir
1984	Arnon, Collins, McCallum
1987	Canny, Roy
1988	Arnon, McCallum
1989	Alonso, Raimondo
1992	Feng, Grigor'ev, Vorobjov
1993	Hong
1994	Heintz, Roy, Solerno
1996	Basu, Pollack, Roy
2008	Hong, Quinn
2011	Safey El Din, Schost
2012	Basu, Roy, Safey El Din, Schost
2013	Basu, Roy

Method: Overview

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Input: $f(x_1, x_2)$, ,

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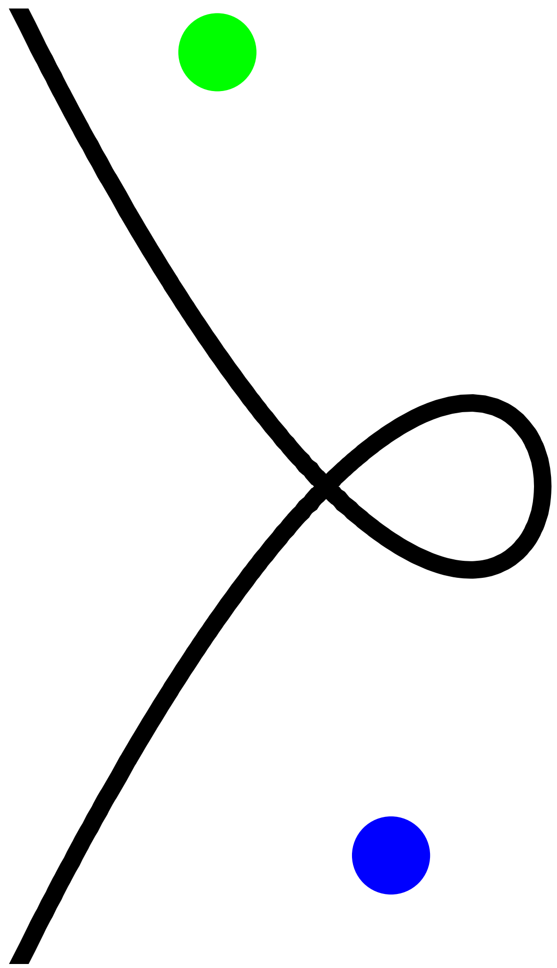

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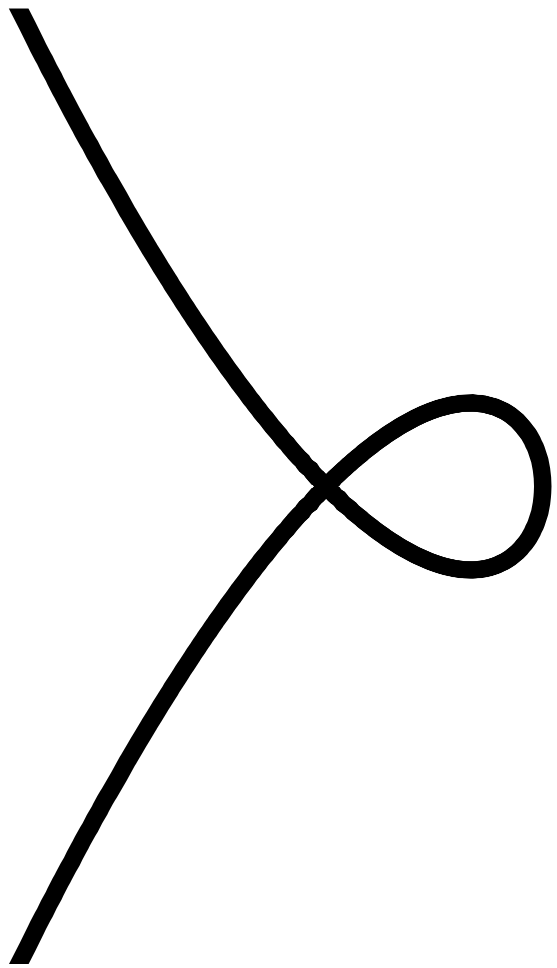

Input: $f(x_1, x_2)$, ●, ●

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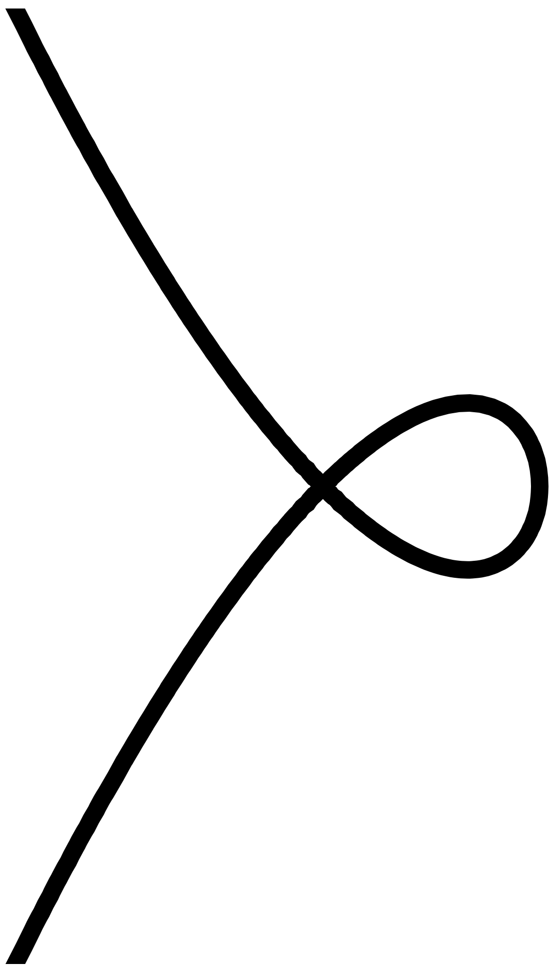

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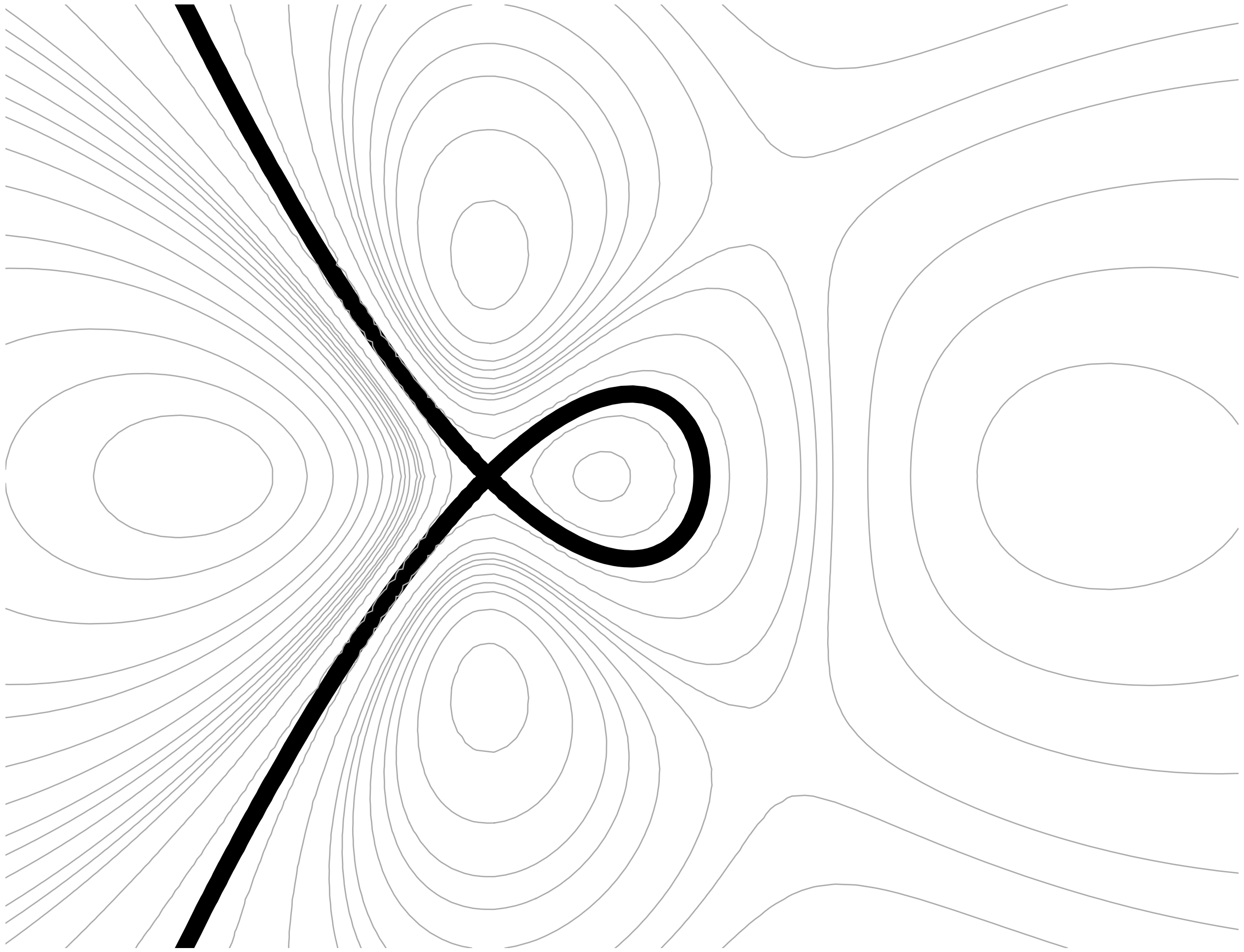

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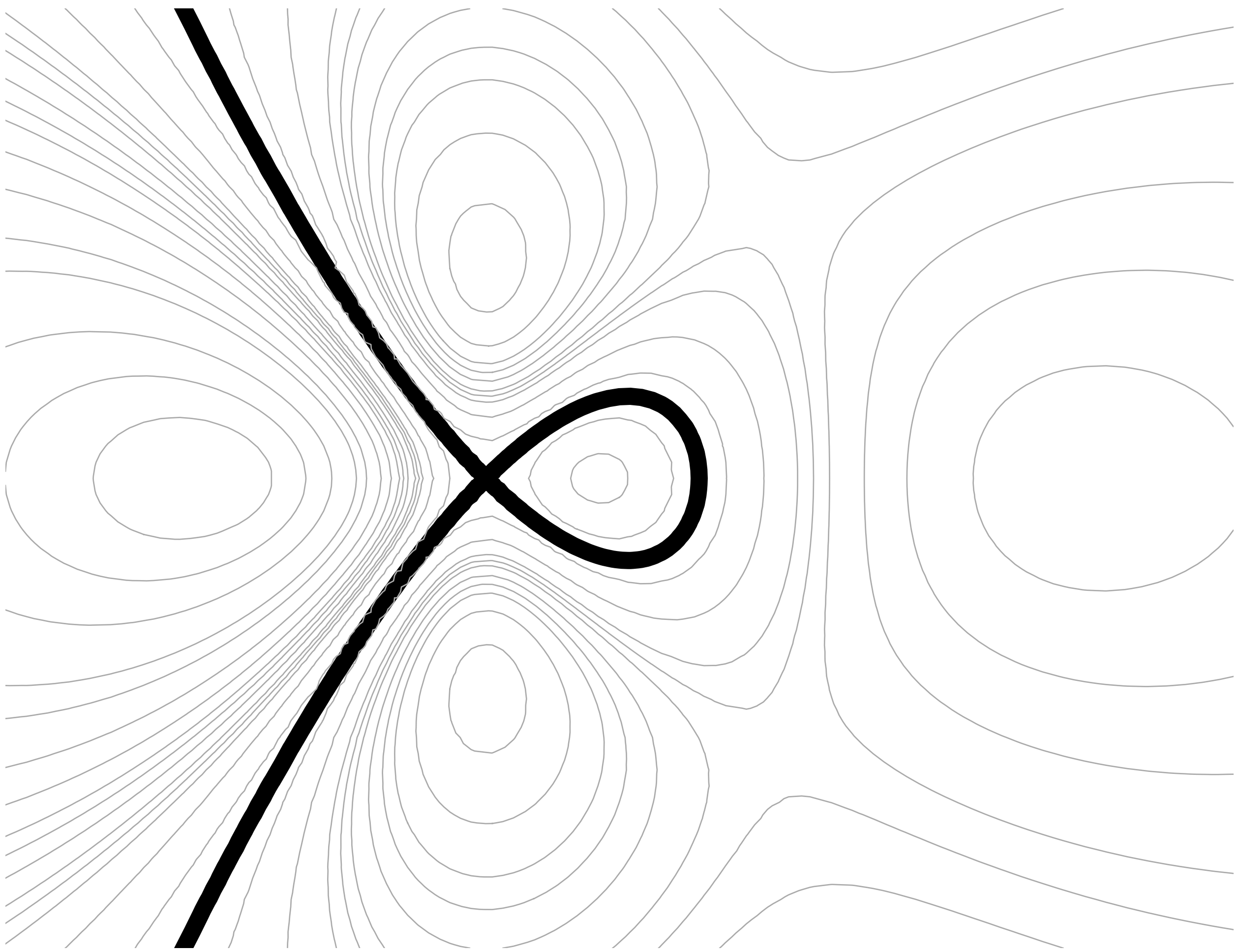

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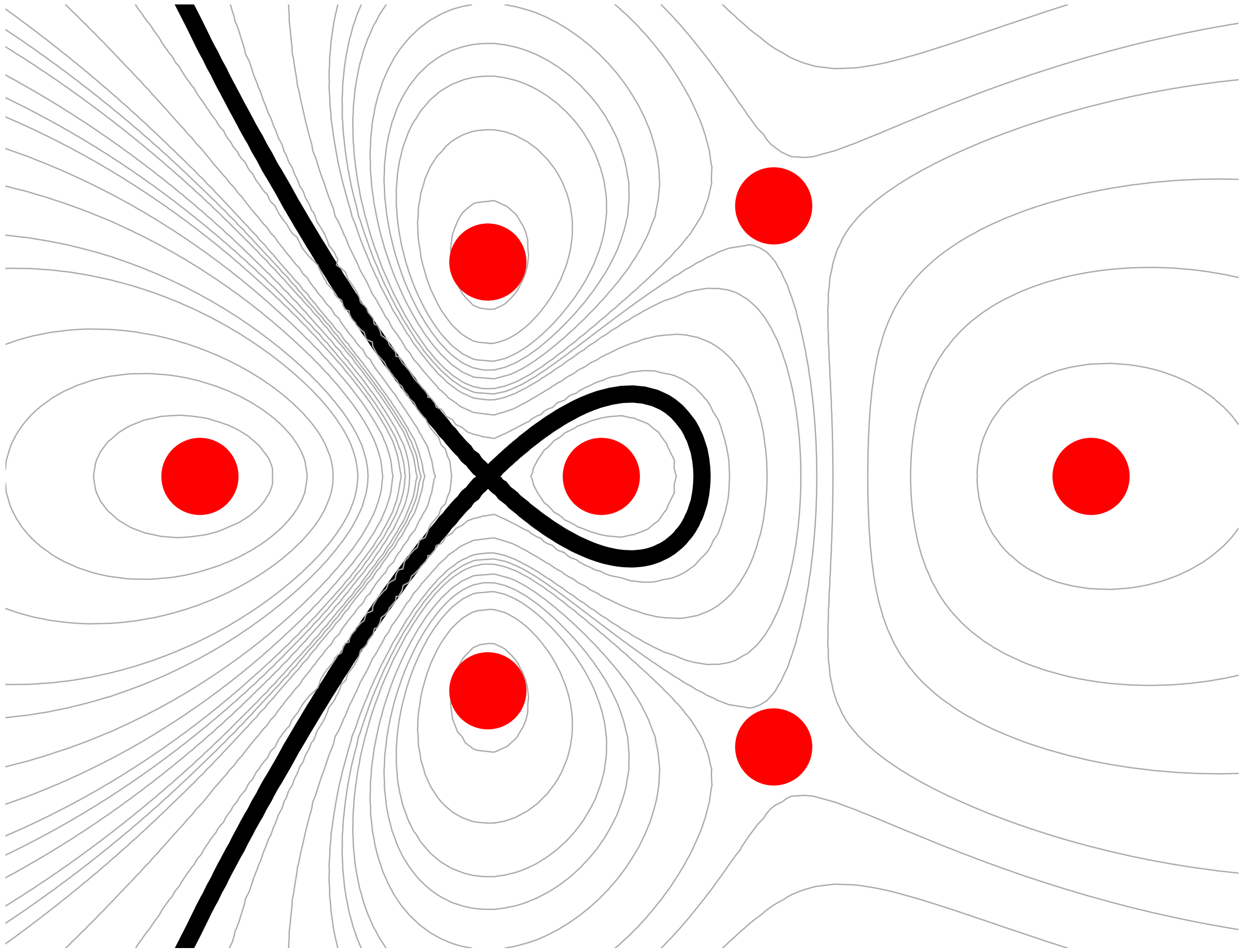

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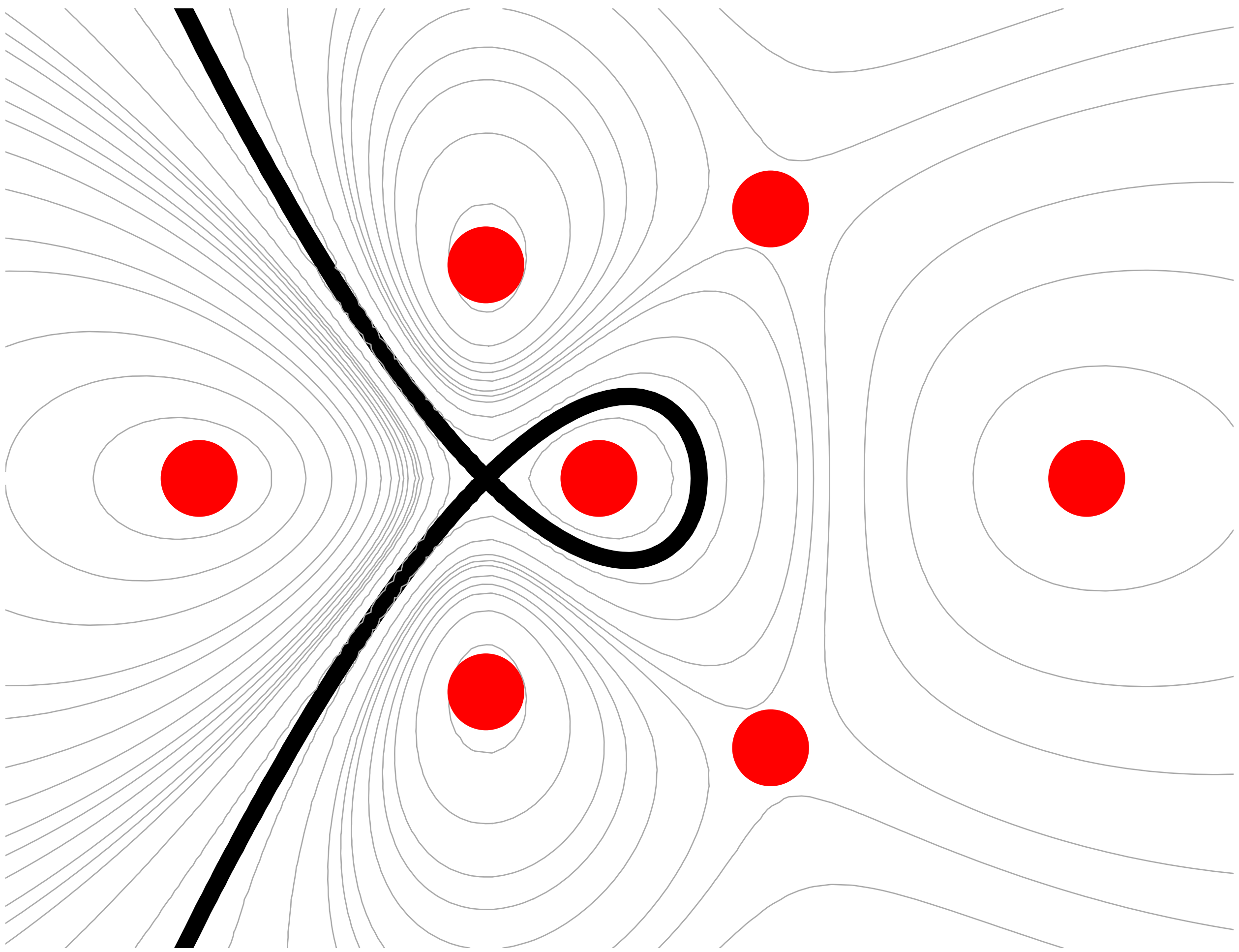

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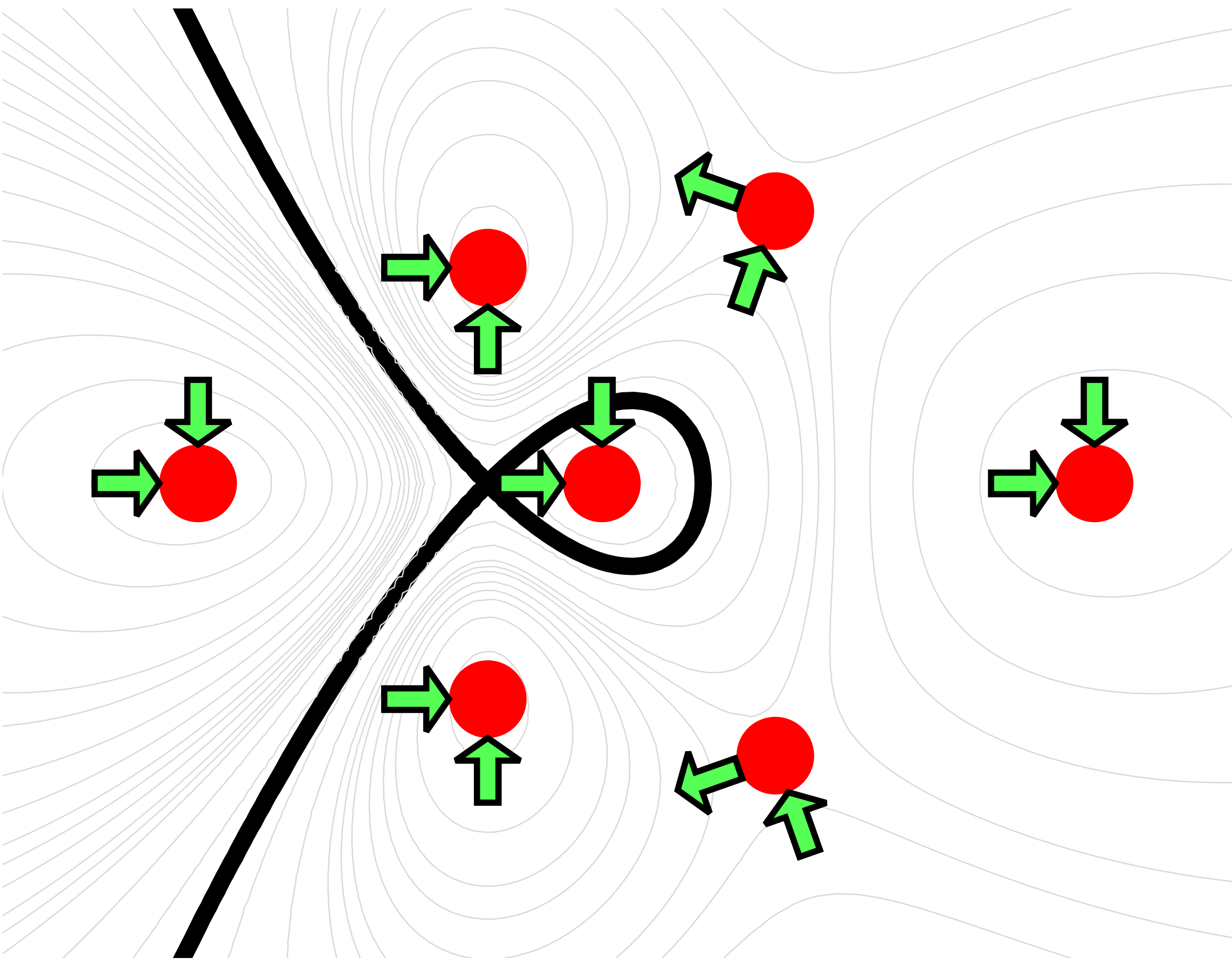

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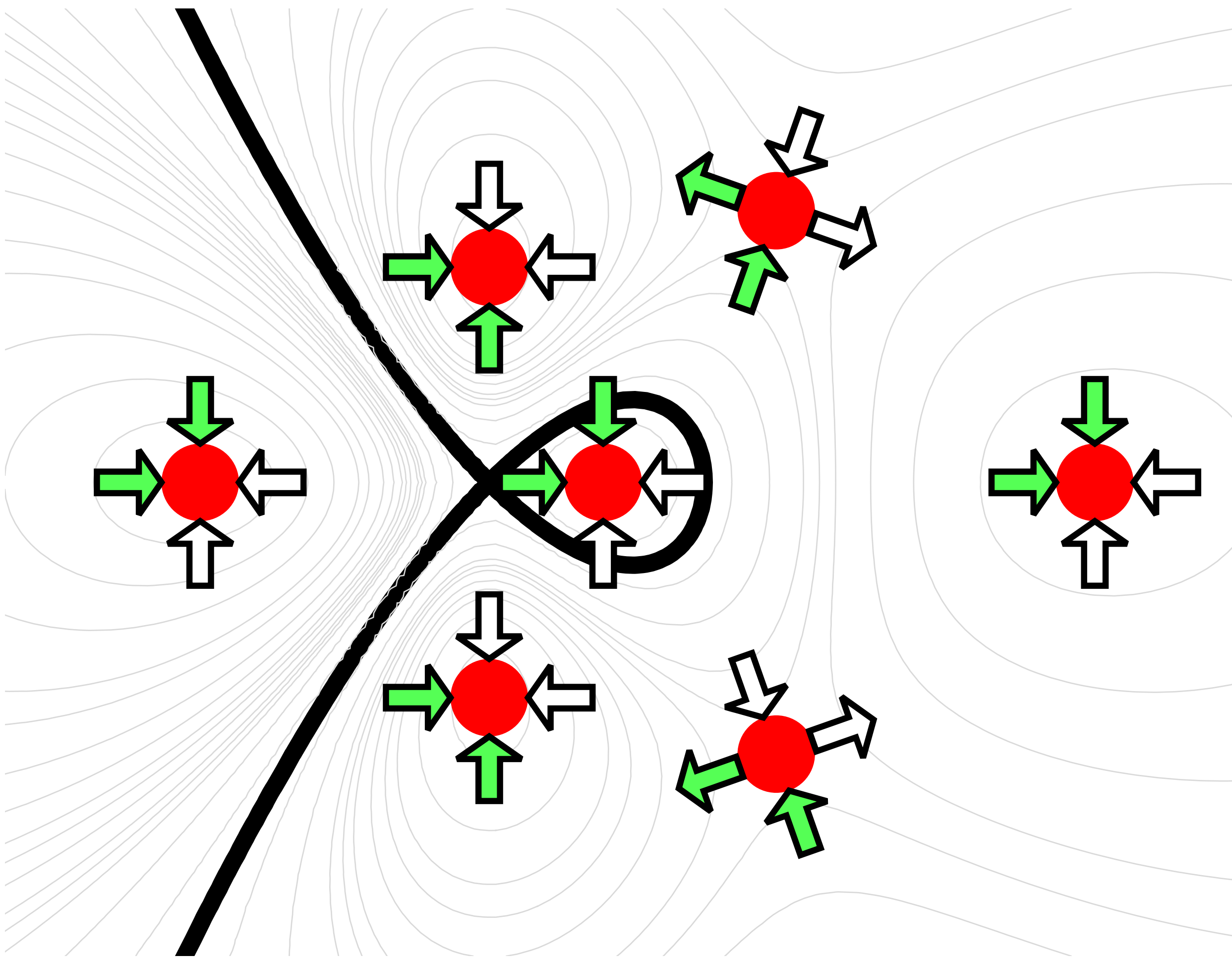

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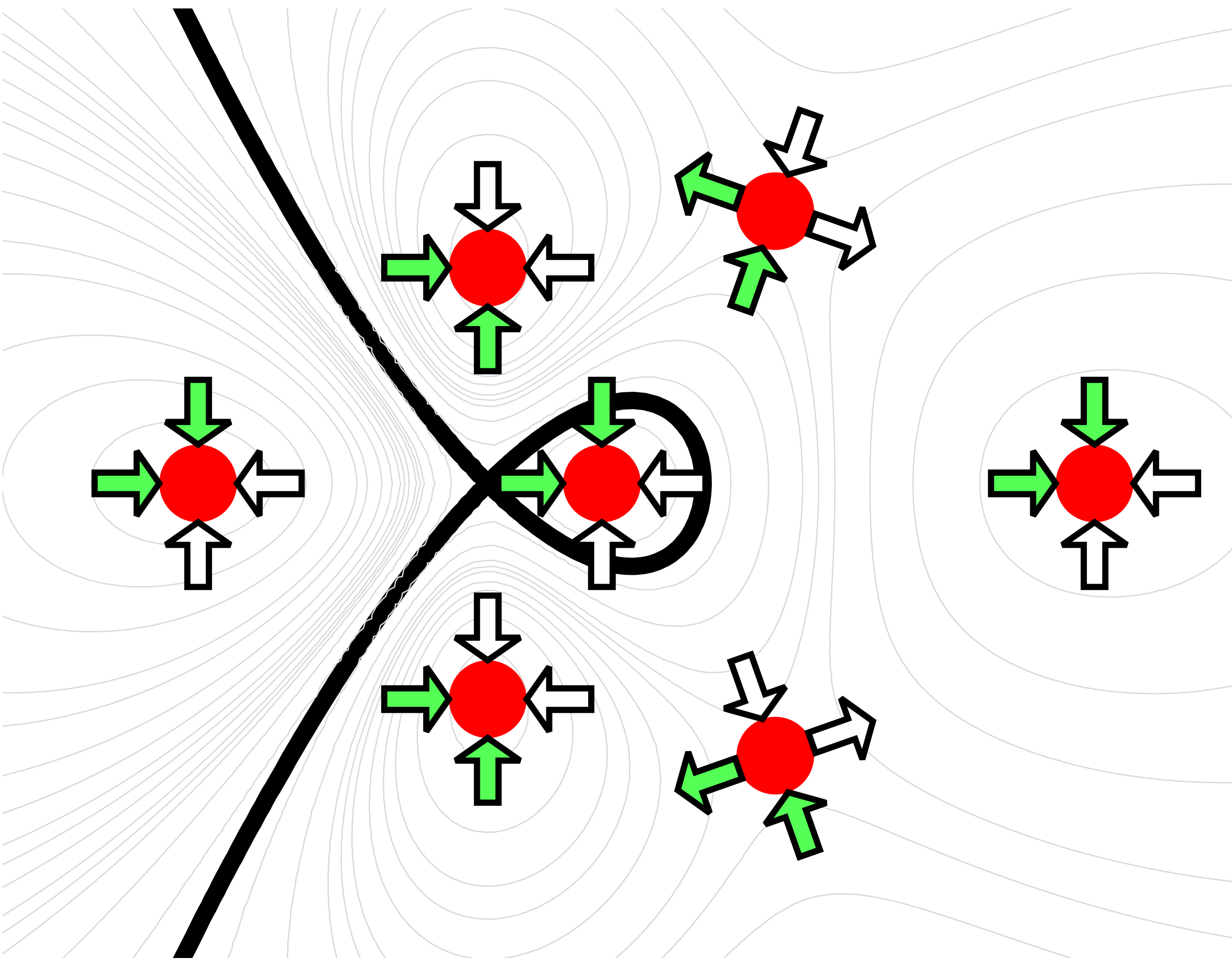

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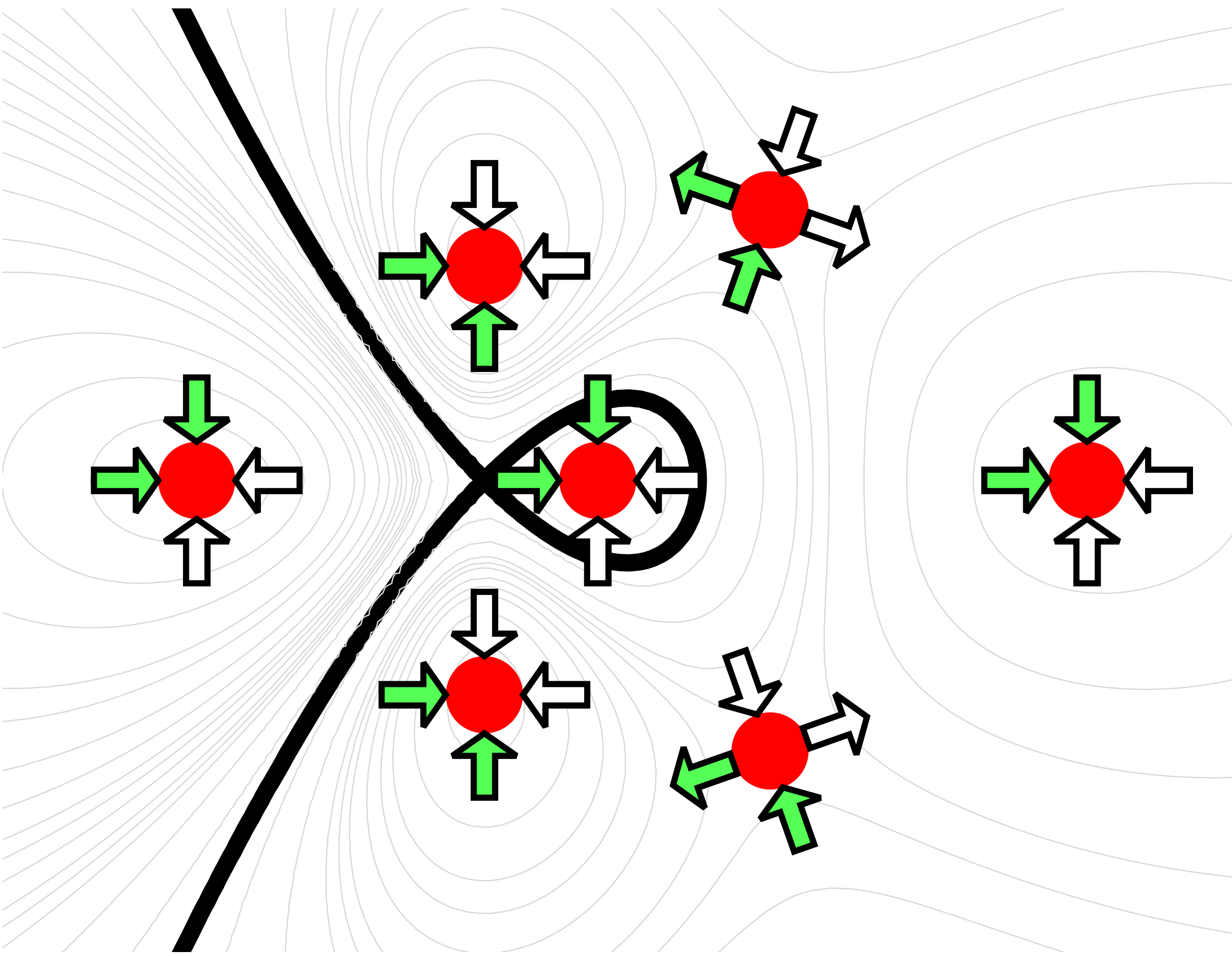

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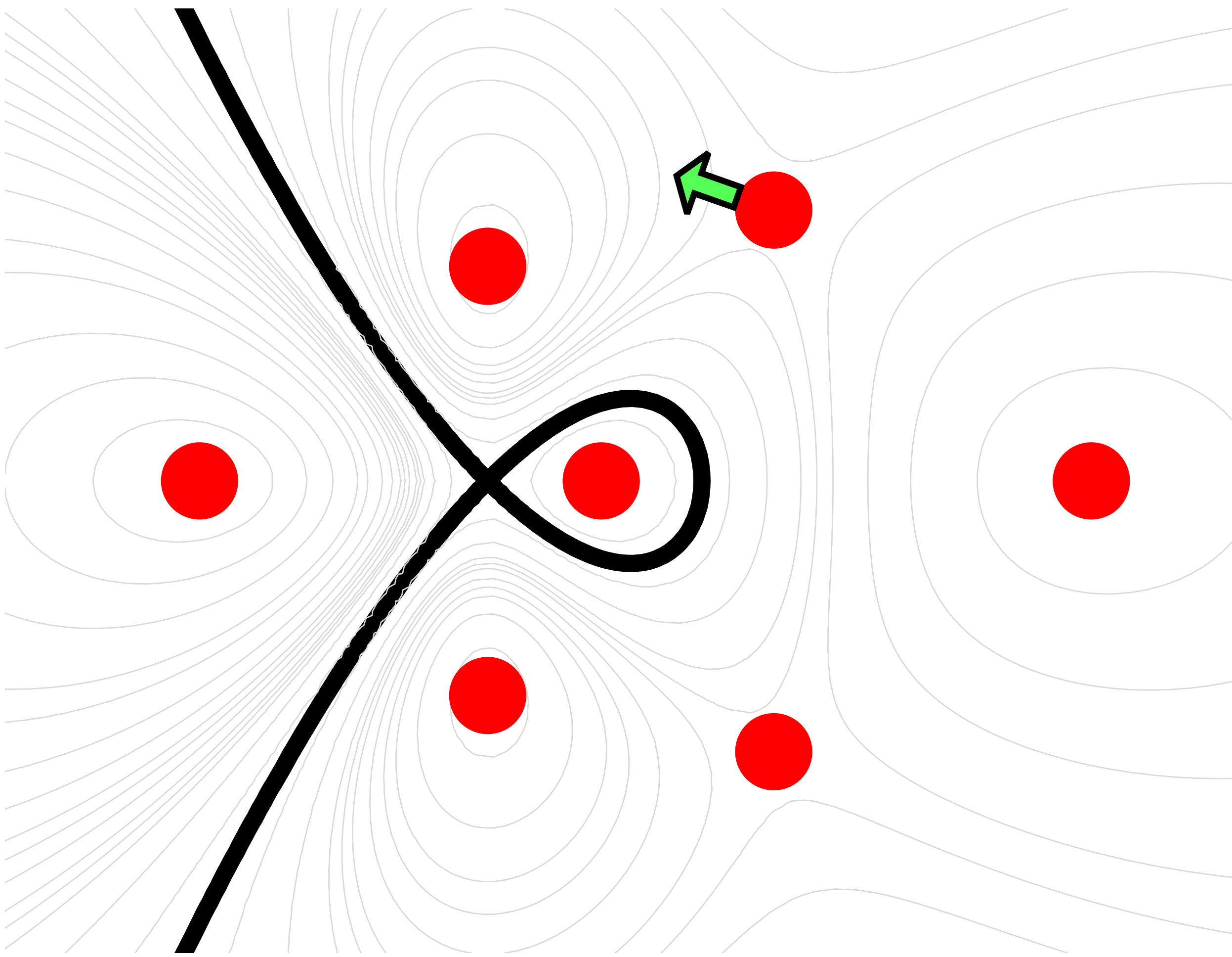

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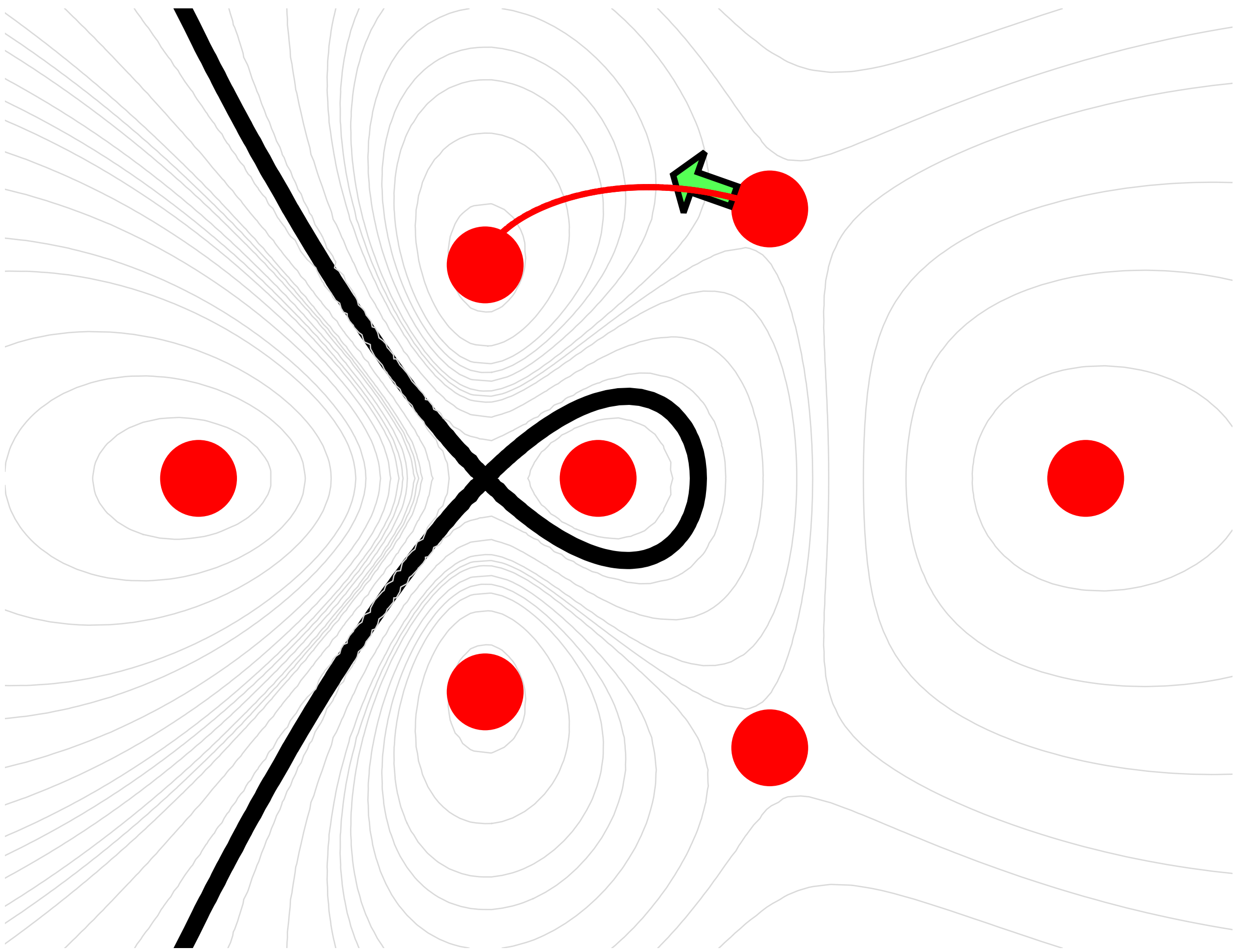

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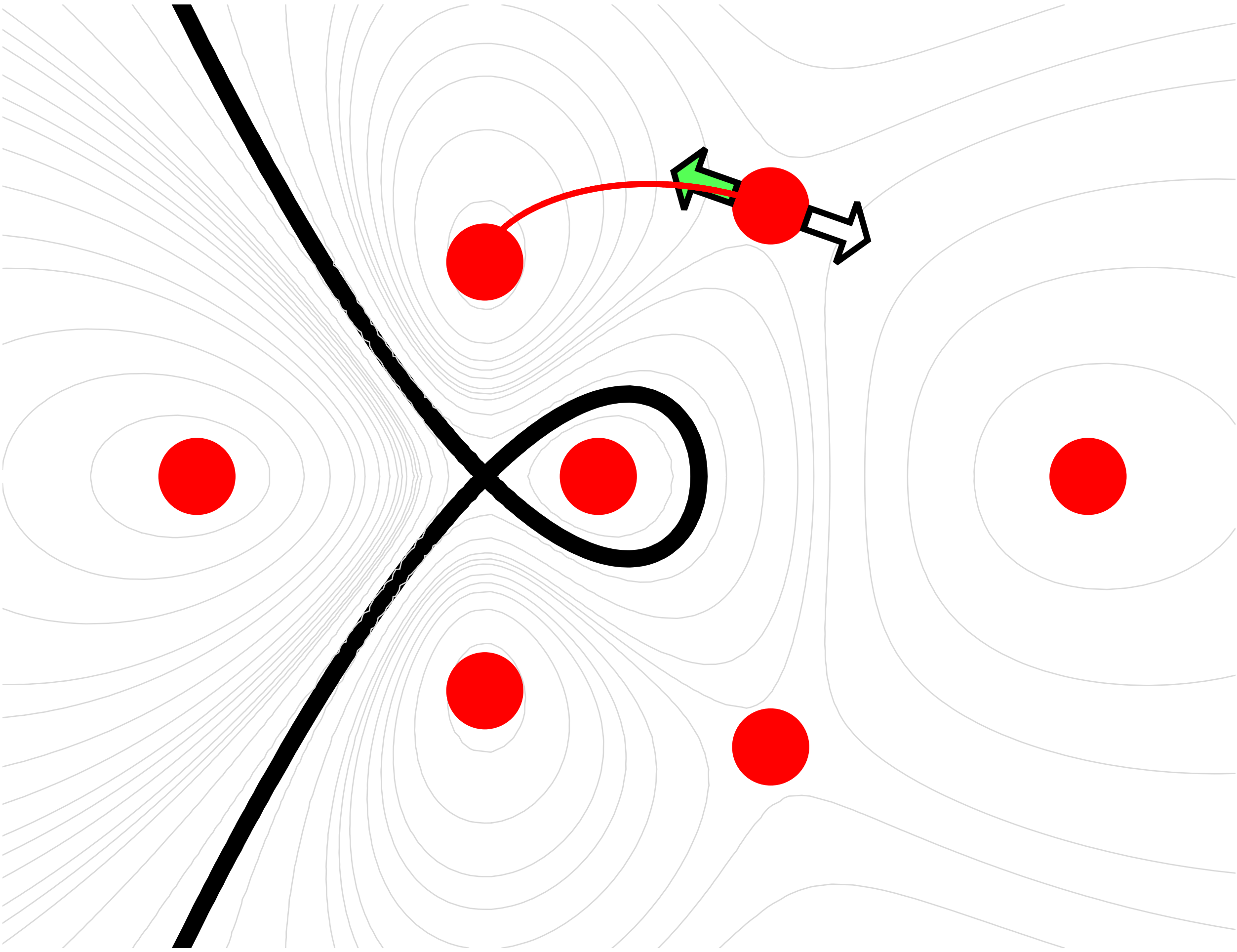

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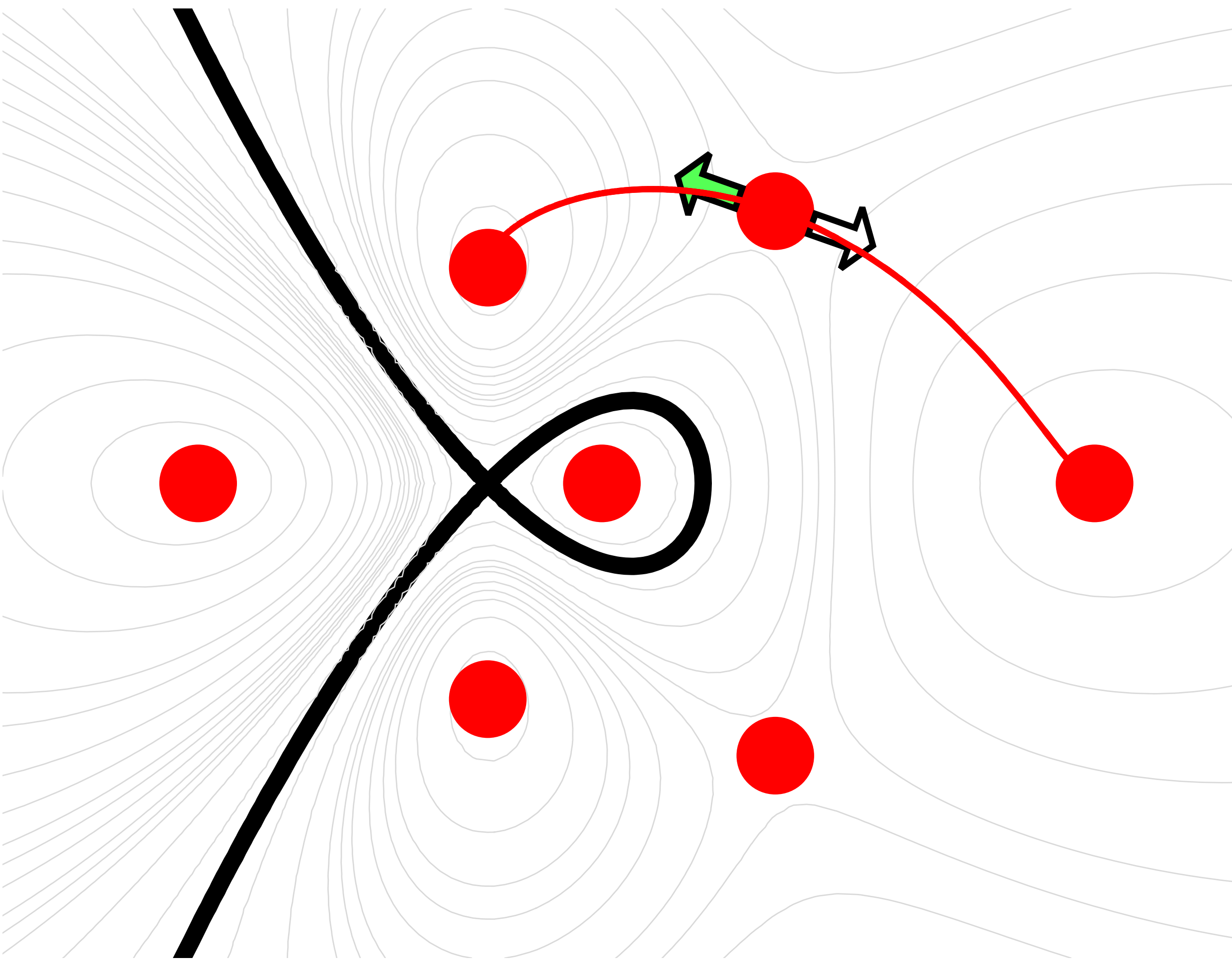

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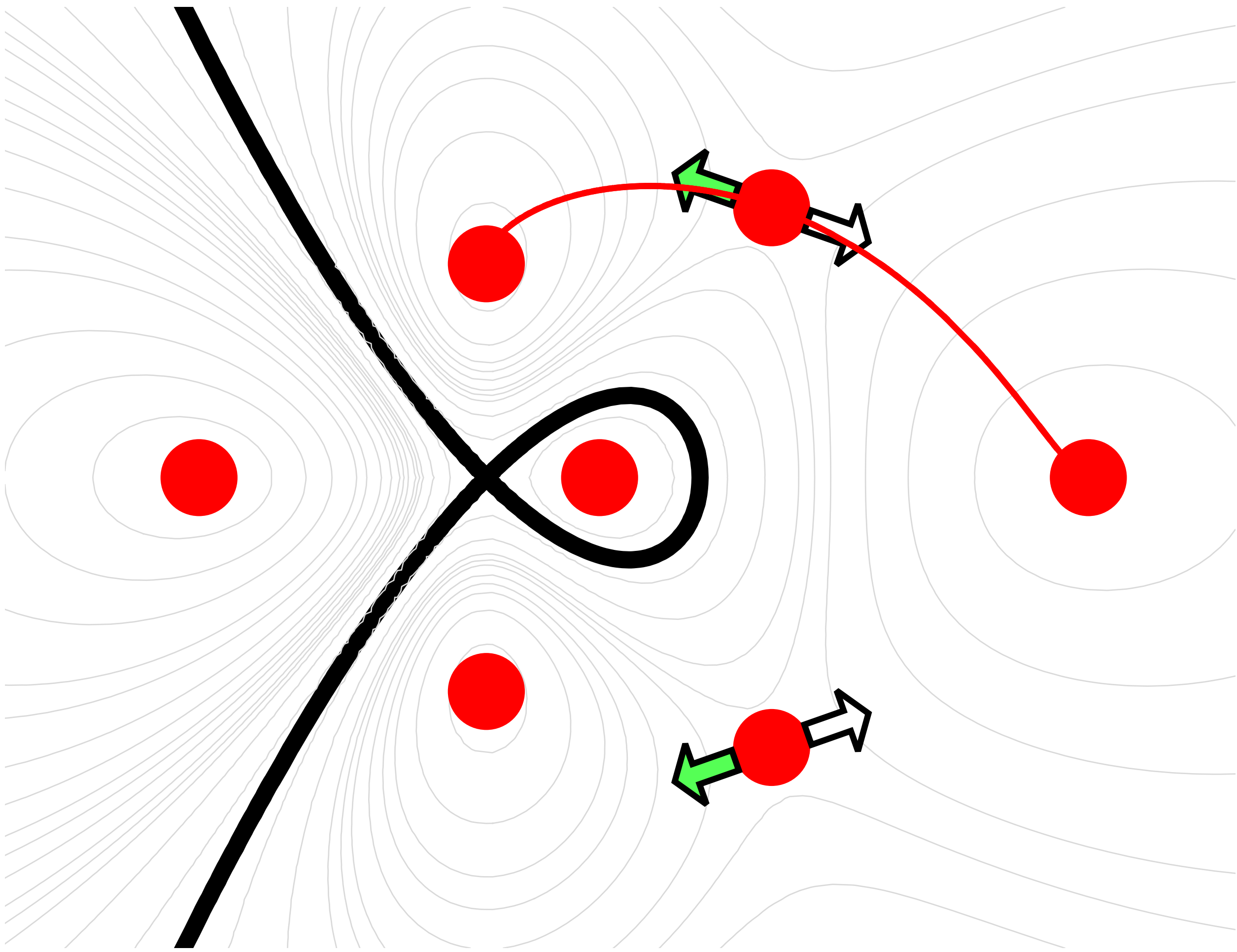

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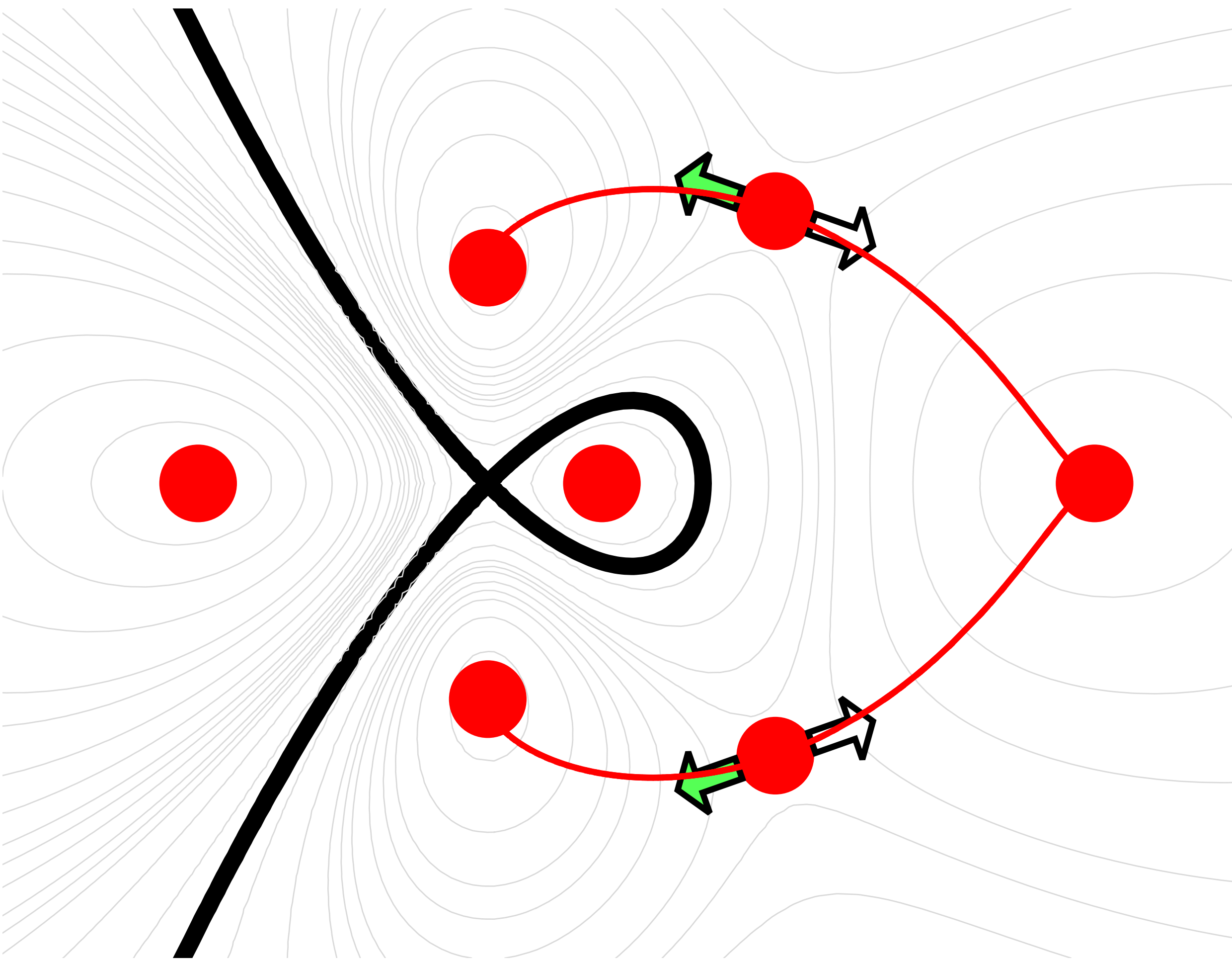

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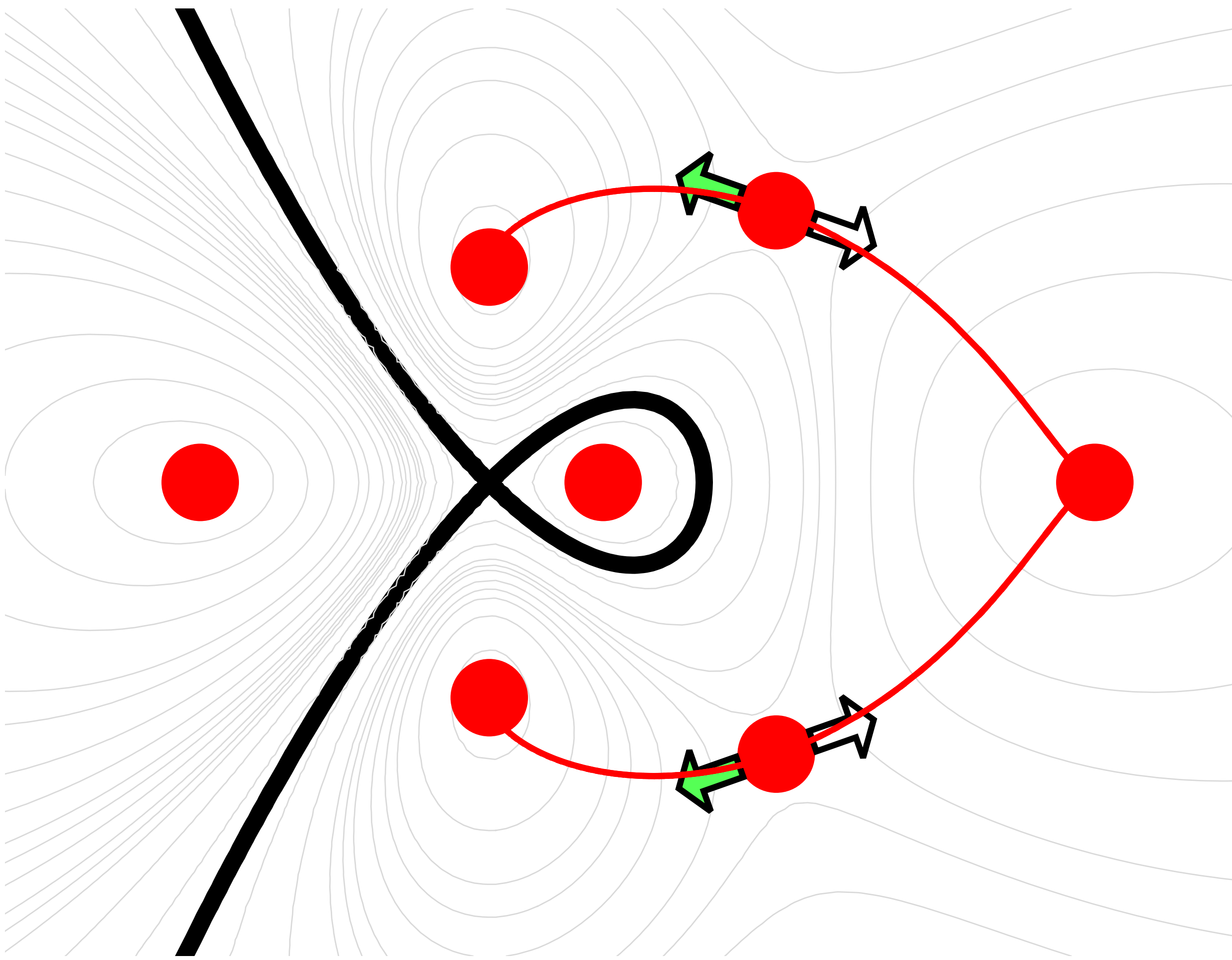

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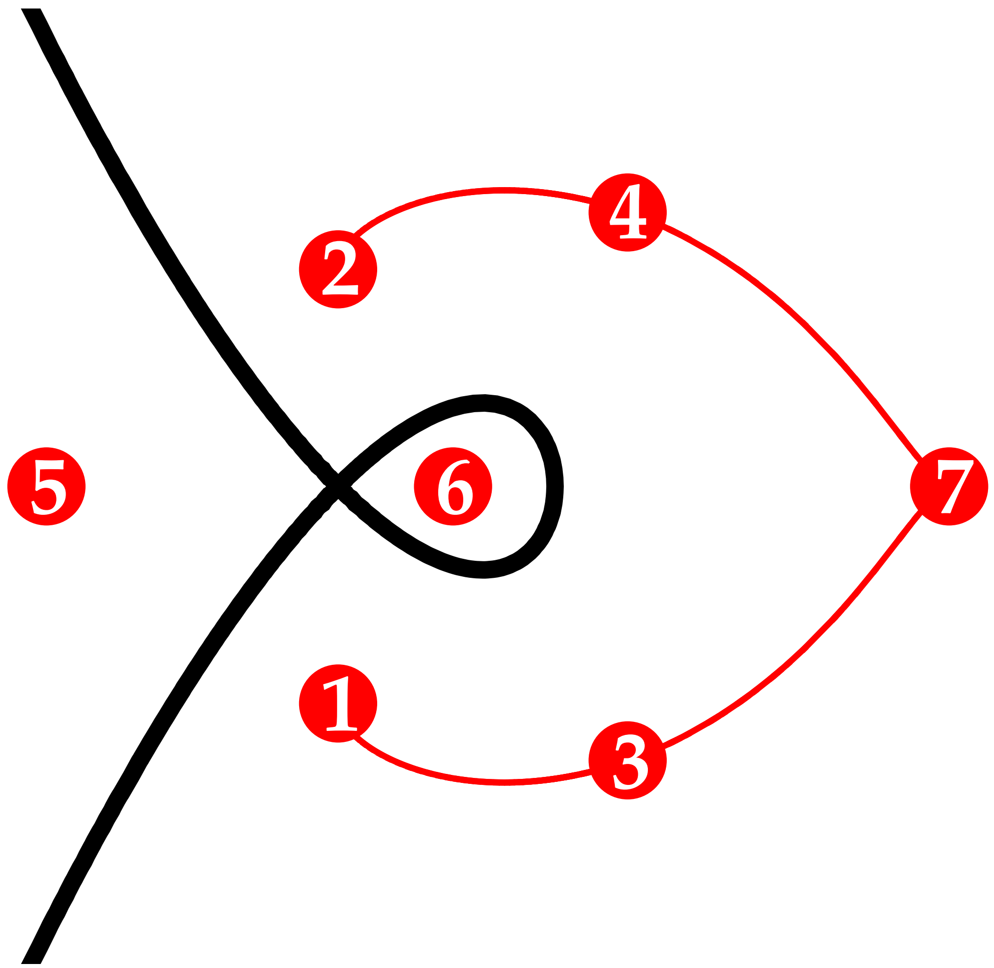
positive eigenvalue

5: Form adjacency matrix

Method: Overview



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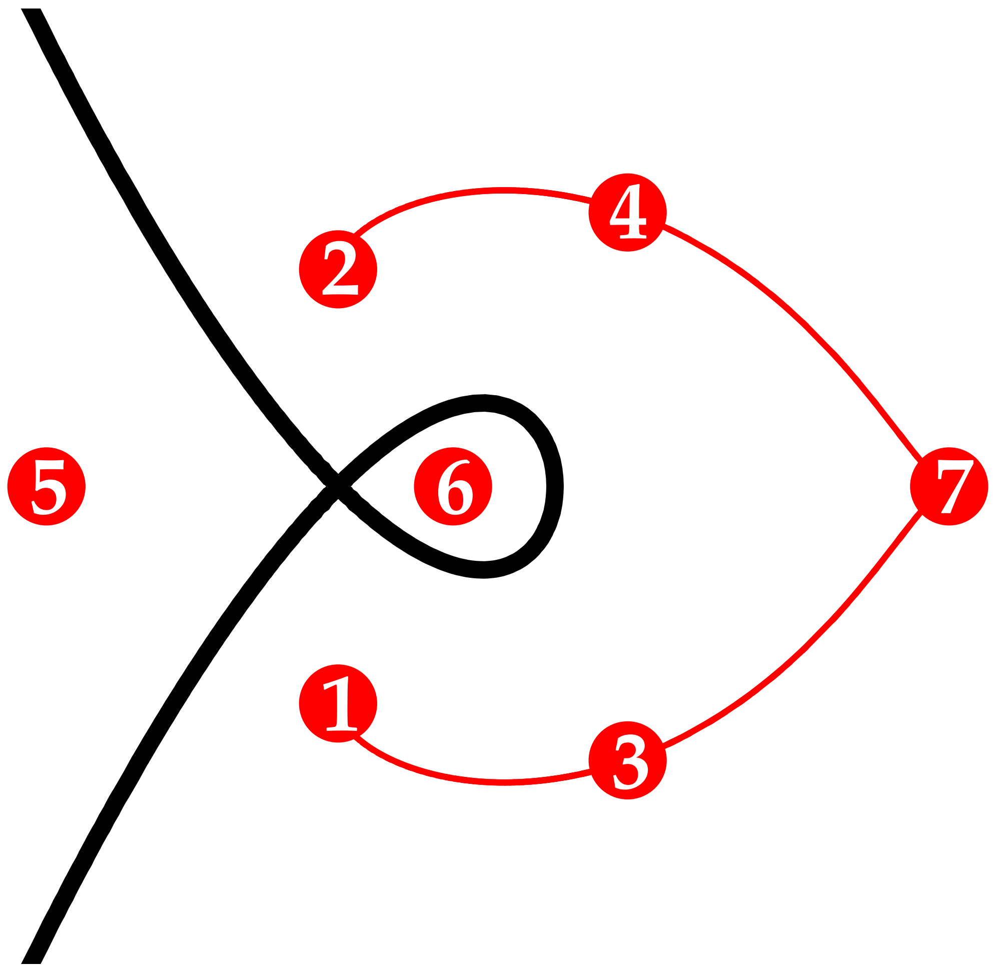

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\begin{bmatrix}
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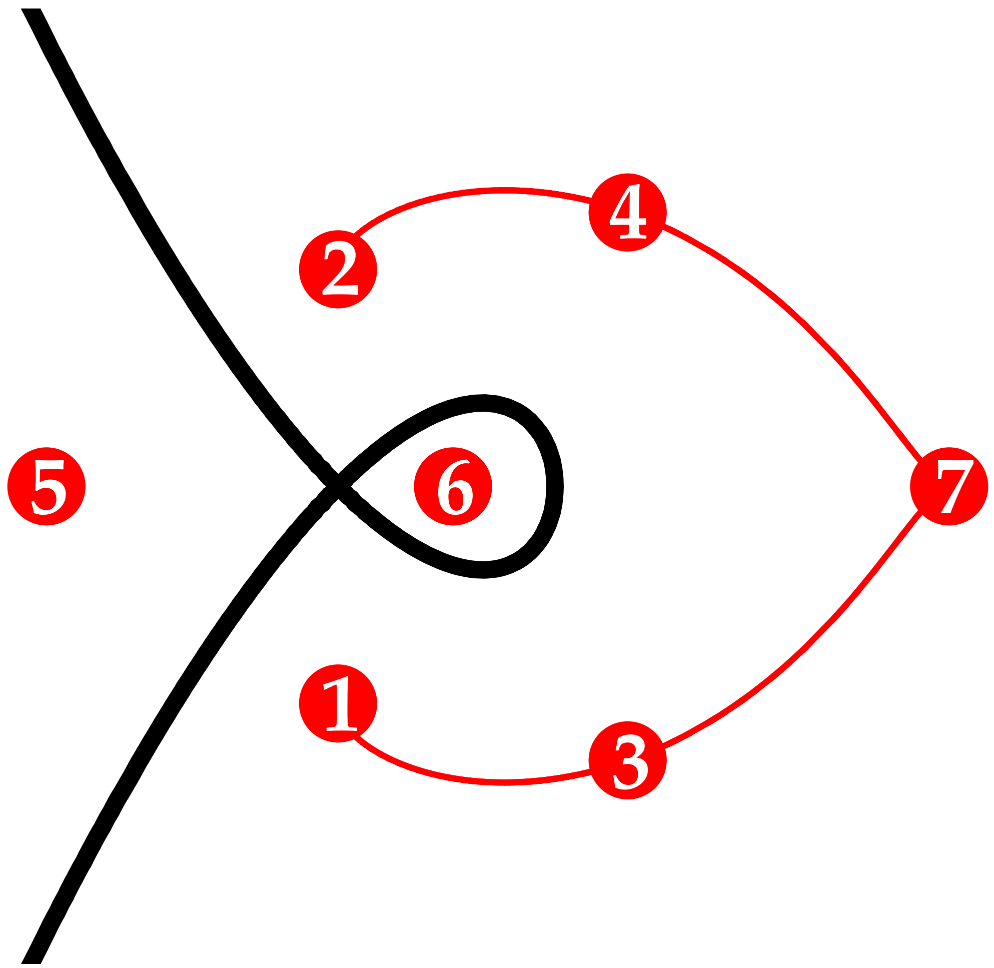

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6: Closure of adjacency matrix

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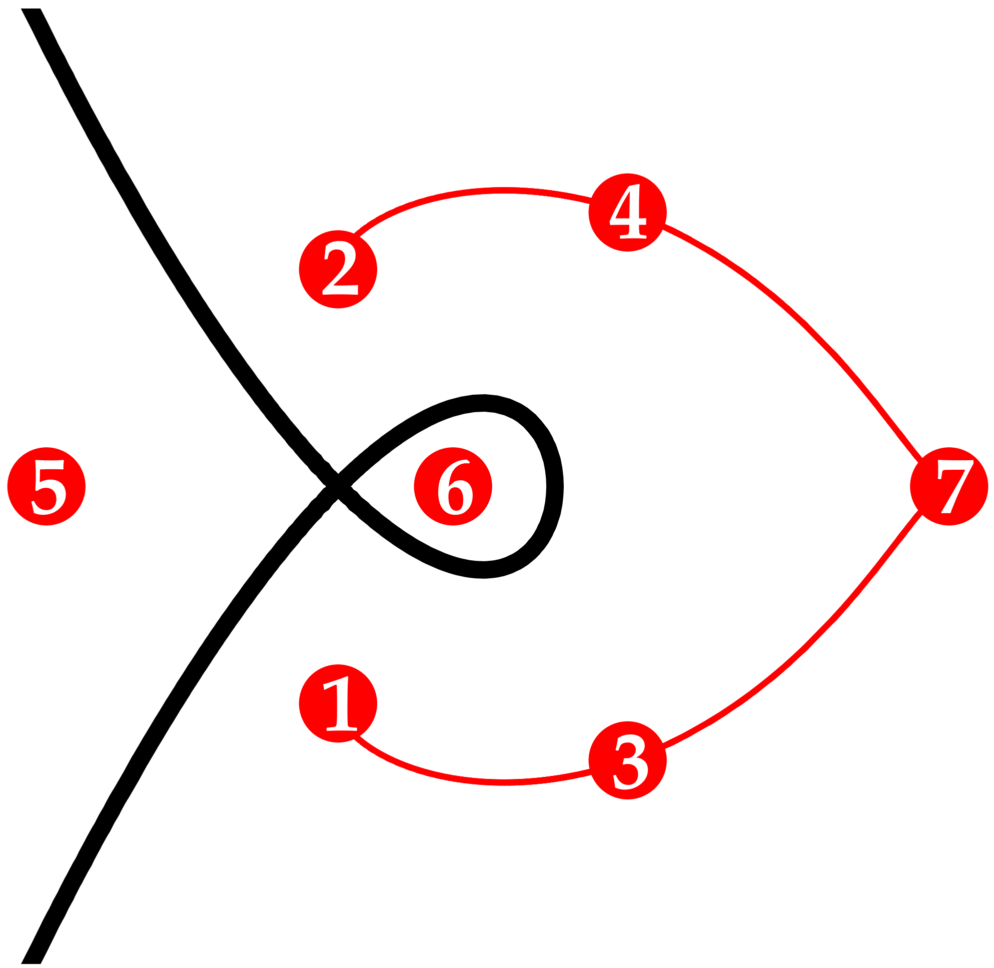

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\begin{array}{c|ccccccc}
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\hline
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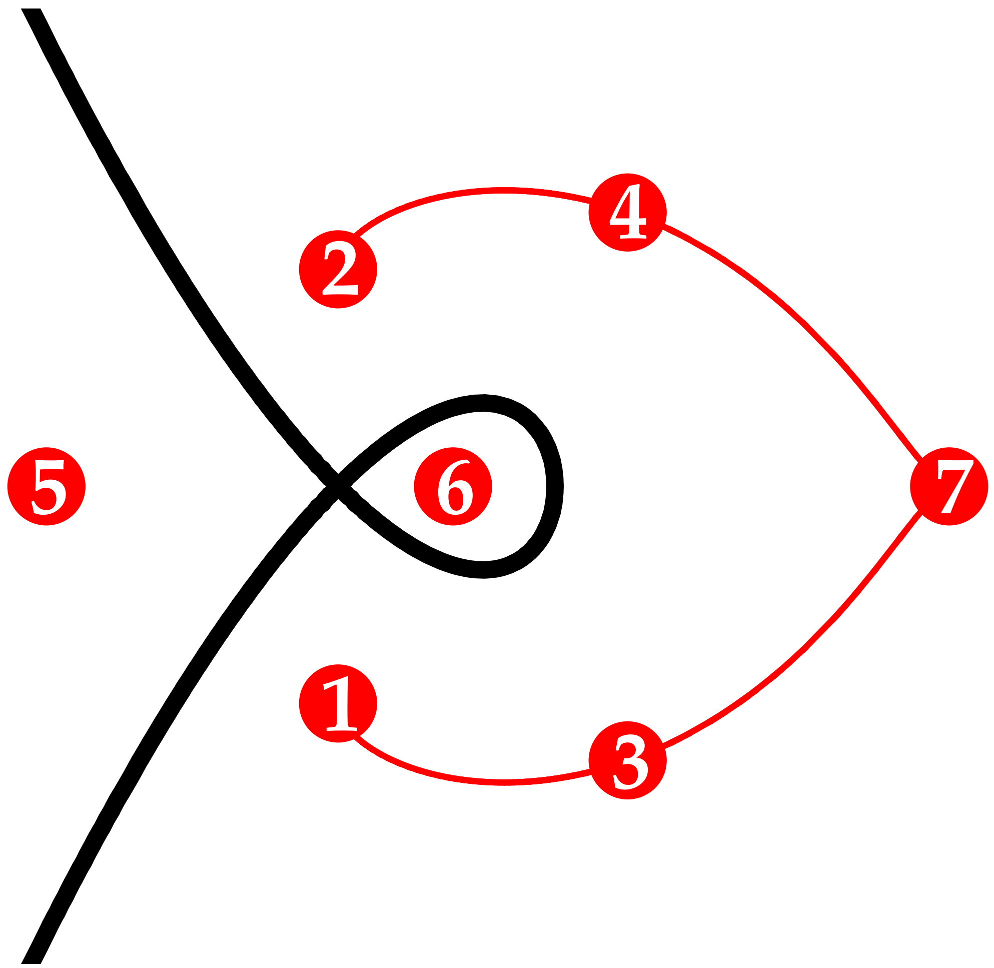

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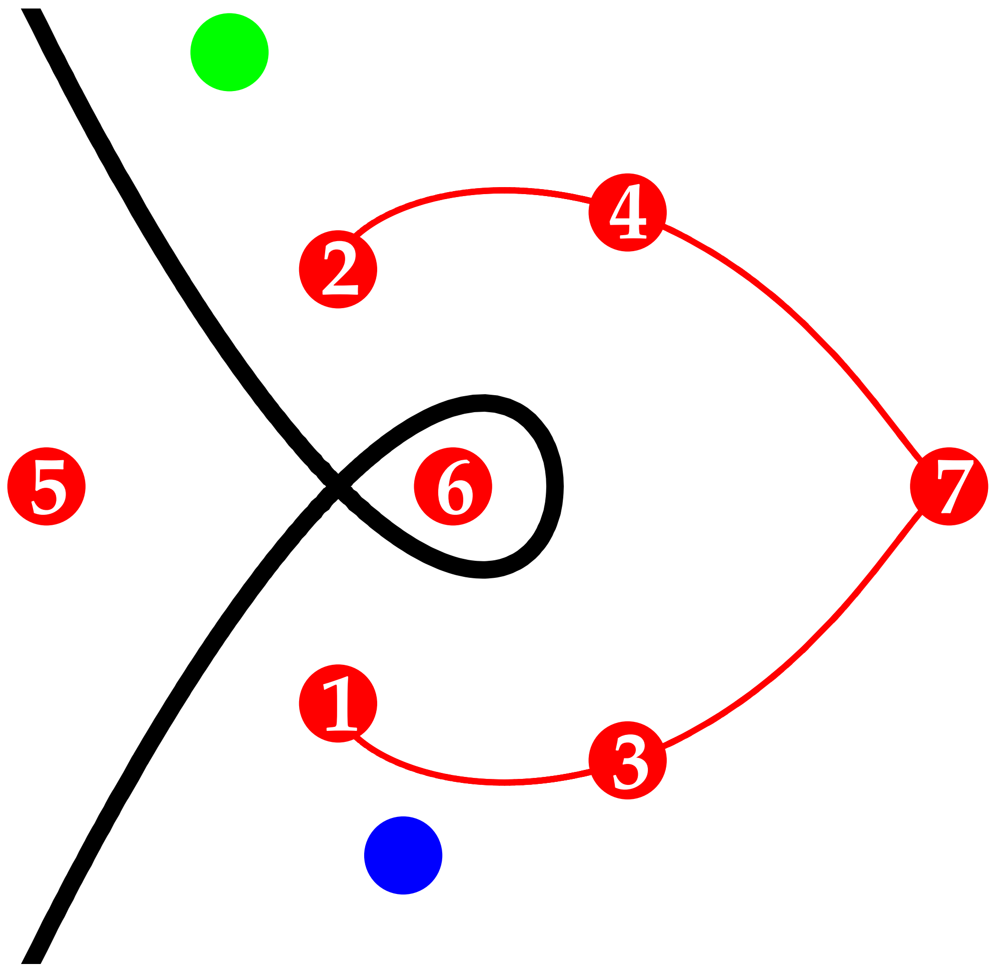

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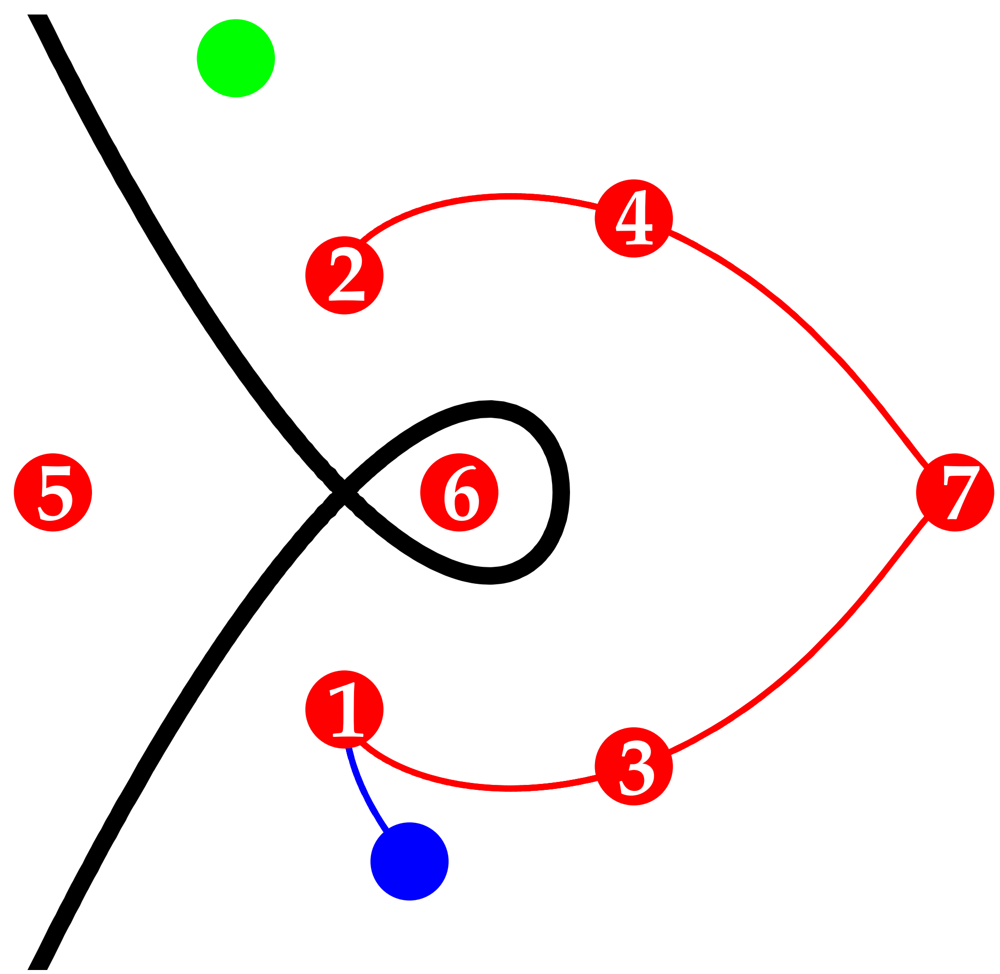

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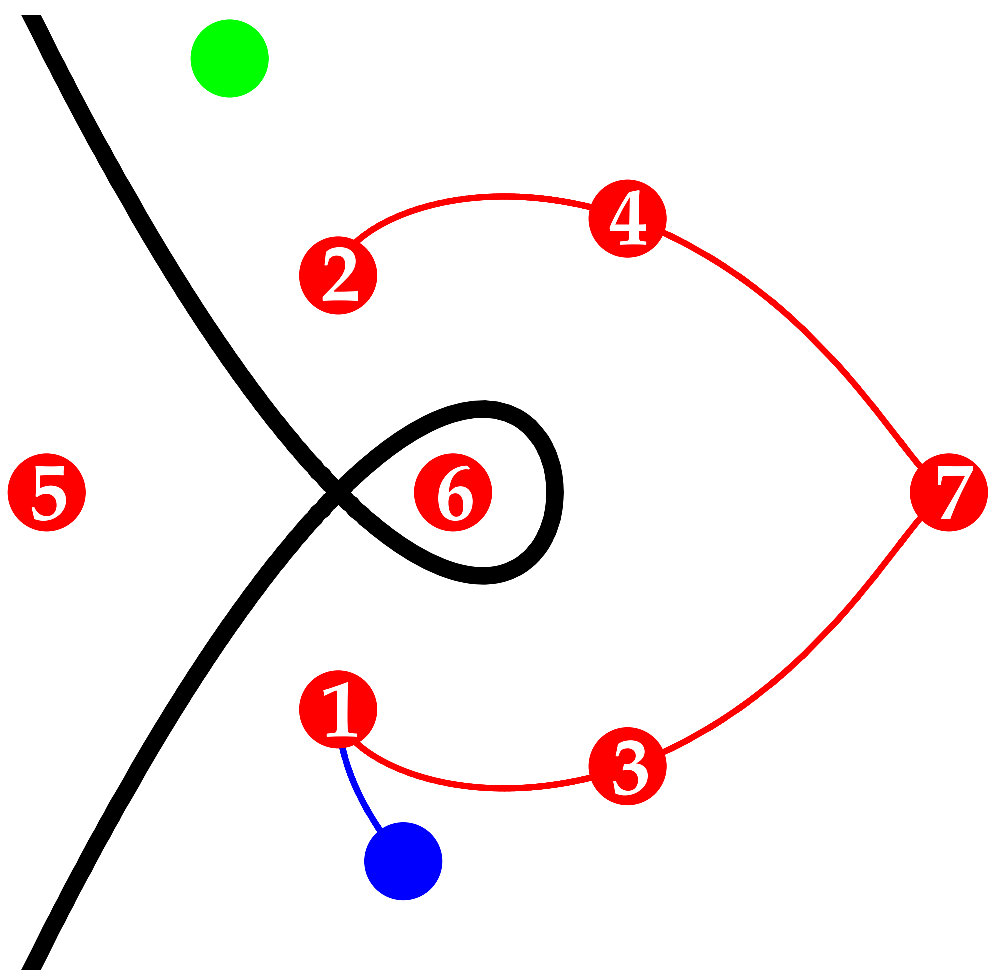

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1 & 1 & 1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

Input: $f(x_1, x_2)$, ,

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2: Solve $\nabla g(x) = 0 \wedge g(x) \neq 0$

3: Find eigenvectors of (Hess g)()

4: Steepest ascent using outgoing eigenvectors ← *positive eigenvalue*

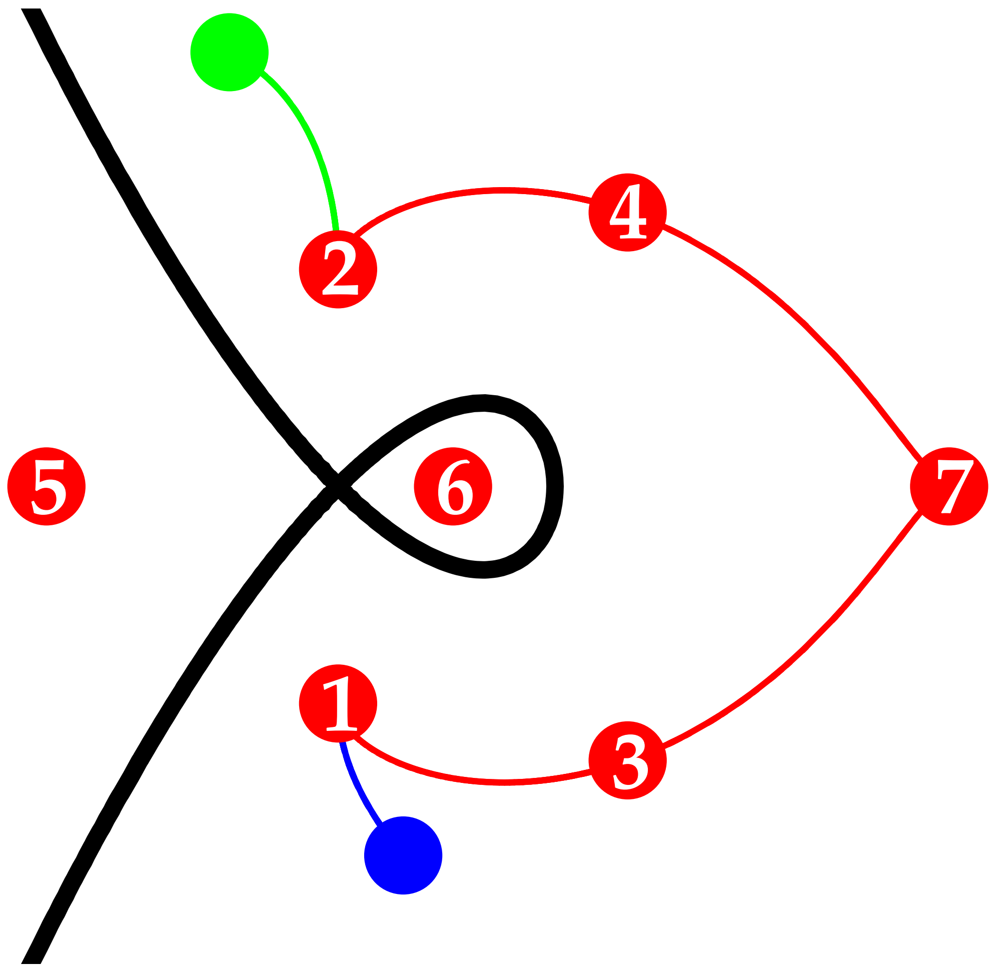
5: Form adjacency matrix

6: Closure of adjacency matrix

7: Steepest ascent from

8: Steepest ascent from

Method: Overview



$$\begin{array}{c c} & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

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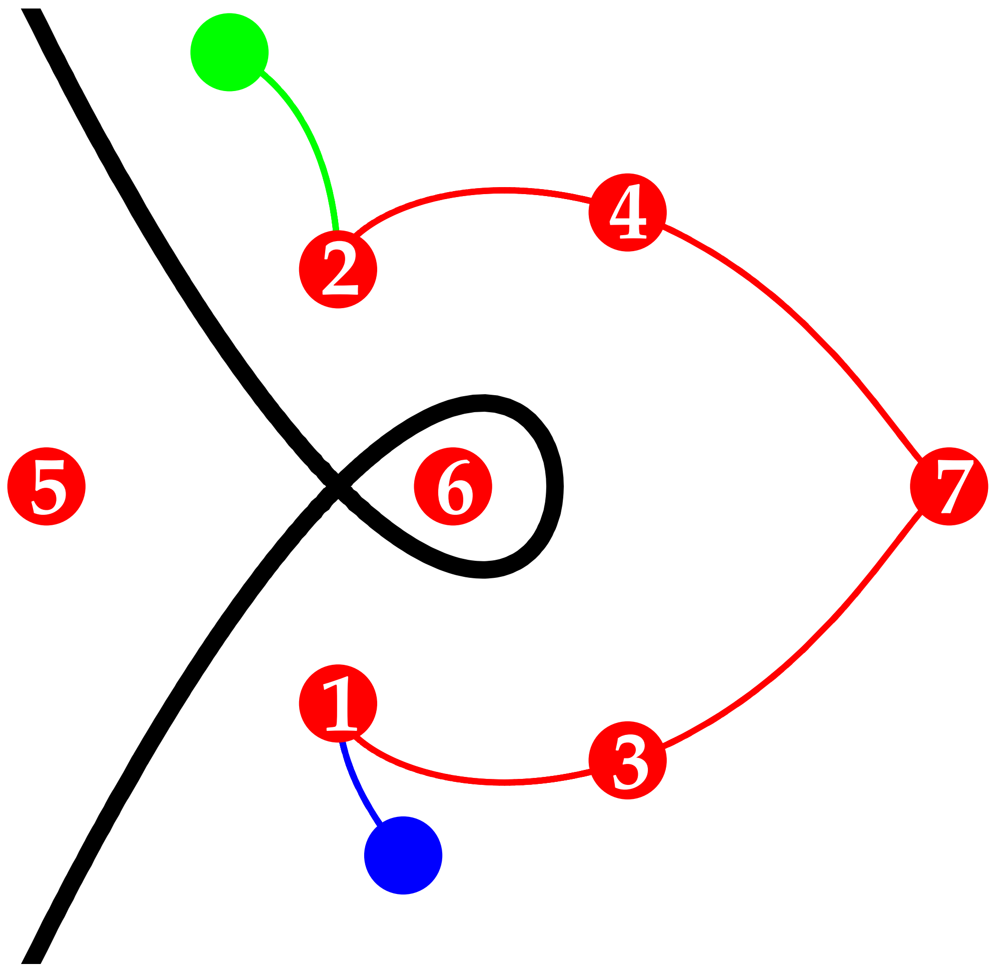

$$\begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 3 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 4 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 5 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 7 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \end{array}$$

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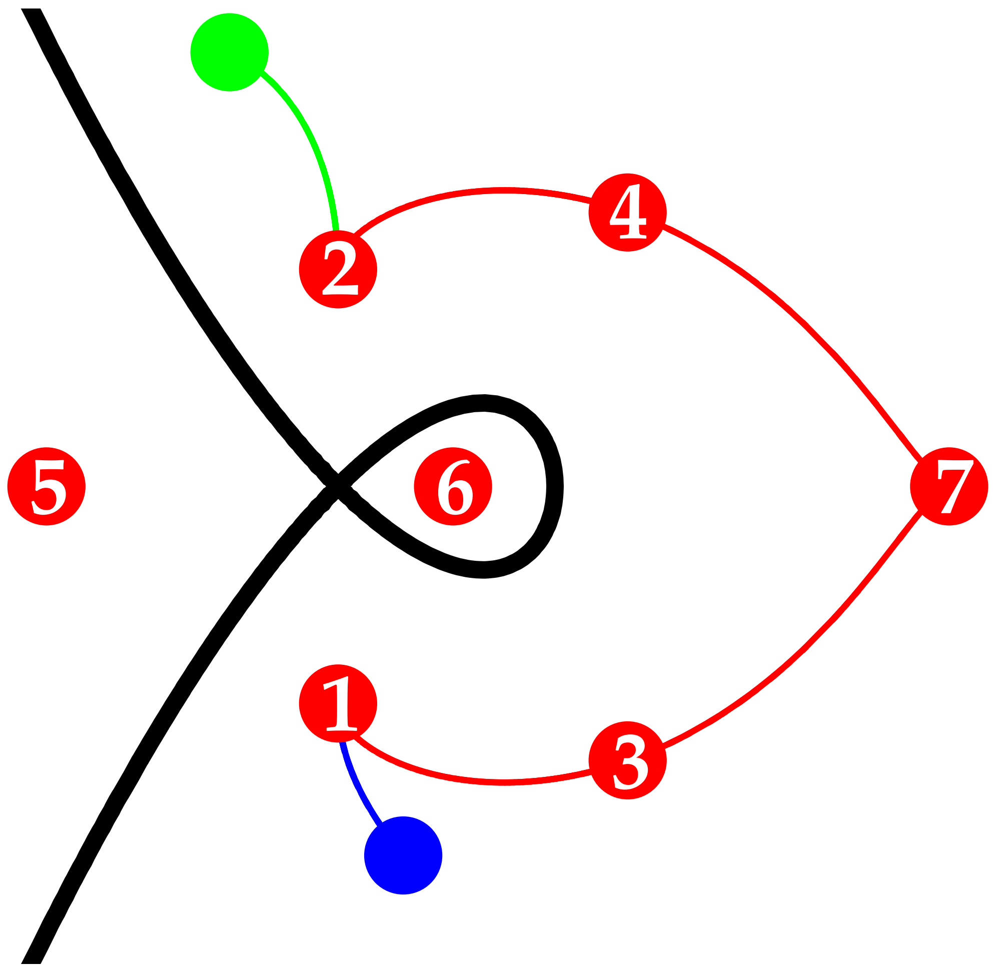
Method: Overview



$$
\begin{array}{c|ccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
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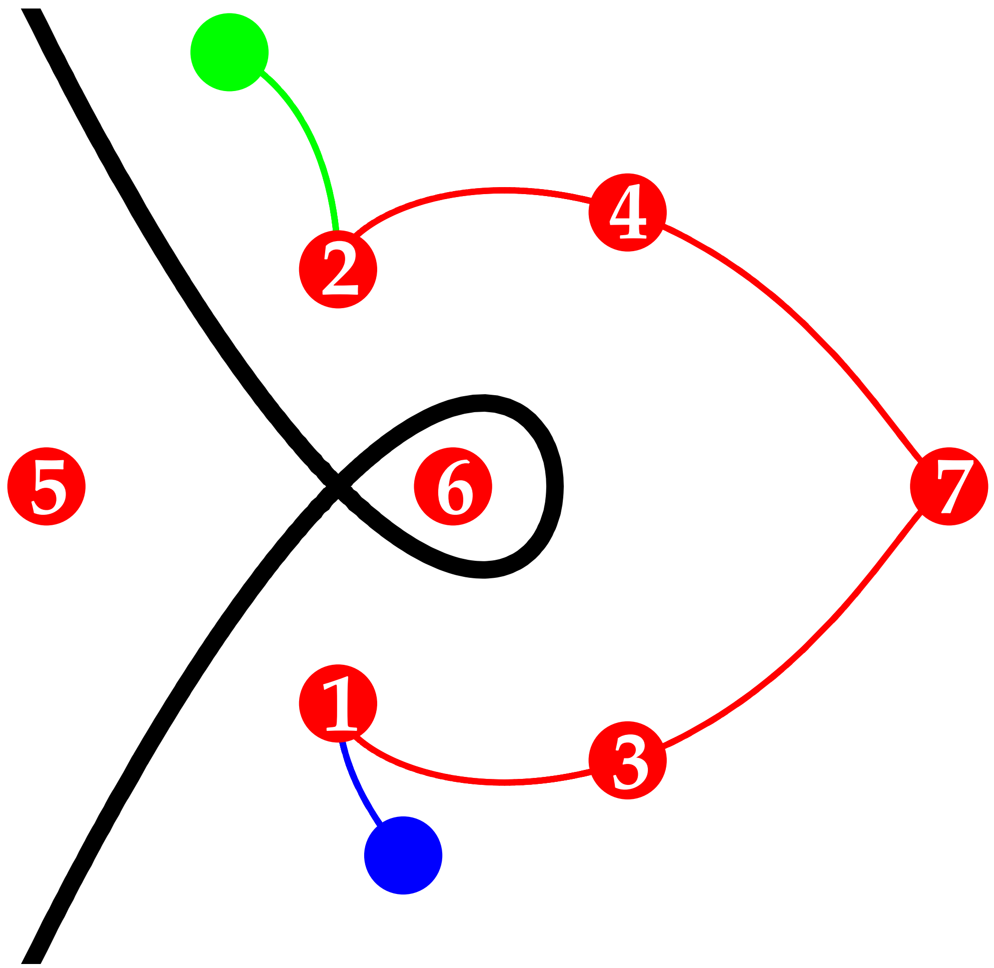

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Output: True

Method: Demo

Research Challenges

Research Challenges

1. Correctness

Research Challenges

1. Correctness
2. Termination

Research Challenges

1. Correctness
2. Termination
3. Length Bound

1. Correctness: Theorem

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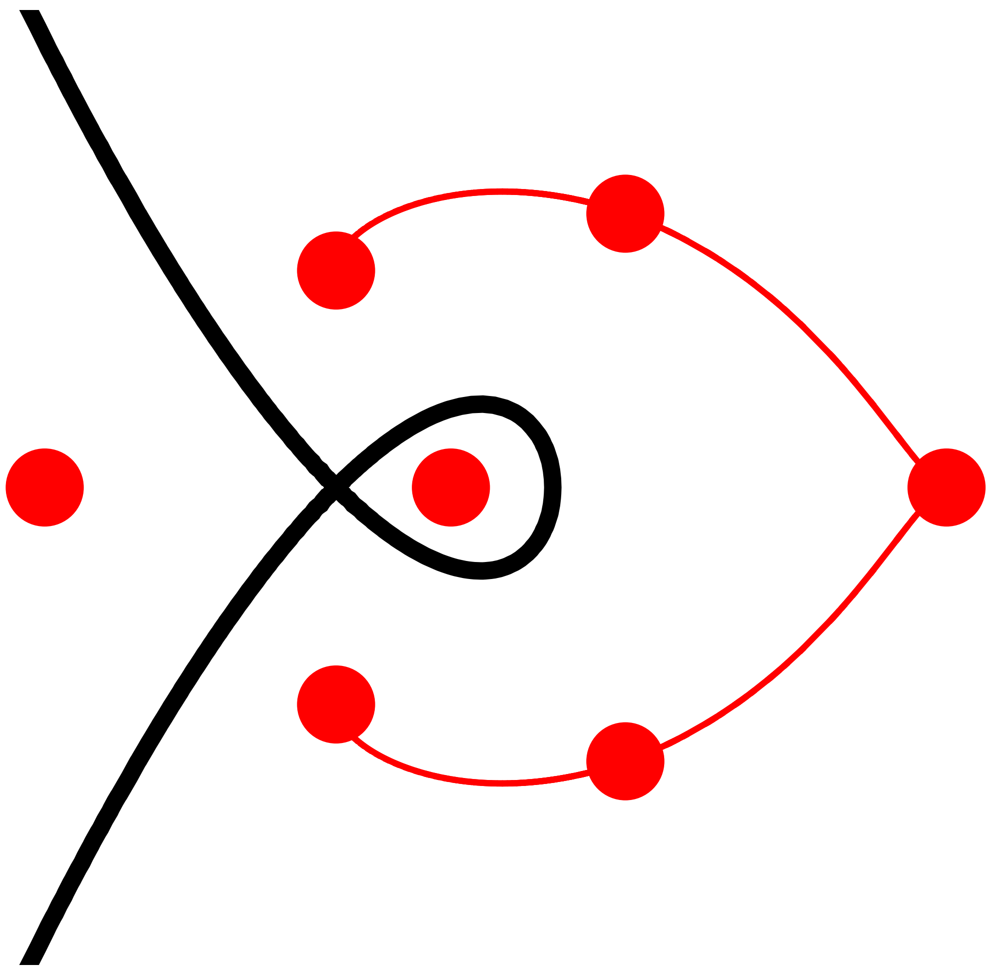
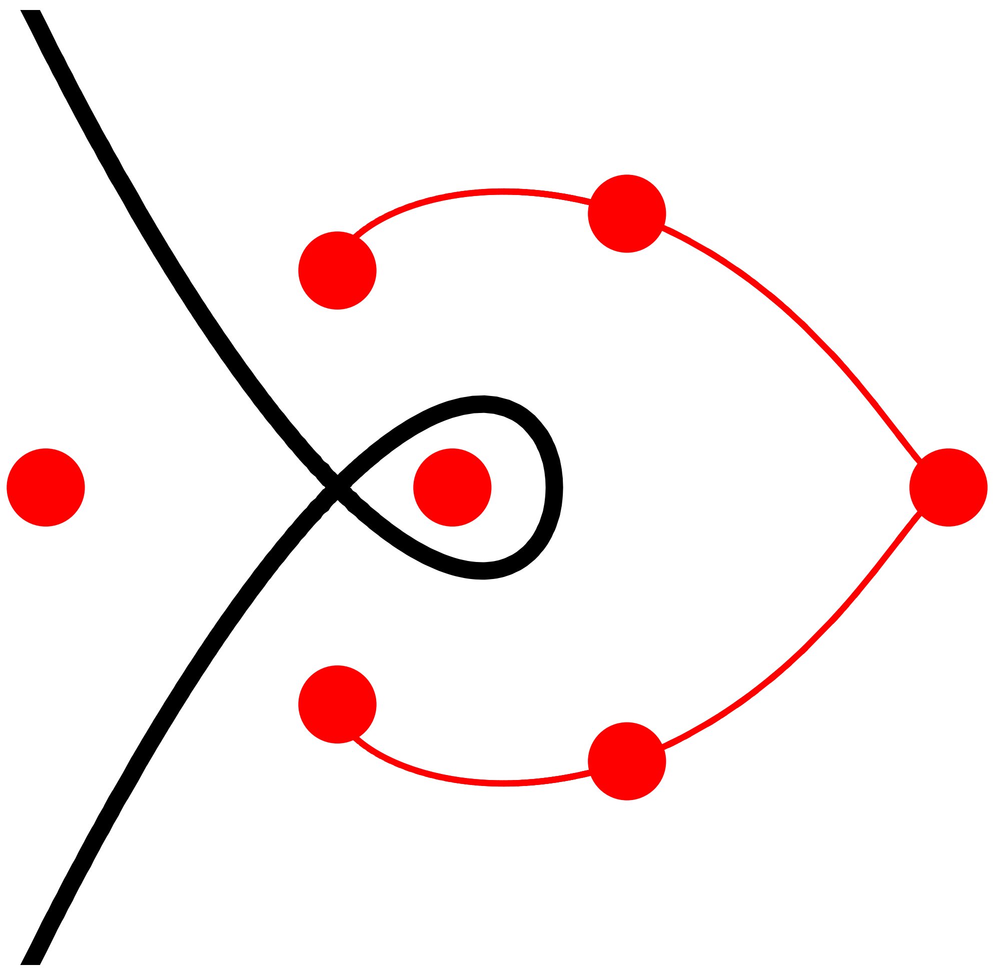


1. Correctness: Theorem

Let g be a routing function.



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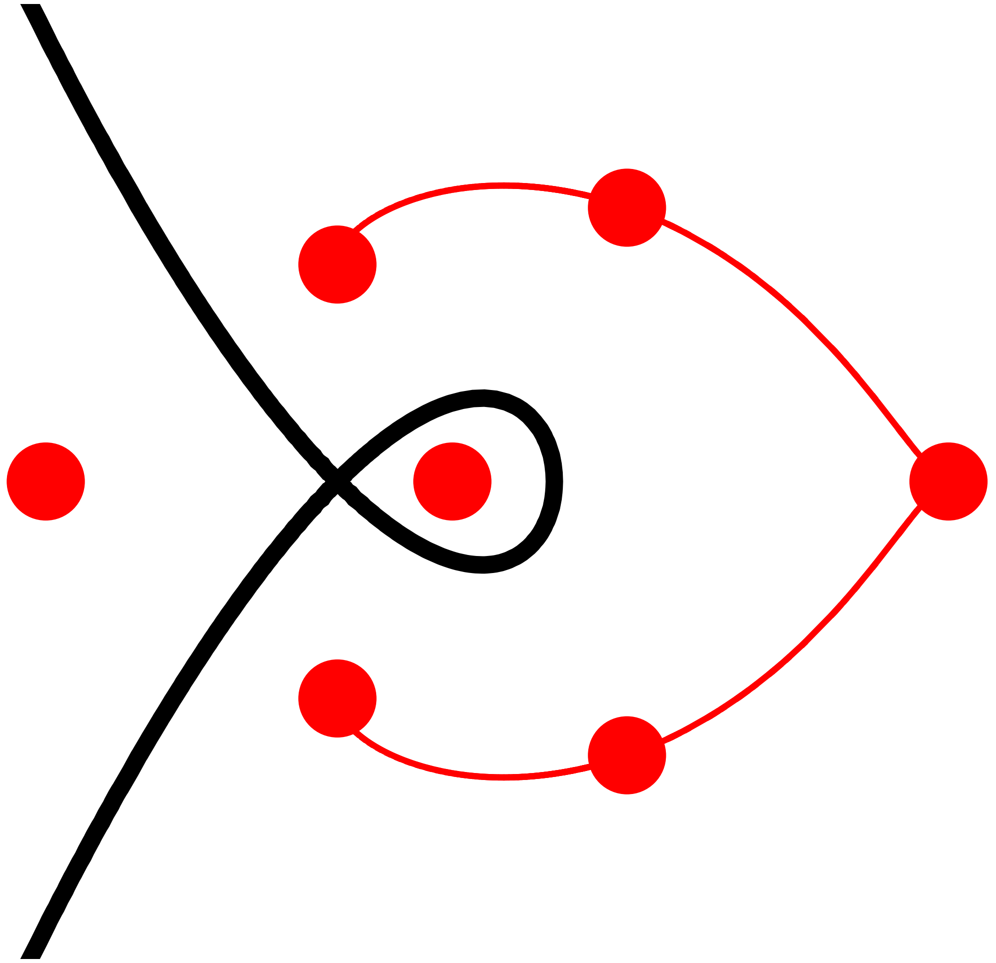

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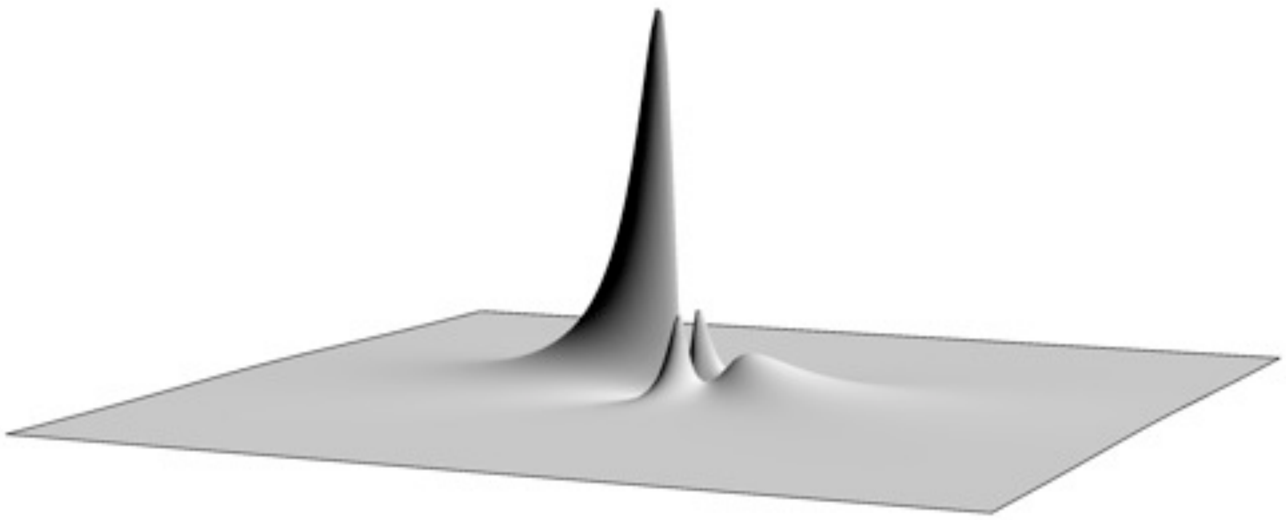

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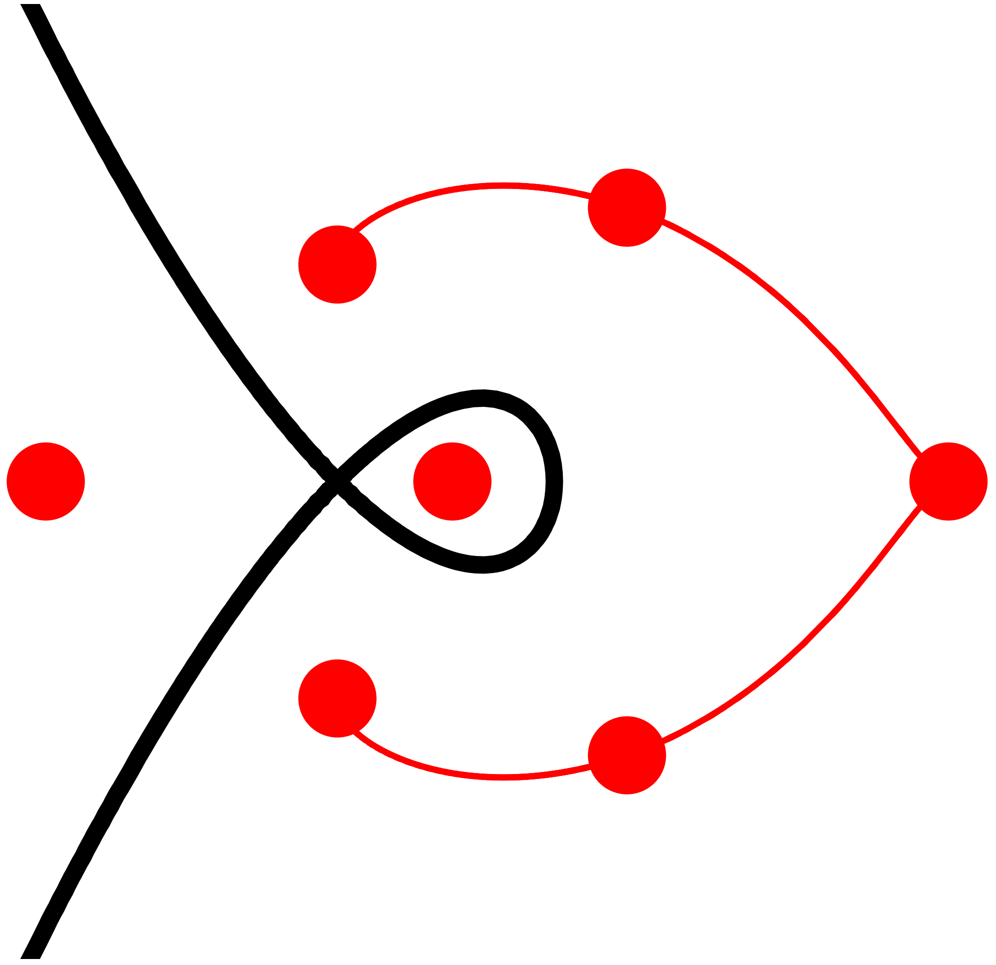

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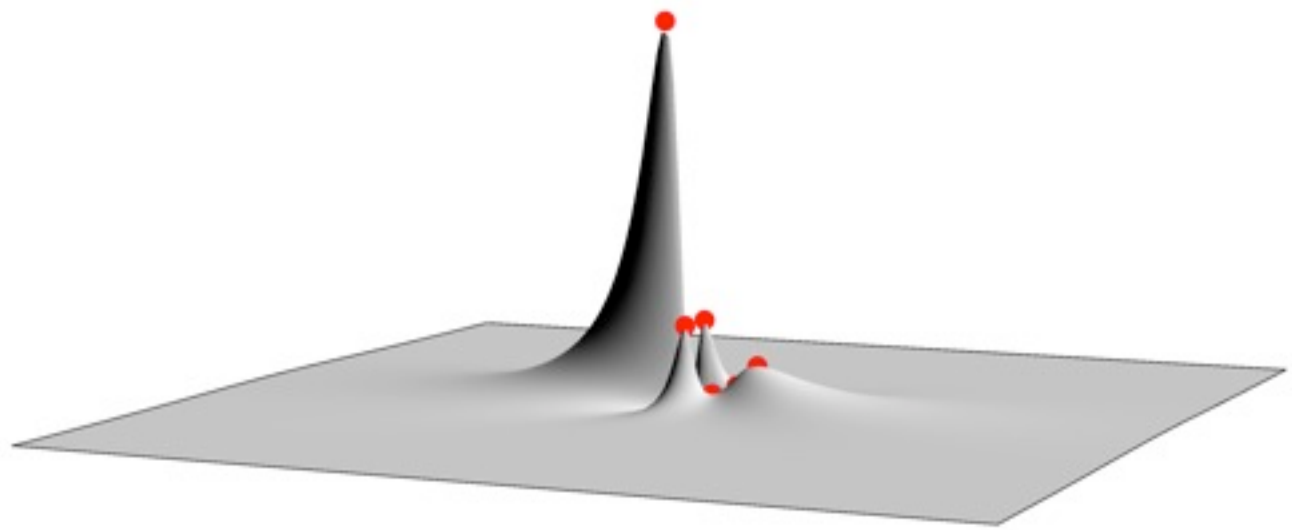

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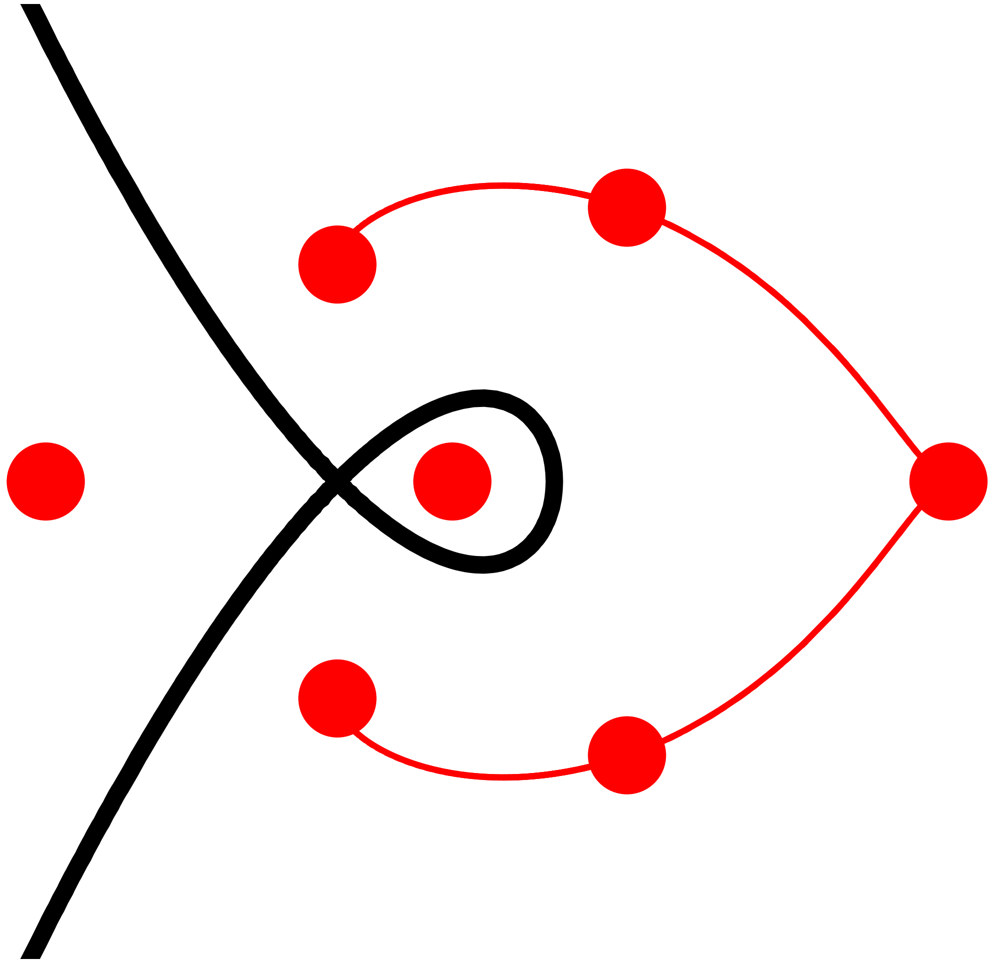

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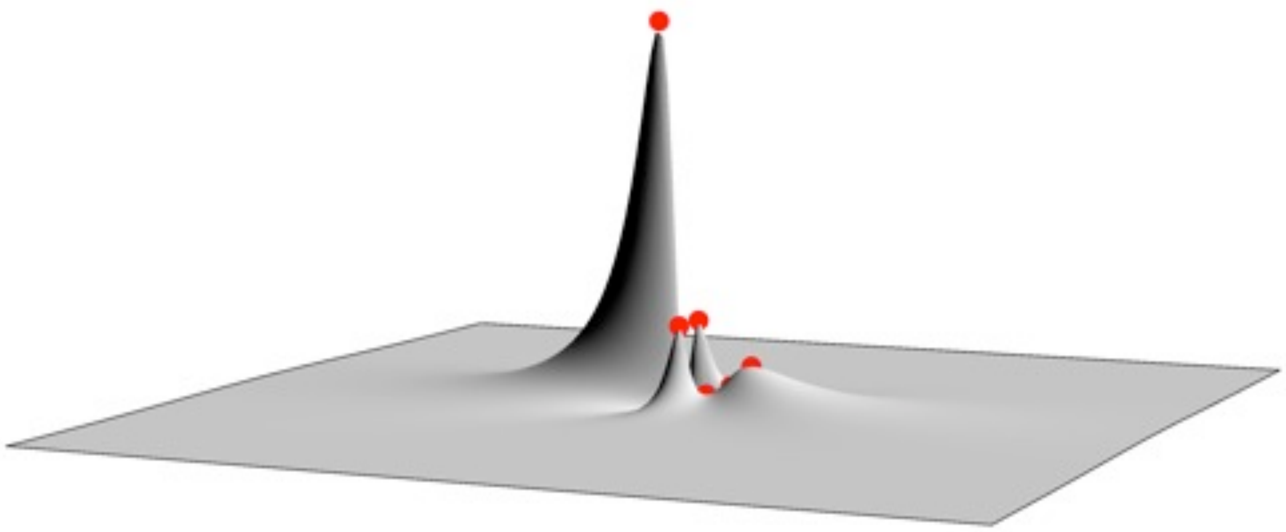

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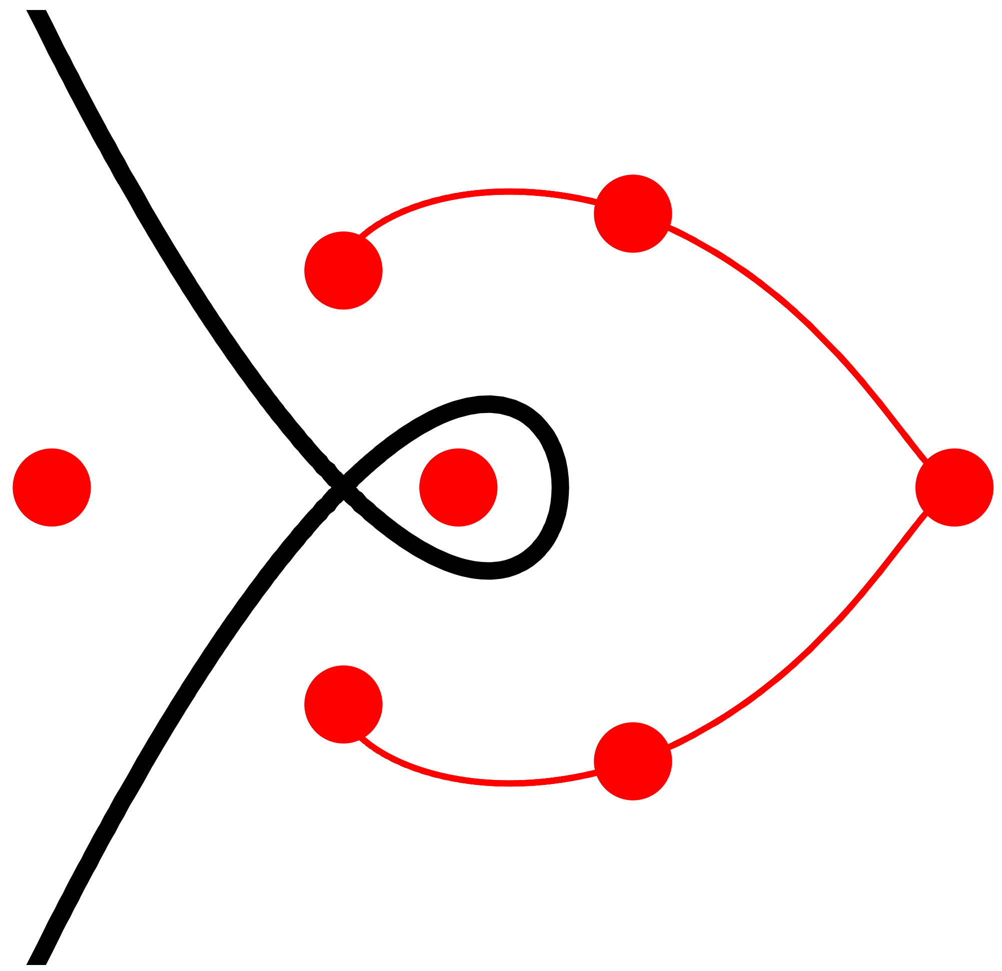

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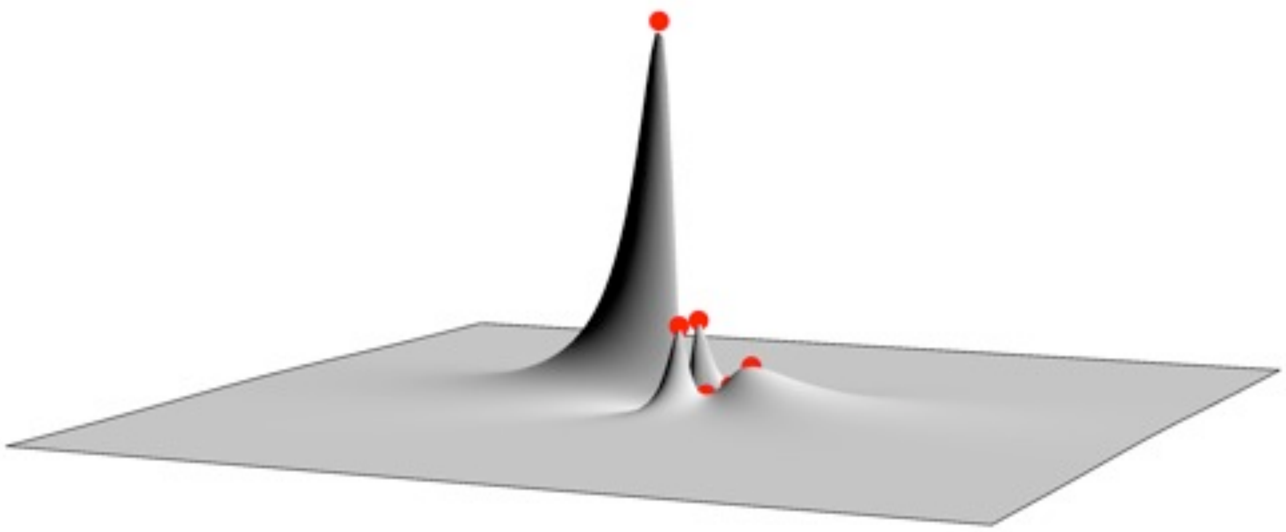

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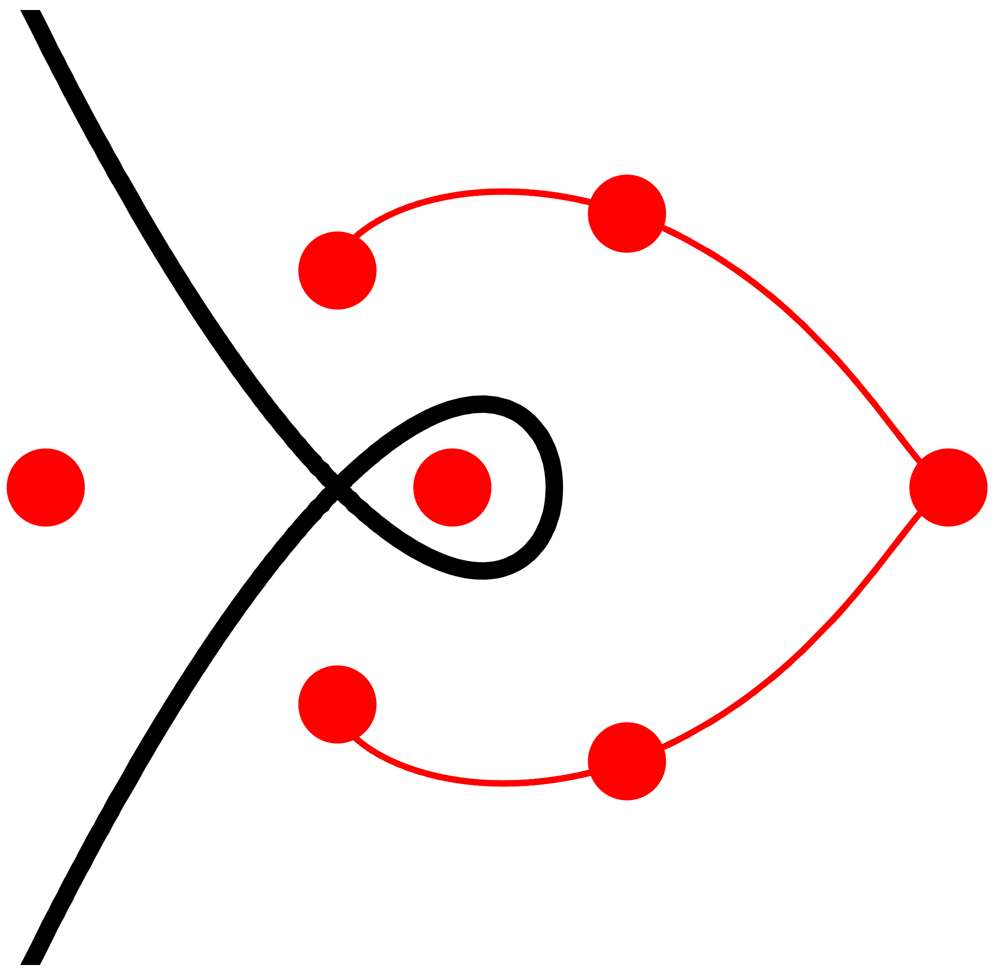

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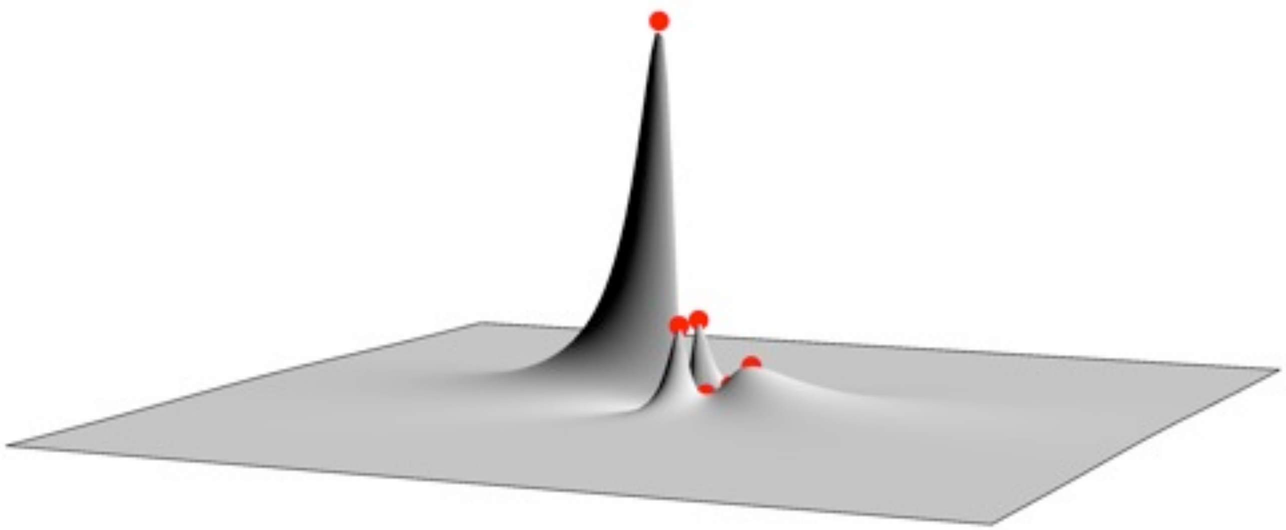

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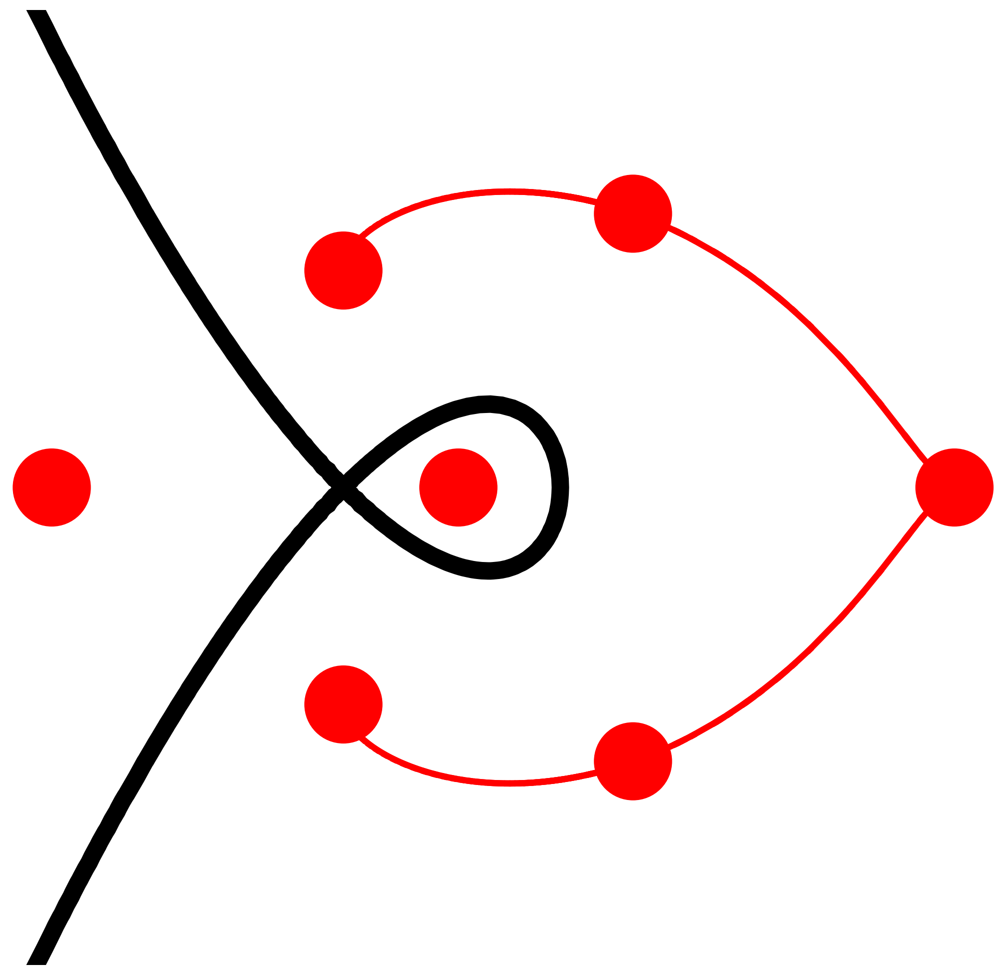

Example: $g = \dfrac{f^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$

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Let g be a routing function.

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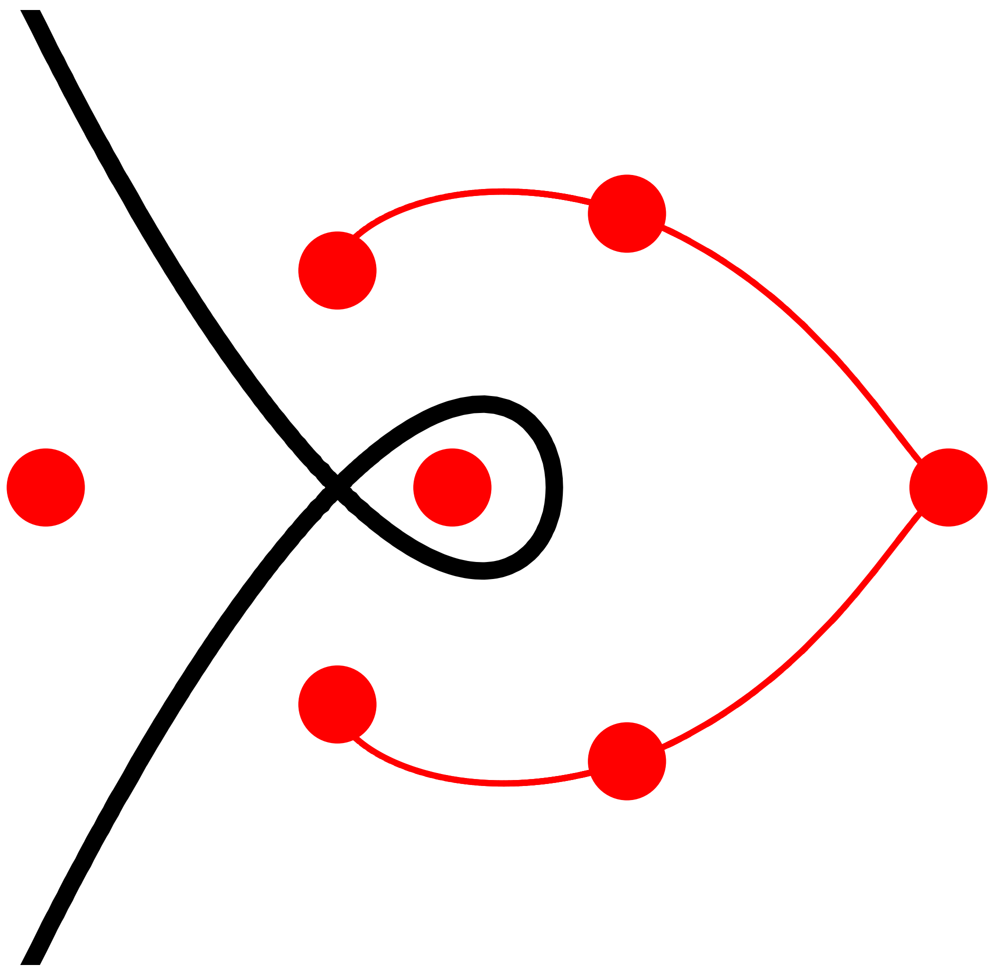

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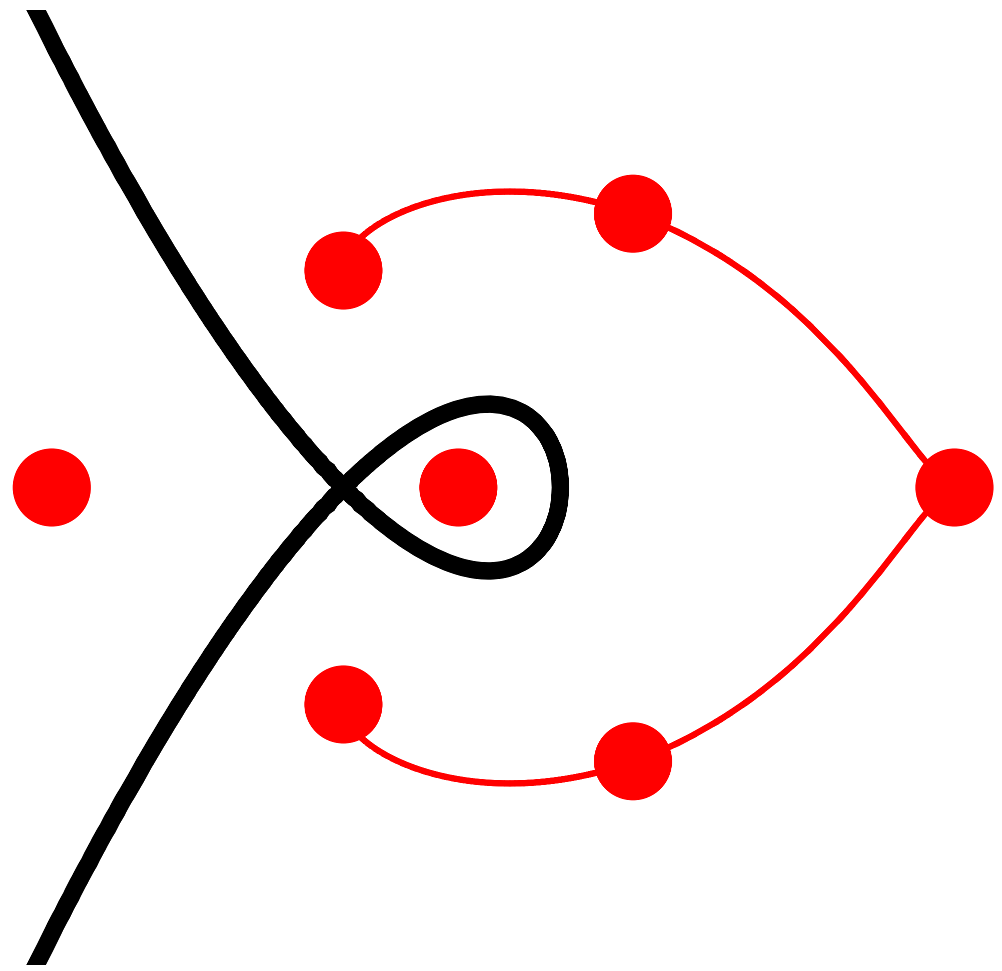

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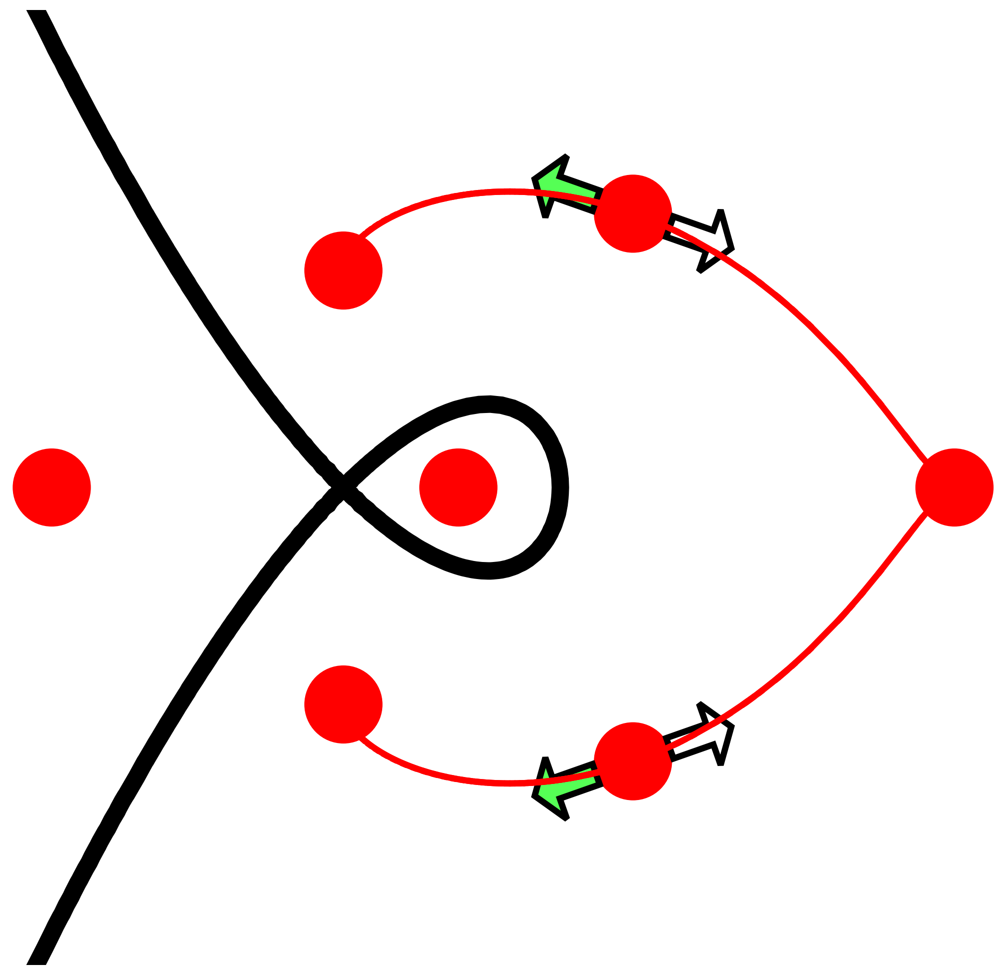

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1. Correctness: Preliminaries

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Trajectory of ∇g through $\bullet$

$$\phi'(t) = \nabla g\big(\phi(t)\big)$$
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Trajectory of ∇g through

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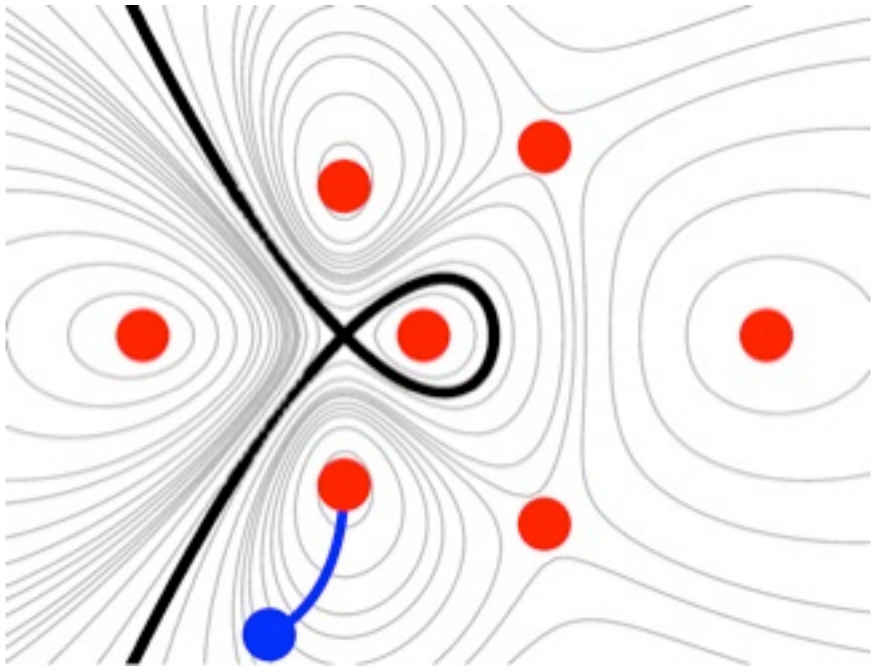
$(0, \infty) \xrightarrow{\phi}$



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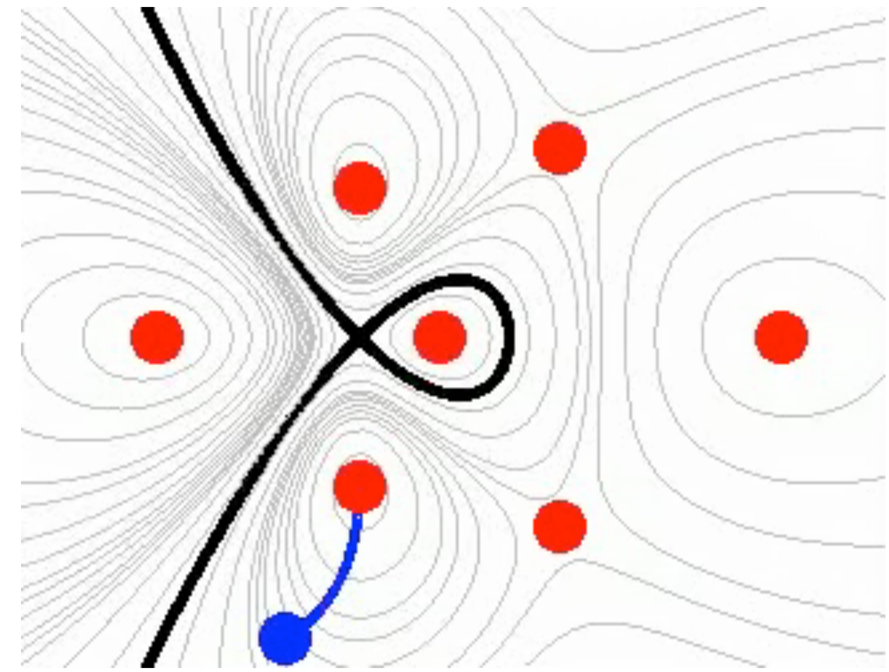

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Trajectory of ∇g through ●

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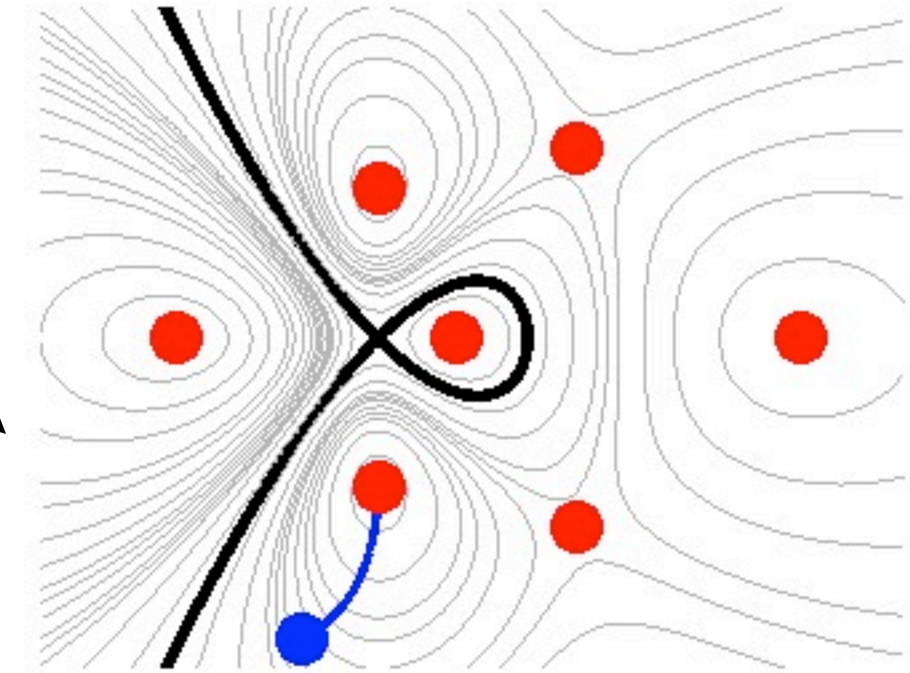

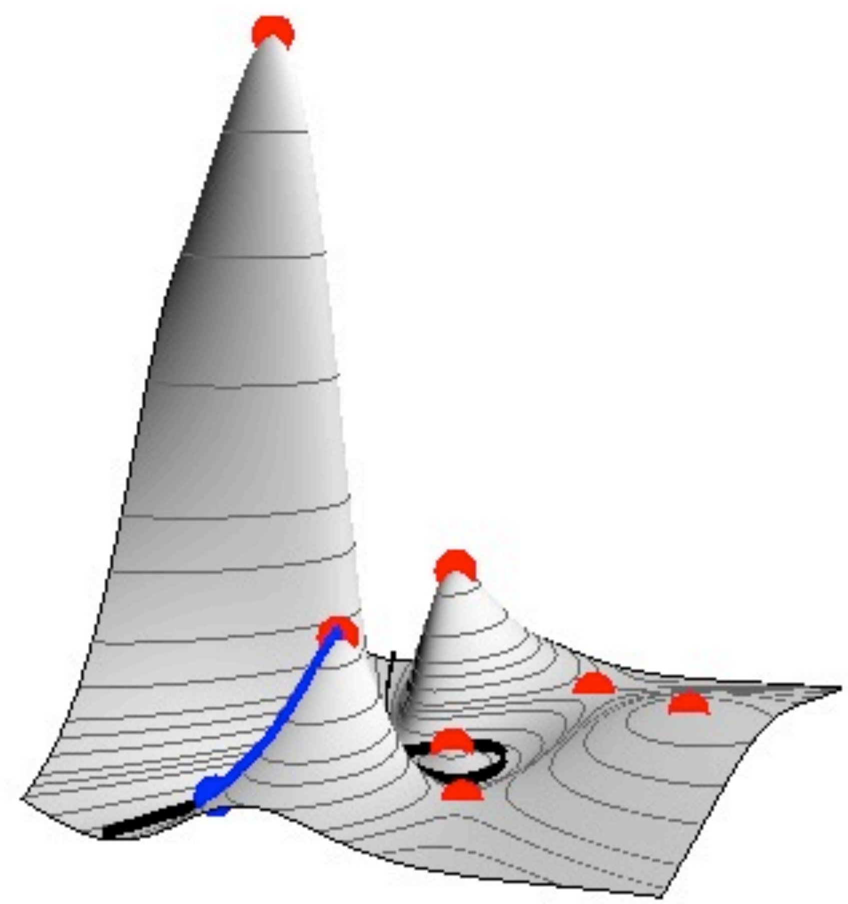

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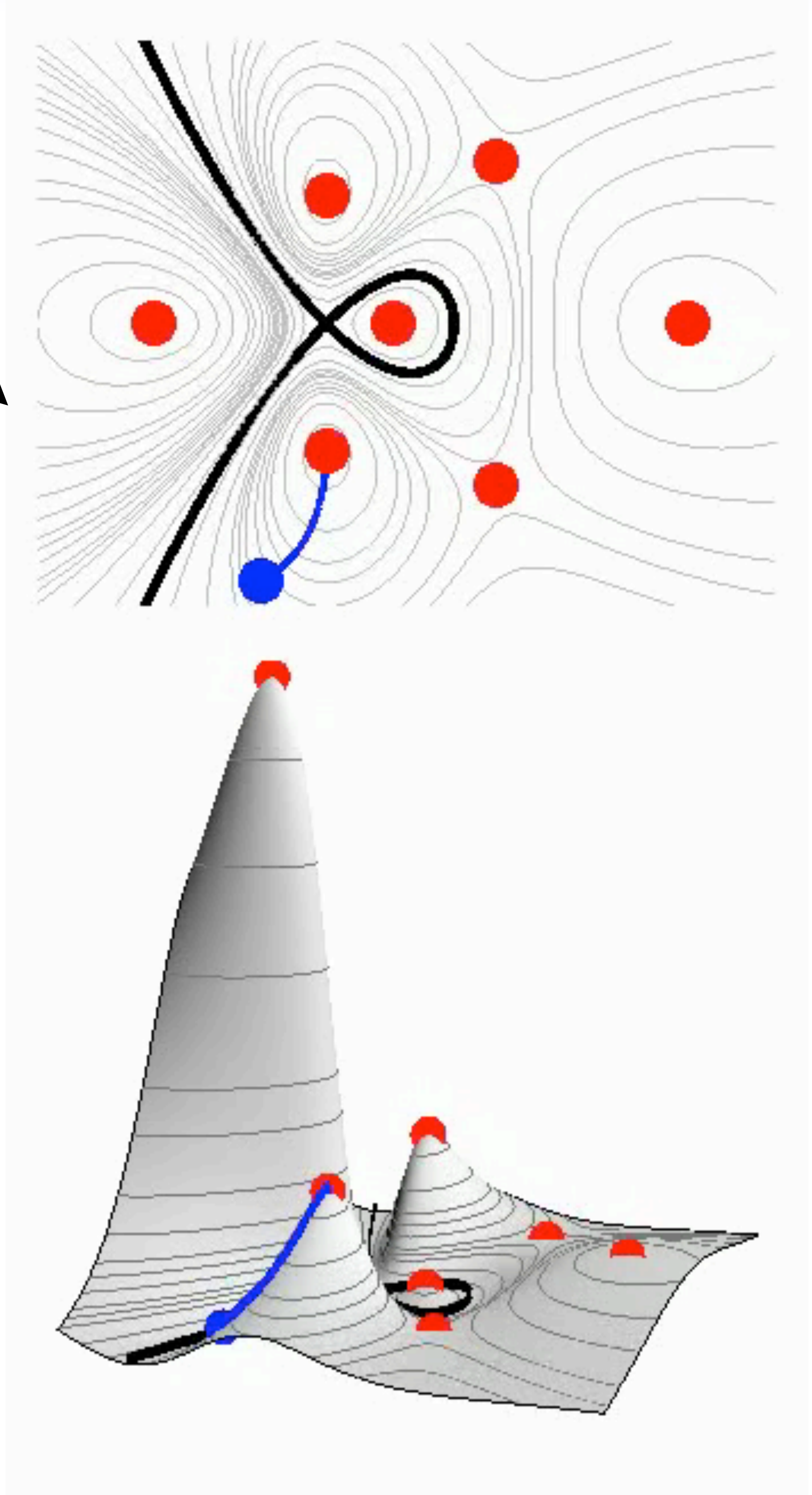

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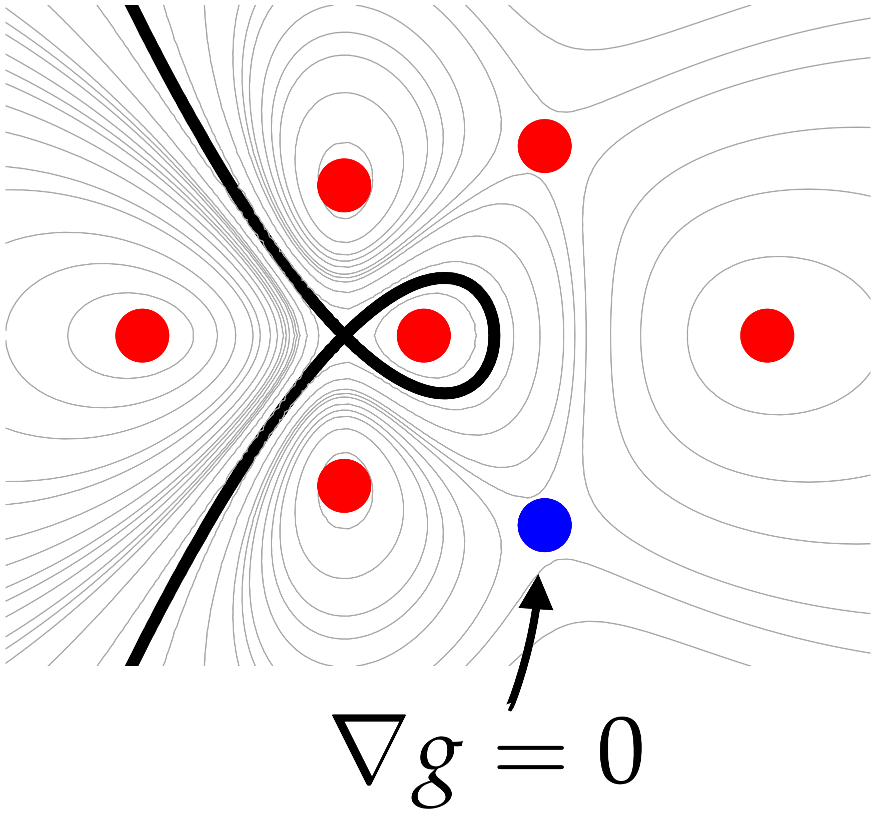

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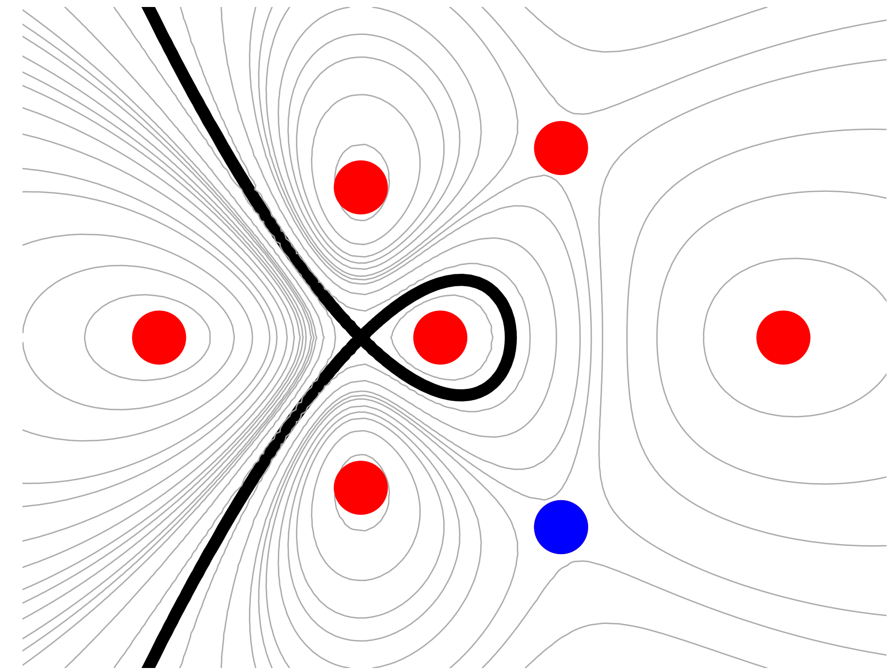
Trajectory of ∇g through using



$$\phi'(t) = \nabla g\big(\phi(t)\big) \text{ when } t \neq 0$$

and

$$\lim_{t \to 0^+} \phi(t) =$$

and

$$\lim_{t \to 0^+} \frac{\phi'(t)}{\|\phi'(t)\|} =$$

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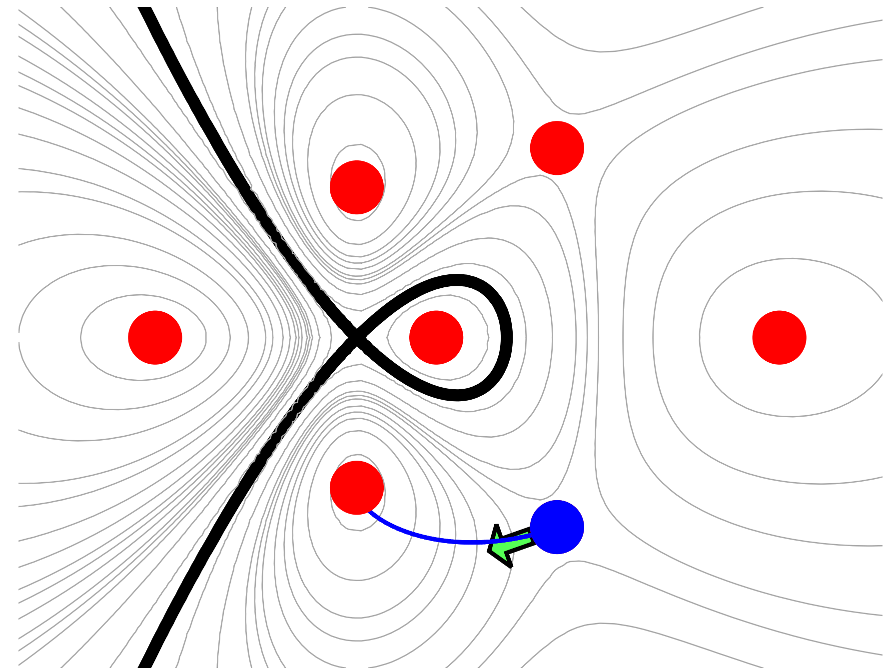

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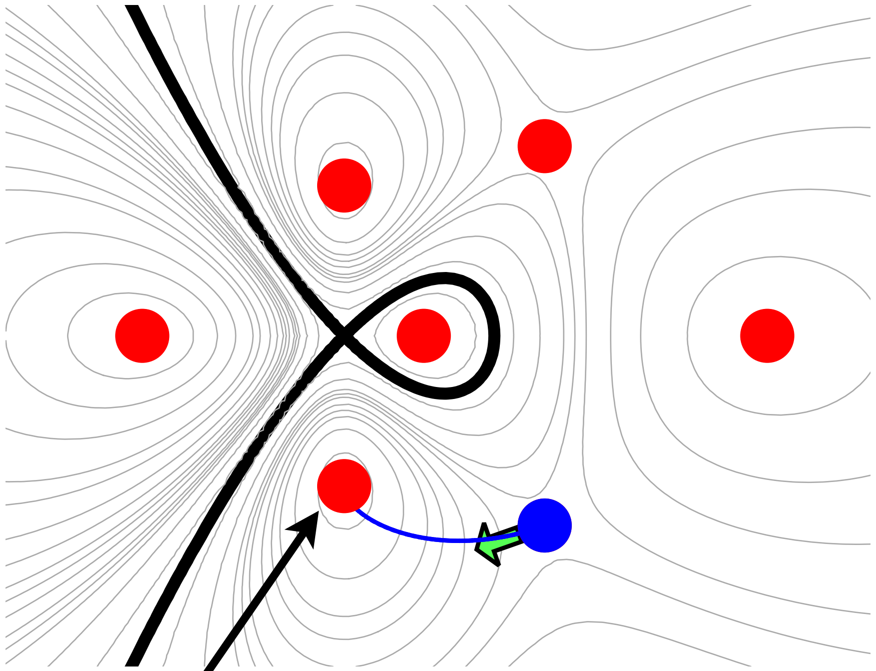

$$\mathrm{dest}(\phi) = \lim_{t \to \infty} \phi(t)$$

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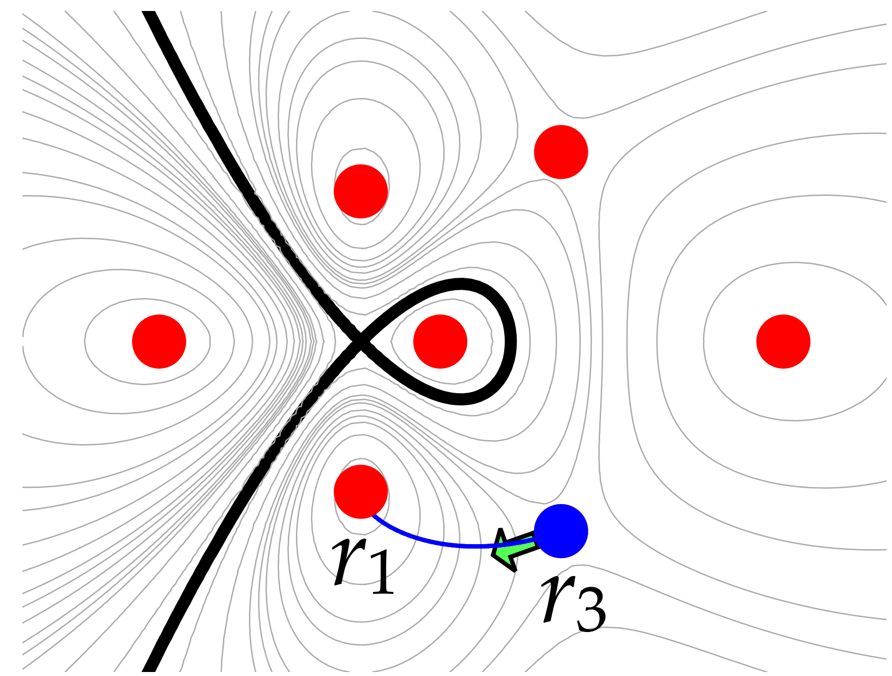
Trajectory of ∇g through using



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= outgoing evec. of
of (Hess g)(r_3)

r_1, r_3 are **connected by steepest ascent paths using outgoing eigenvectors**

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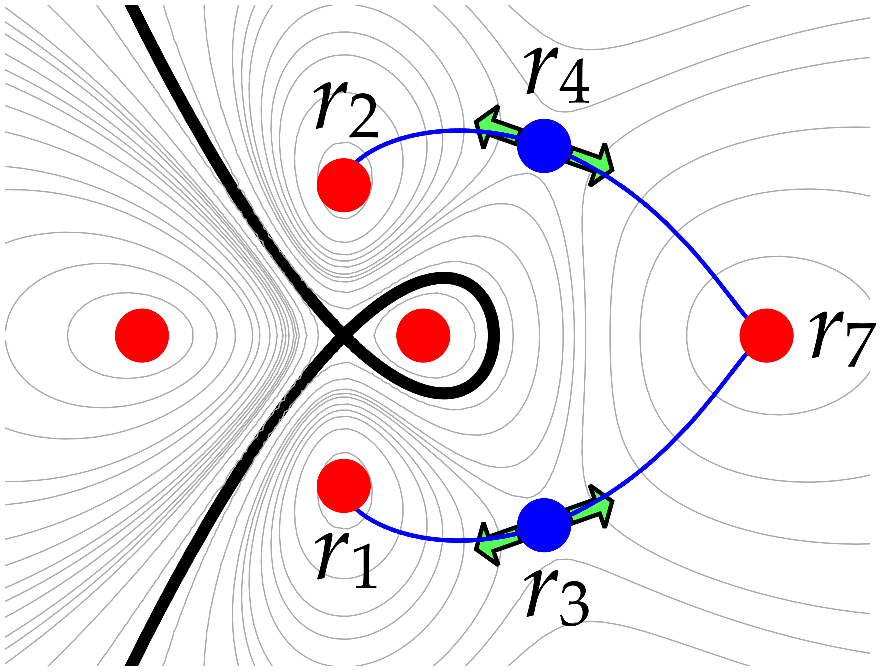

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any two routing points in a connected comp. are **connected by steepest ascent paths using outgoing eigenvectors**

1. Correctness: Preliminaries

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stable manifold of p: $W^s(p) = \{x \in \mathbb{R}^n \mid \text{dest}(\phi_x) = p\} \cup \{p\}$

$$\phi_x = \text{ trajectory of } \nabla g \text{ through } x \text{ using } \widehat{\nabla g(x)}$$

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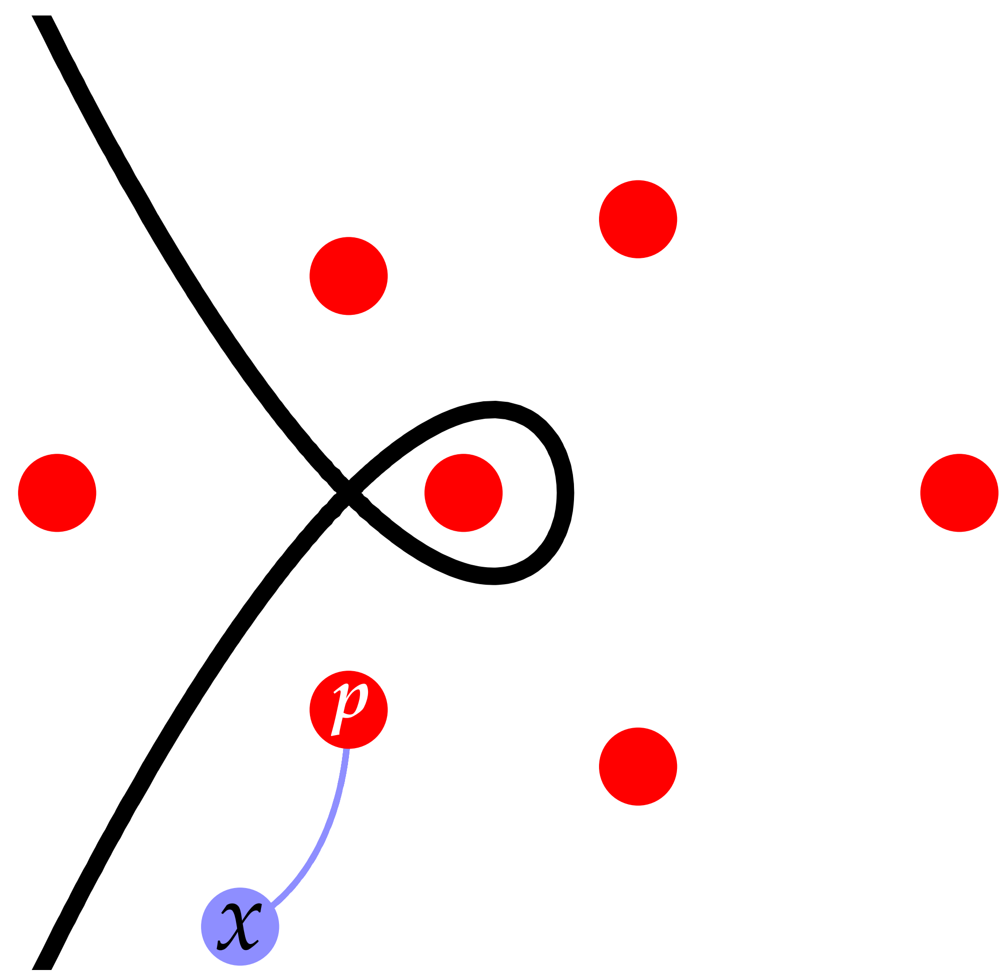

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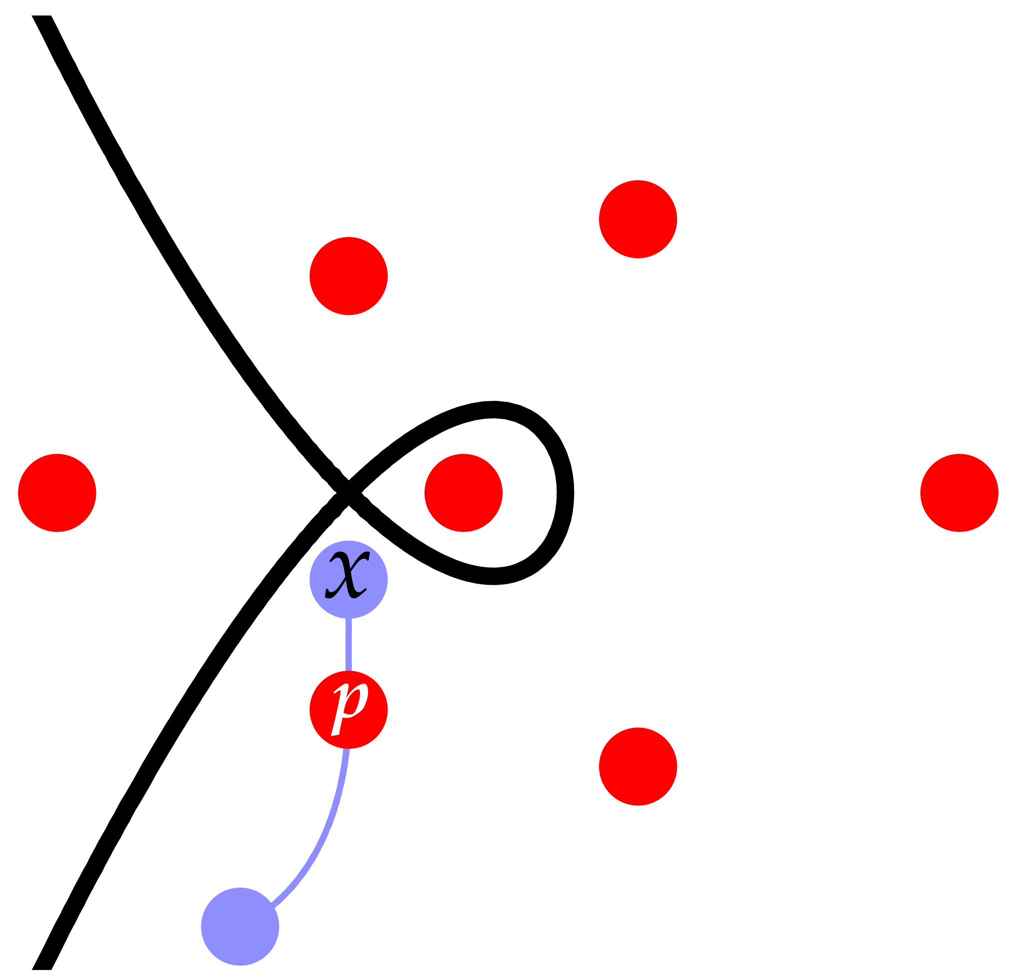

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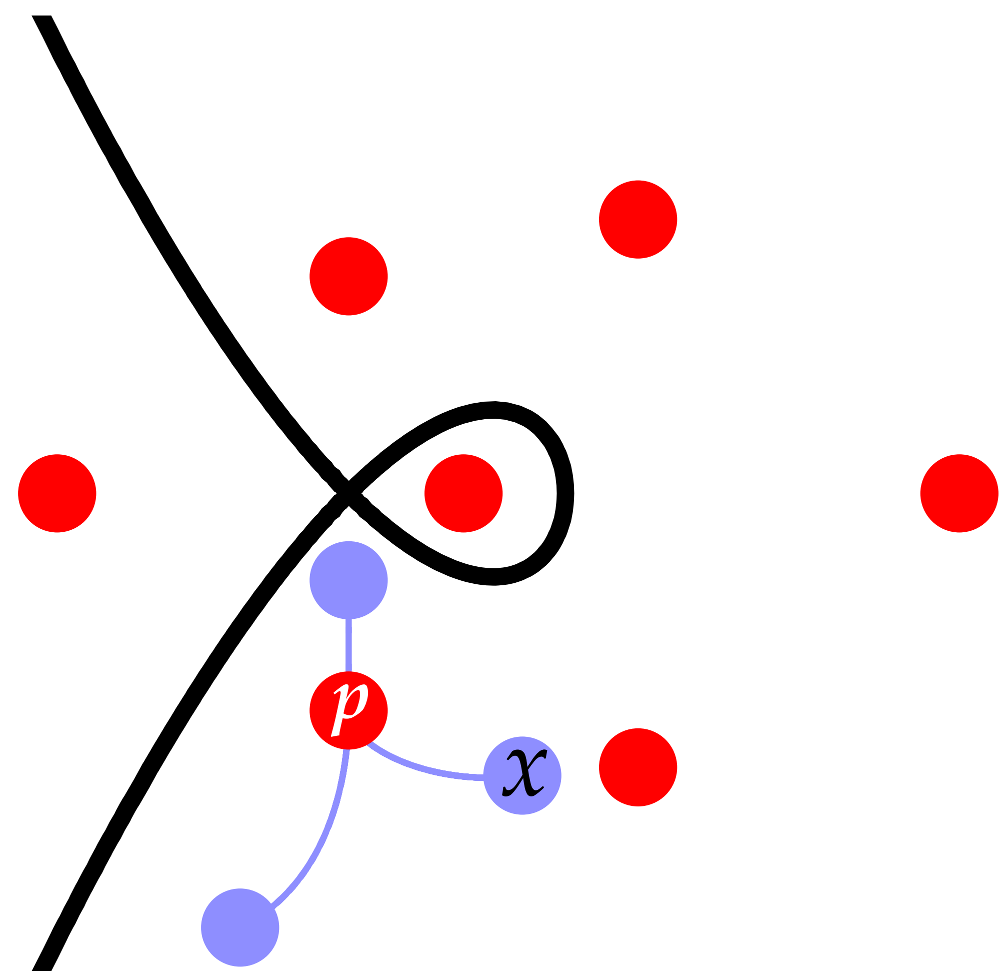

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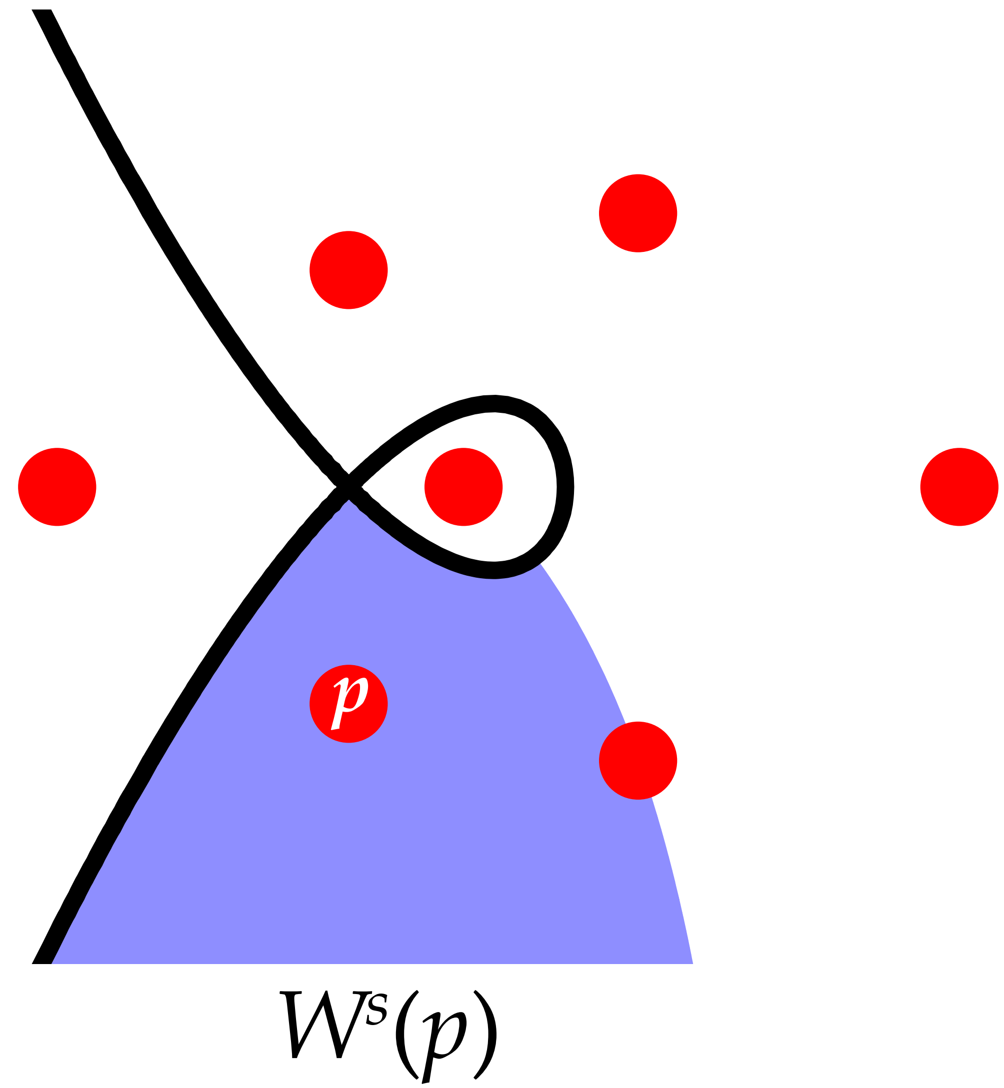

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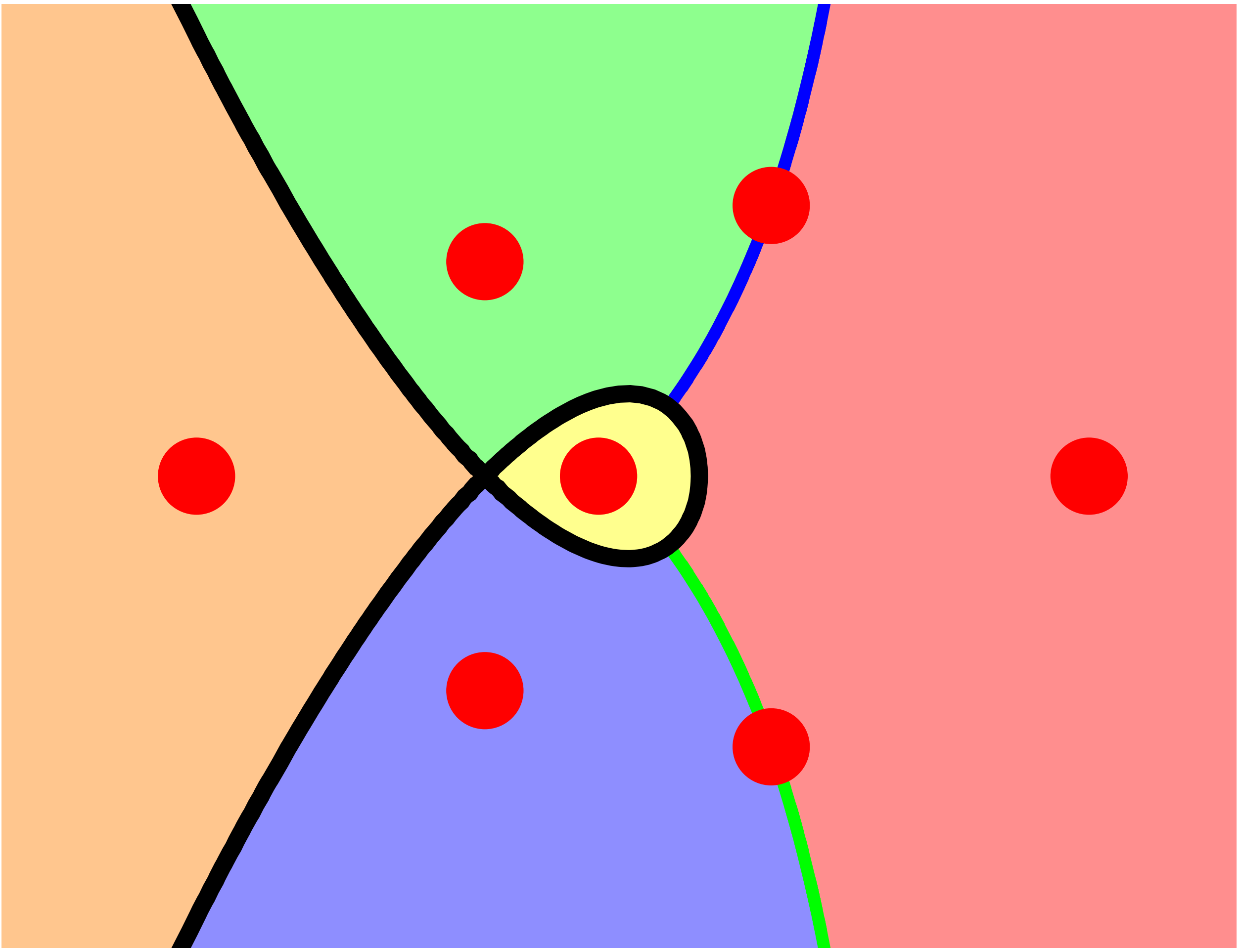

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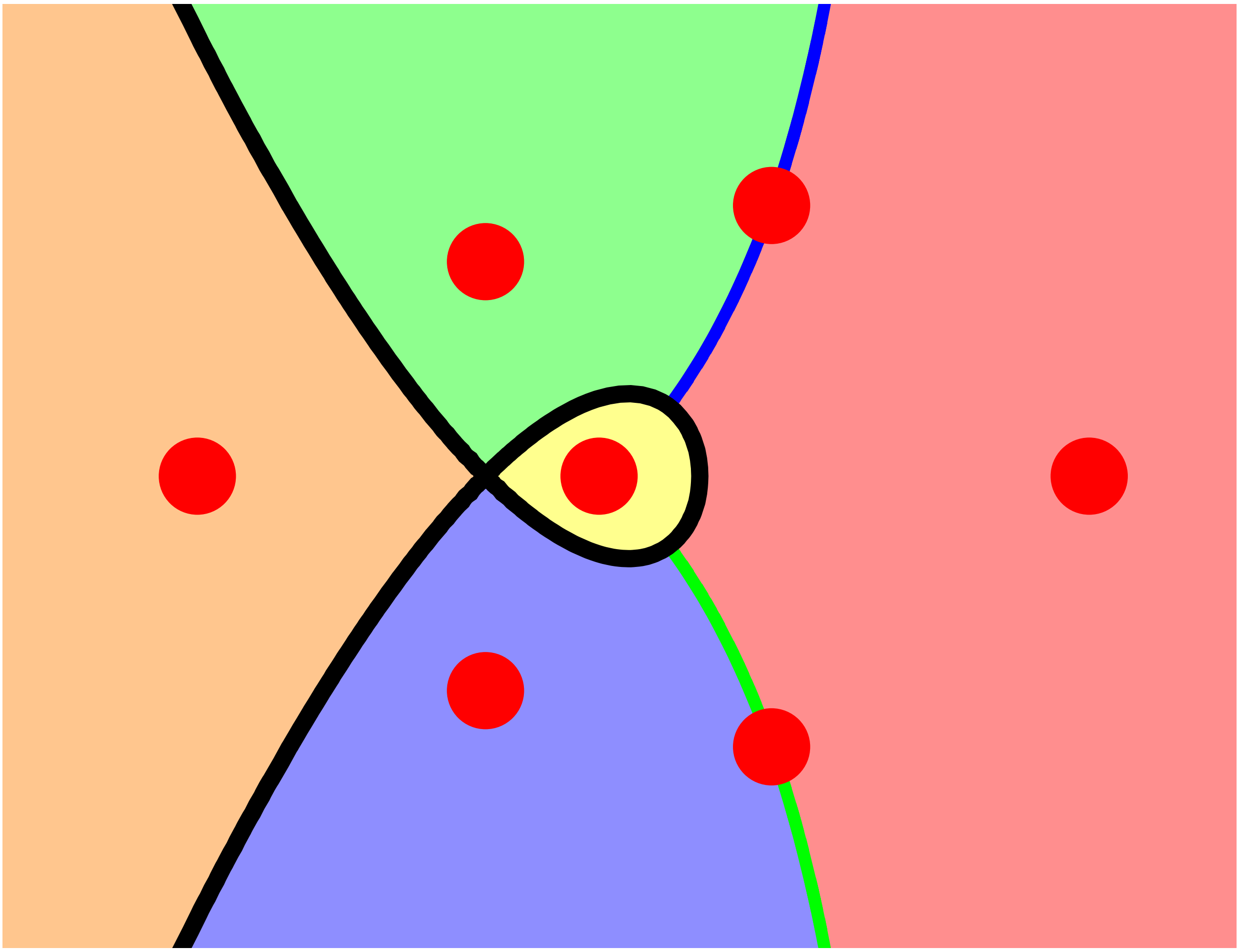
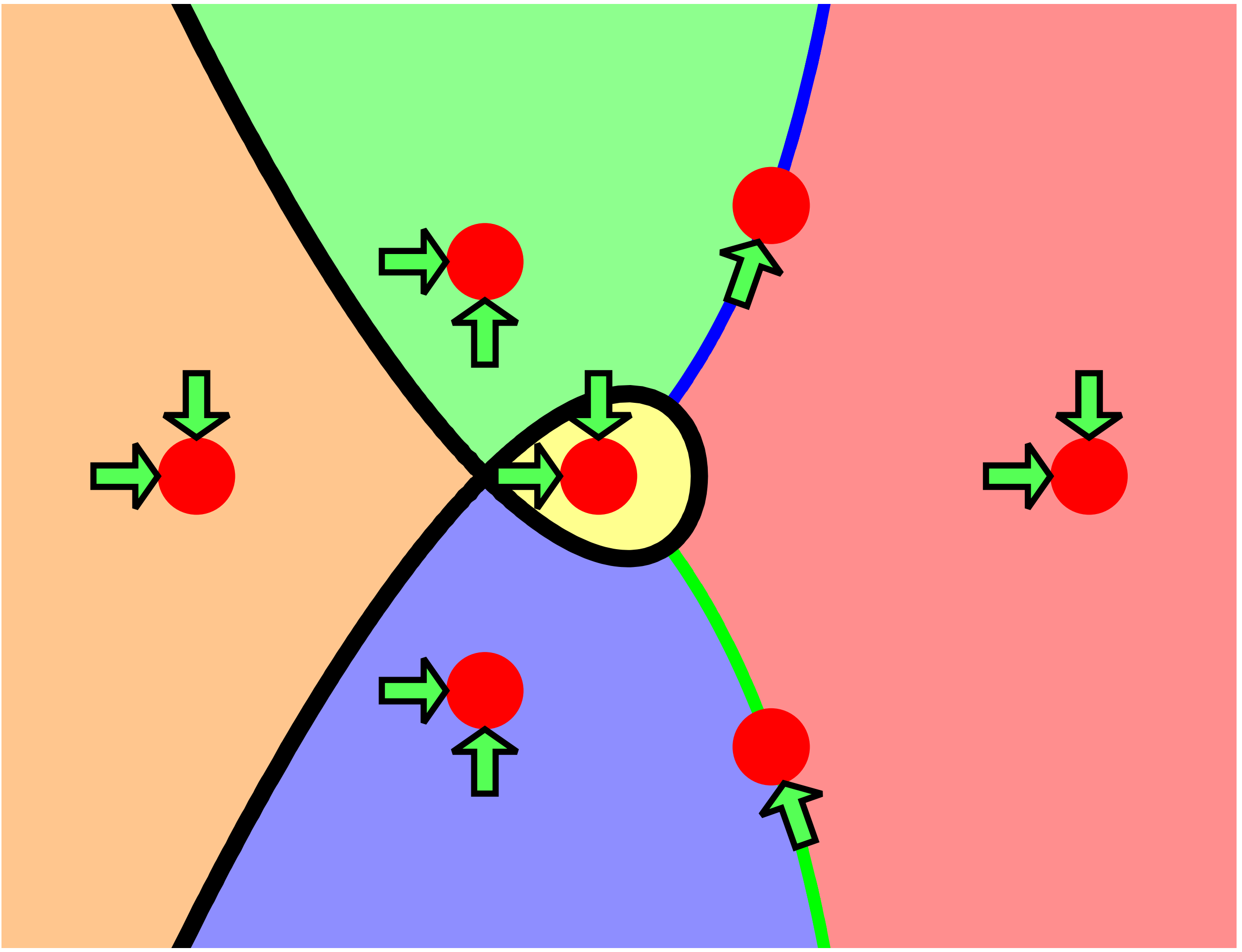


index of *p* = # of negative eigenvalues of (Hess g)(p) = dim $W^s(p)$

1. Correctness: Preliminaries

stable manifold of *p*: $W^s(p) = \{x \in \mathbb{R}^n \mid \mathrm{dest}(\phi_x) = p\} \cup \{p\}$

$\phi_x =$ trajectory of ∇g through x using $\widehat{\nabla g(x)}$



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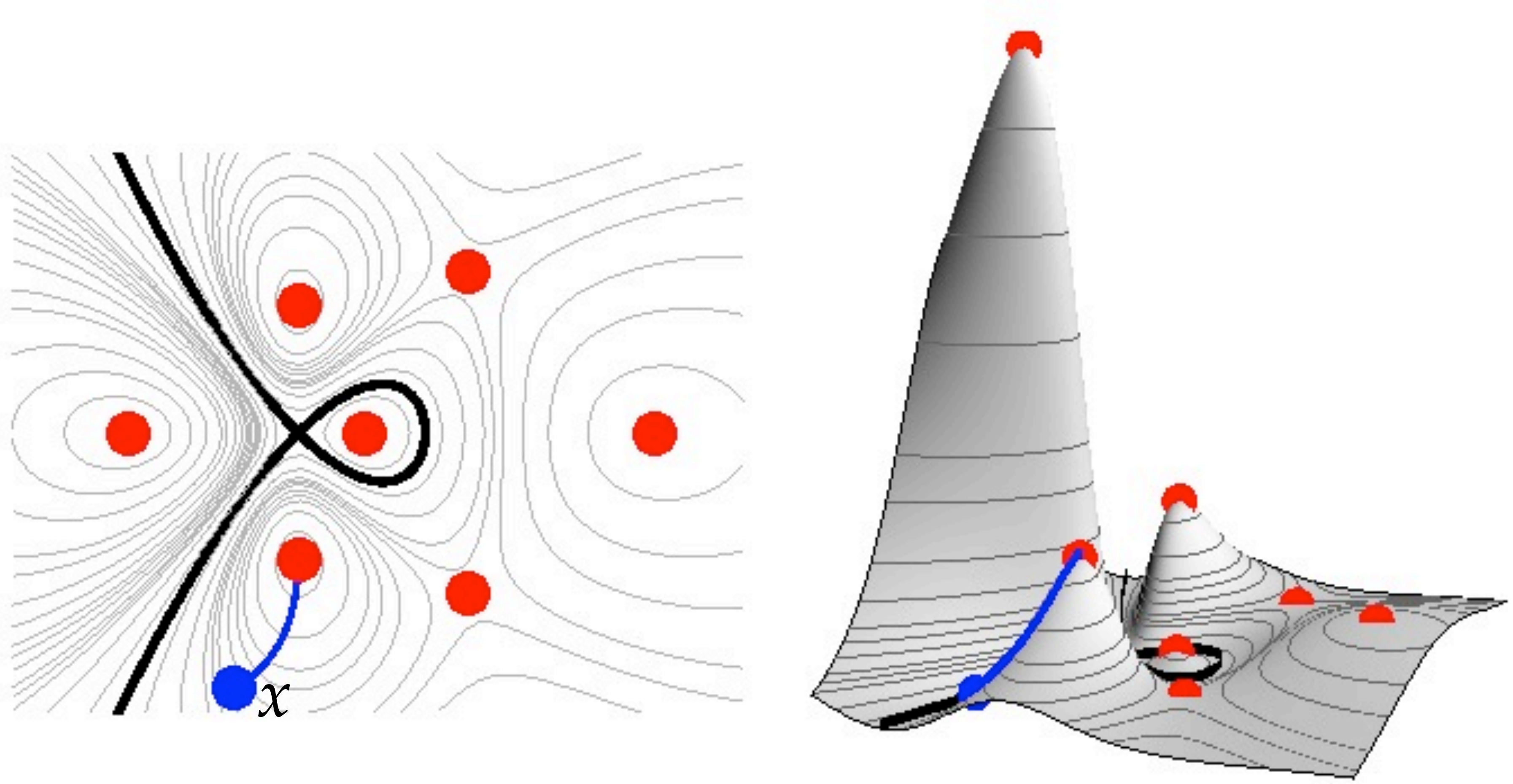

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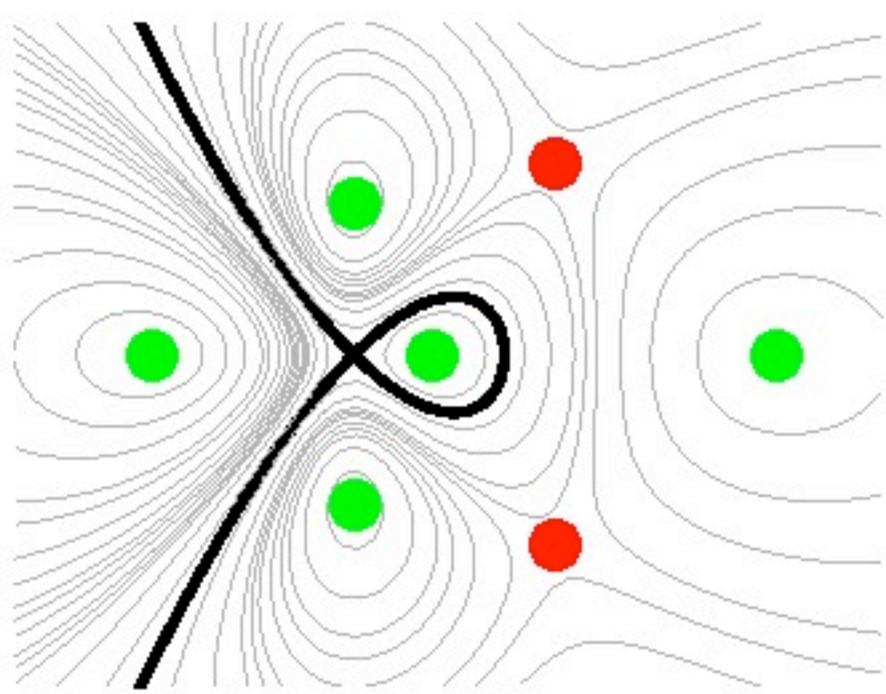

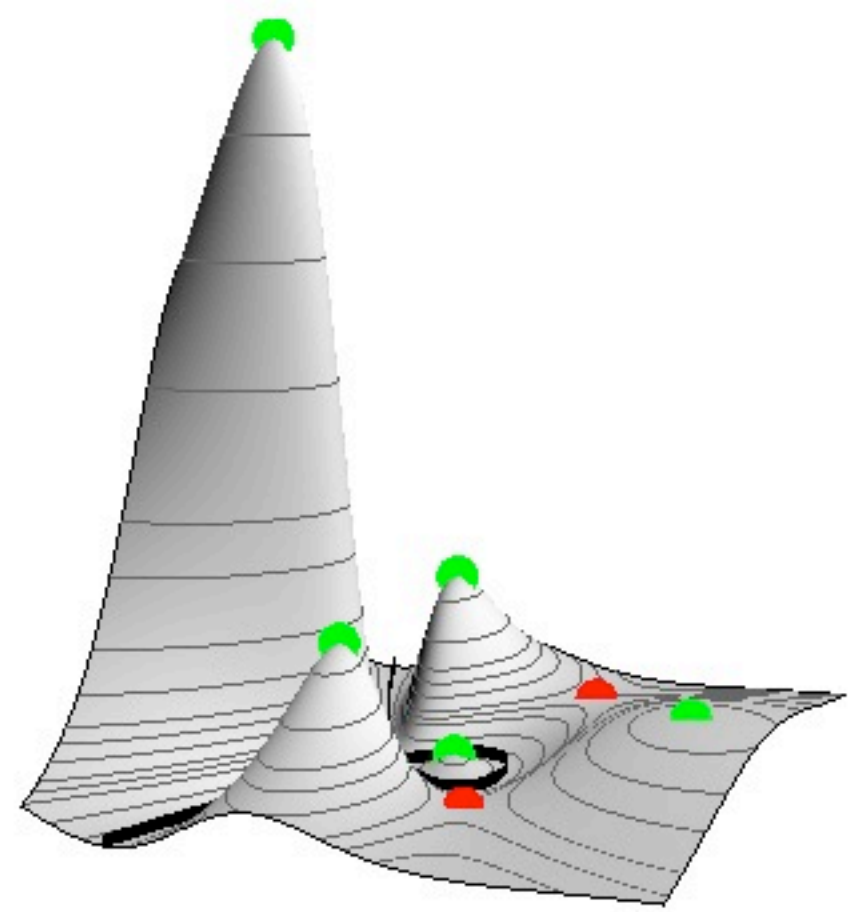

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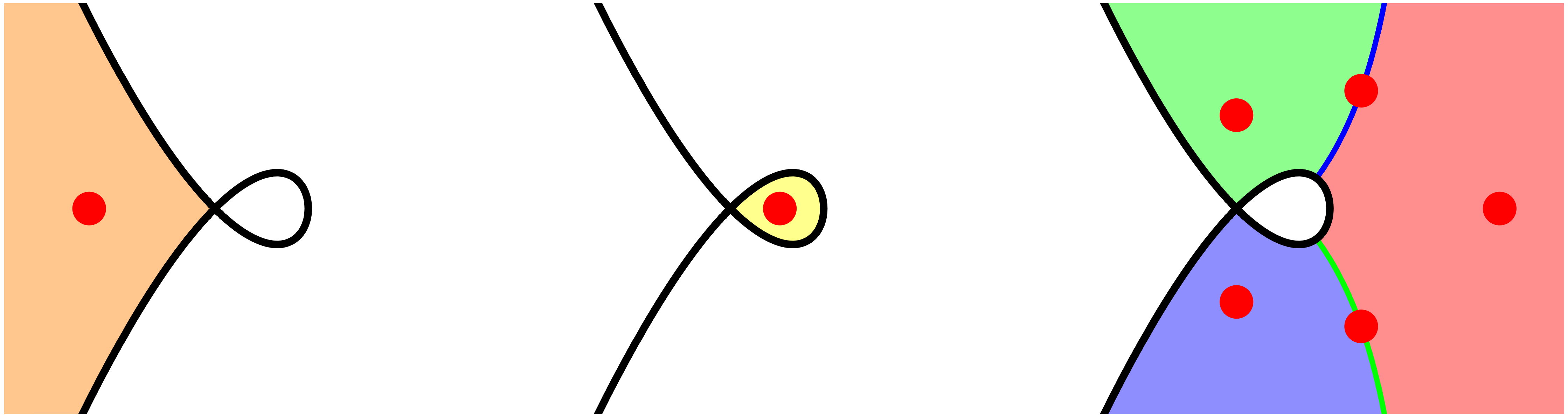

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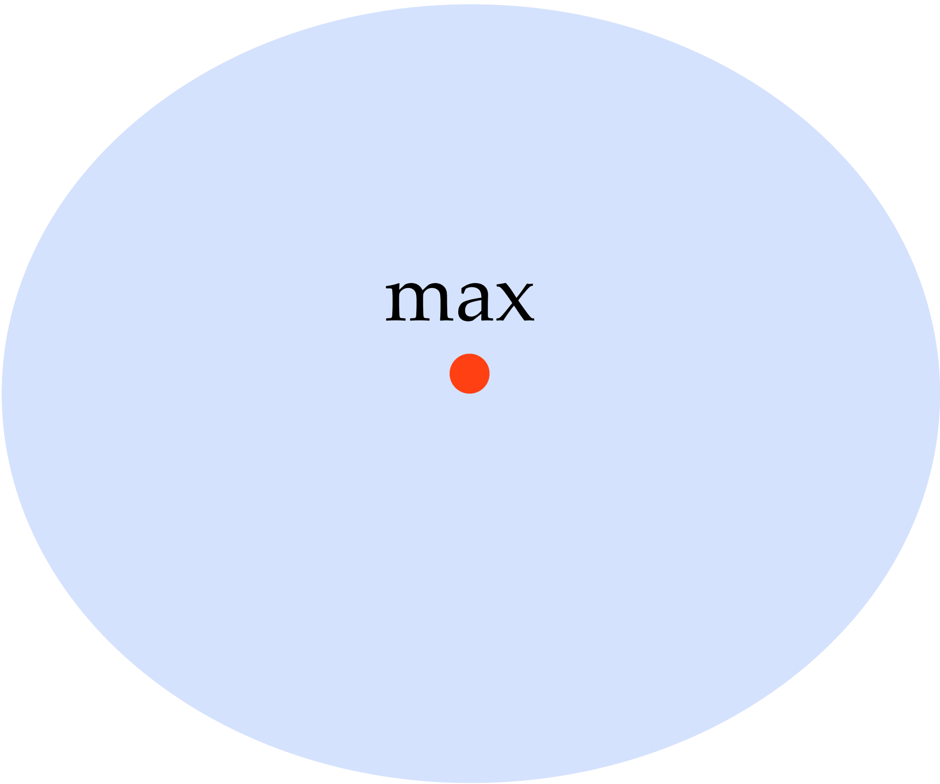

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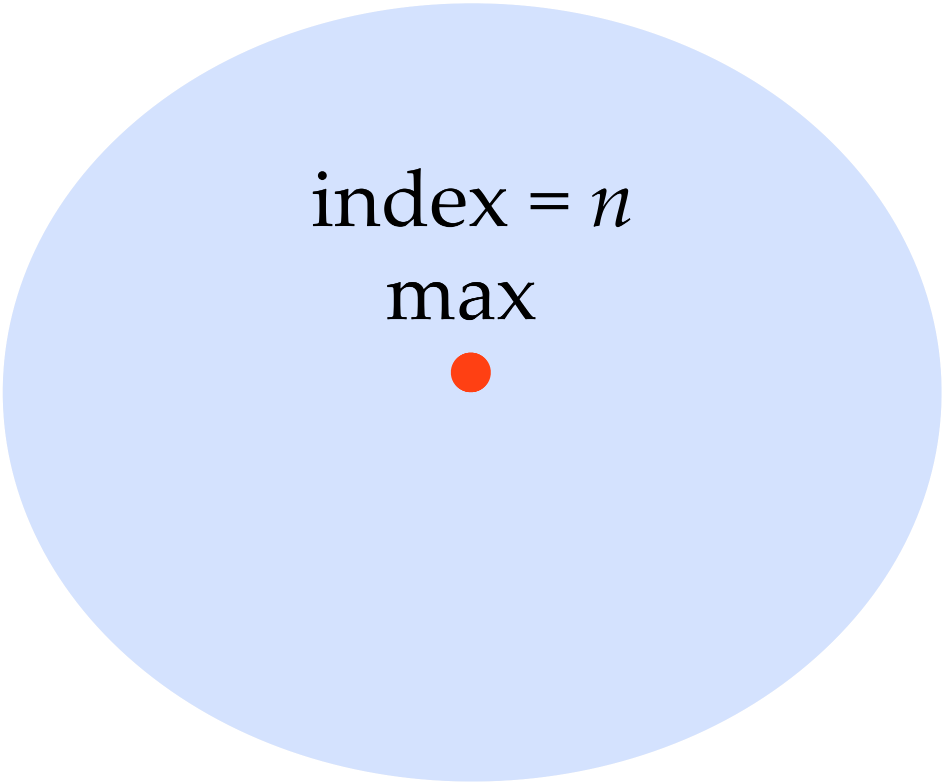

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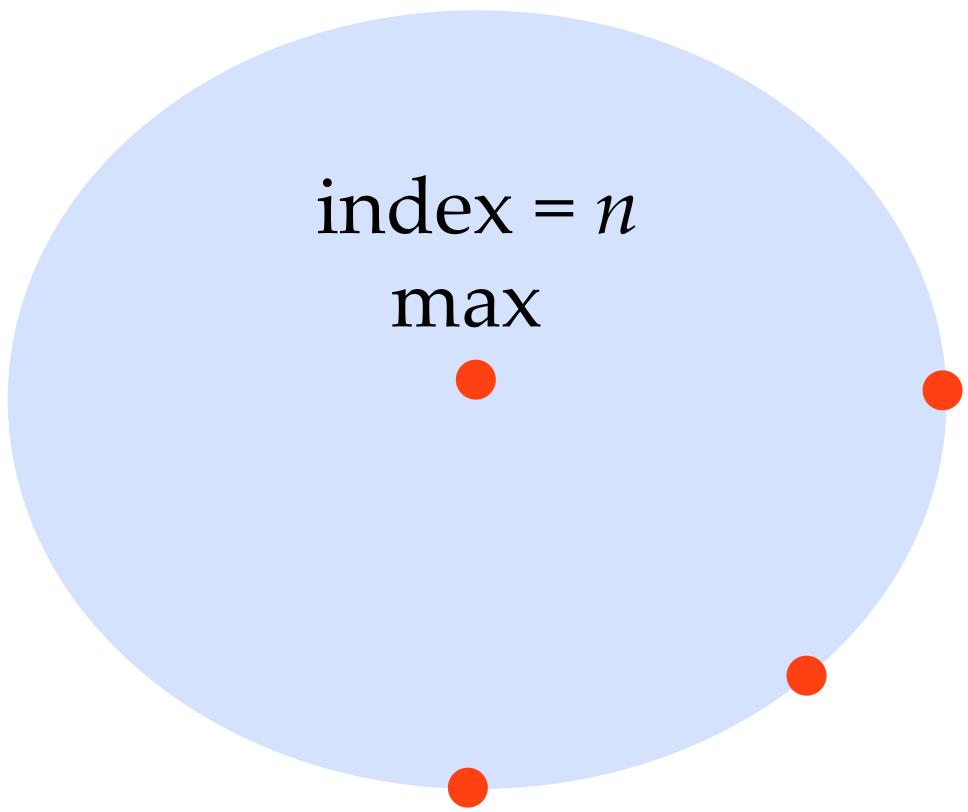

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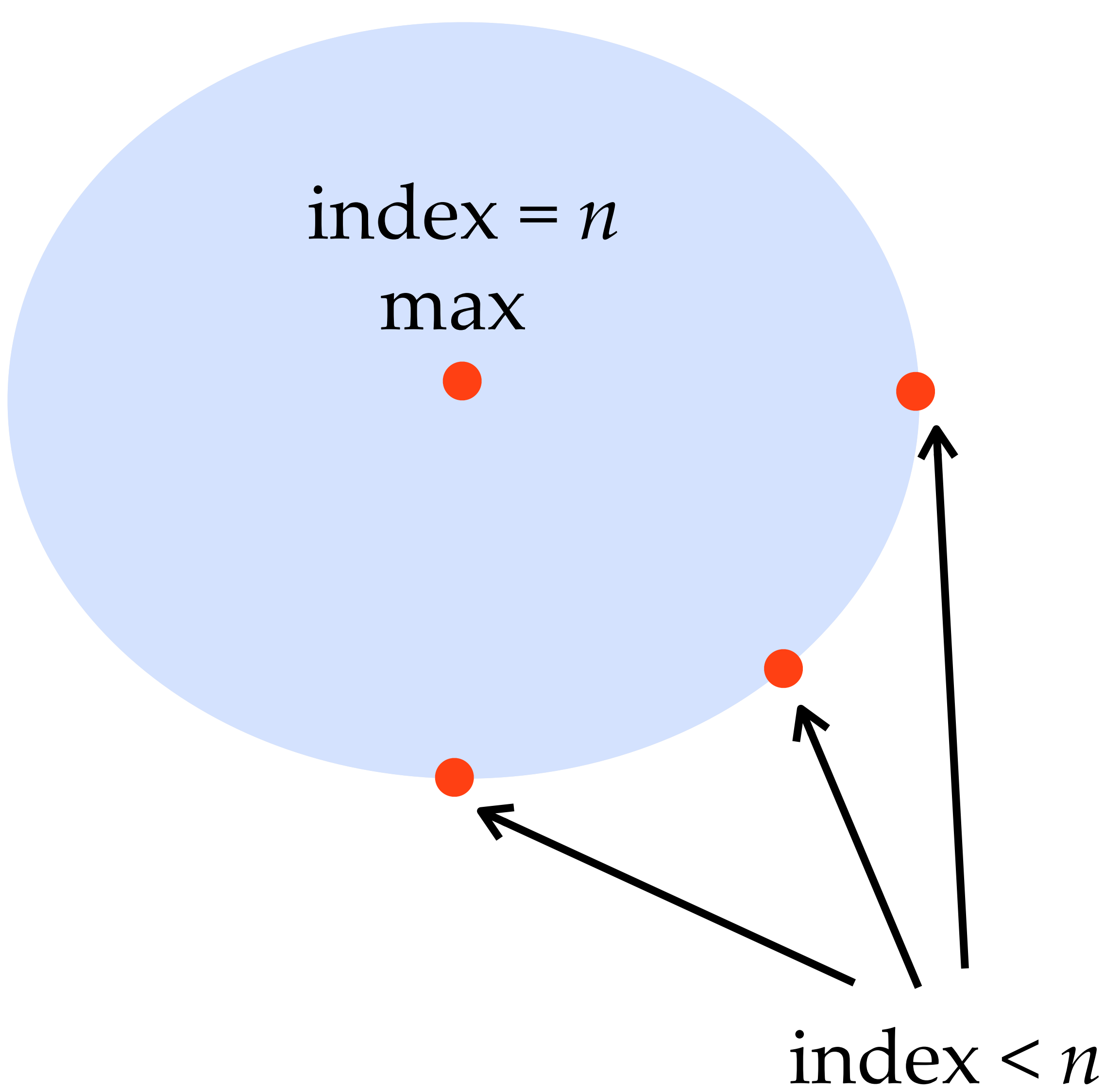

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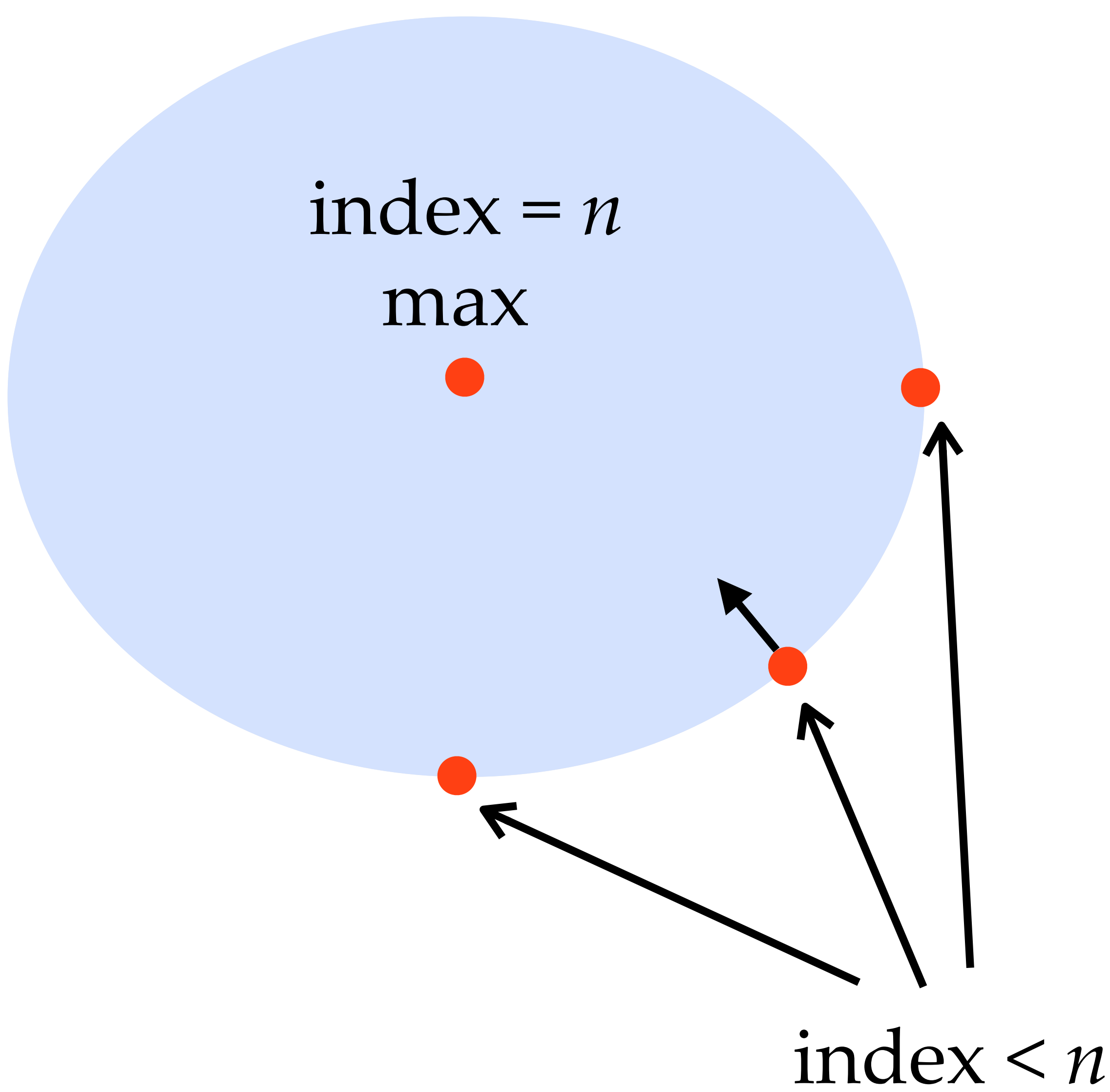

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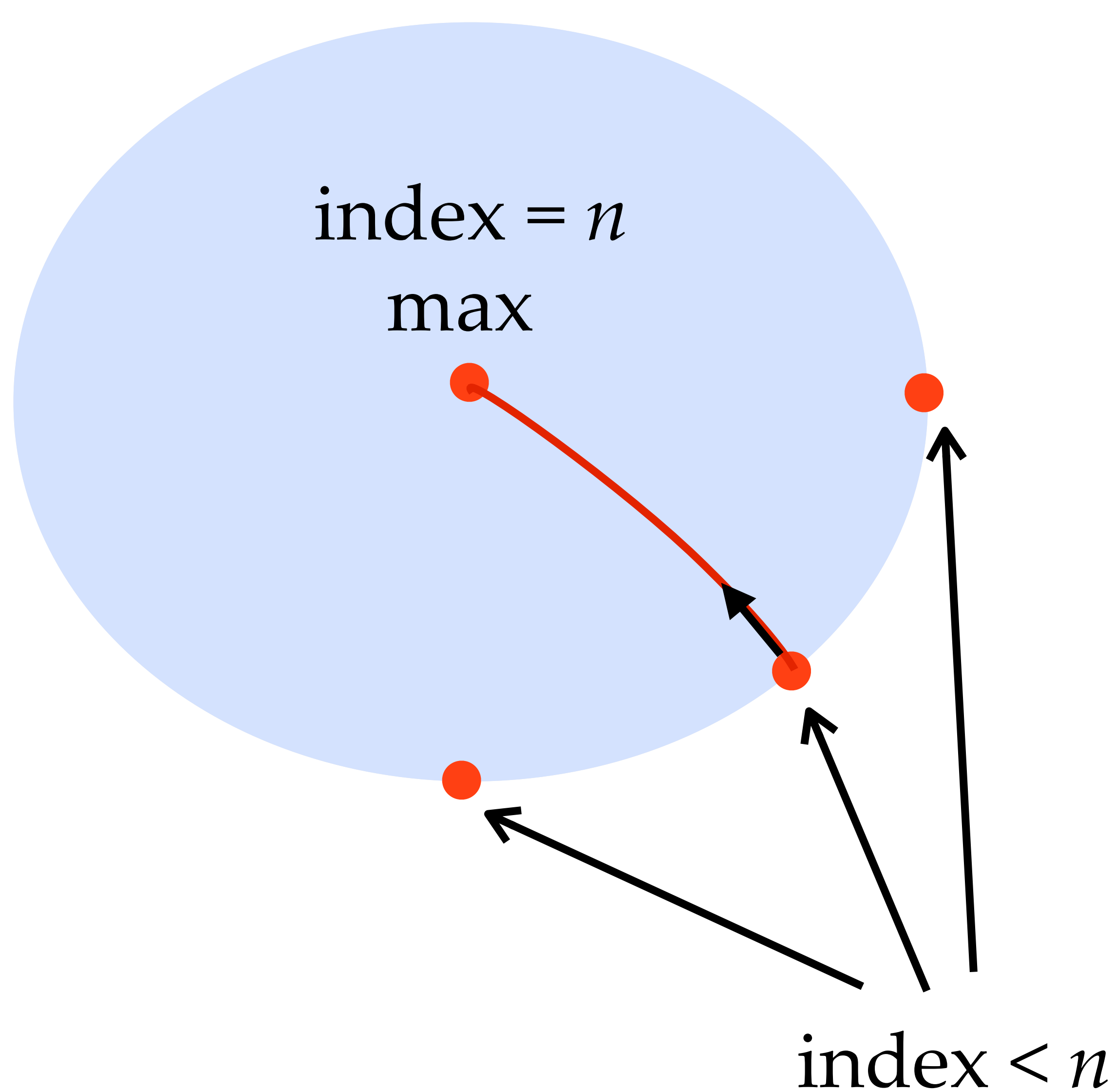

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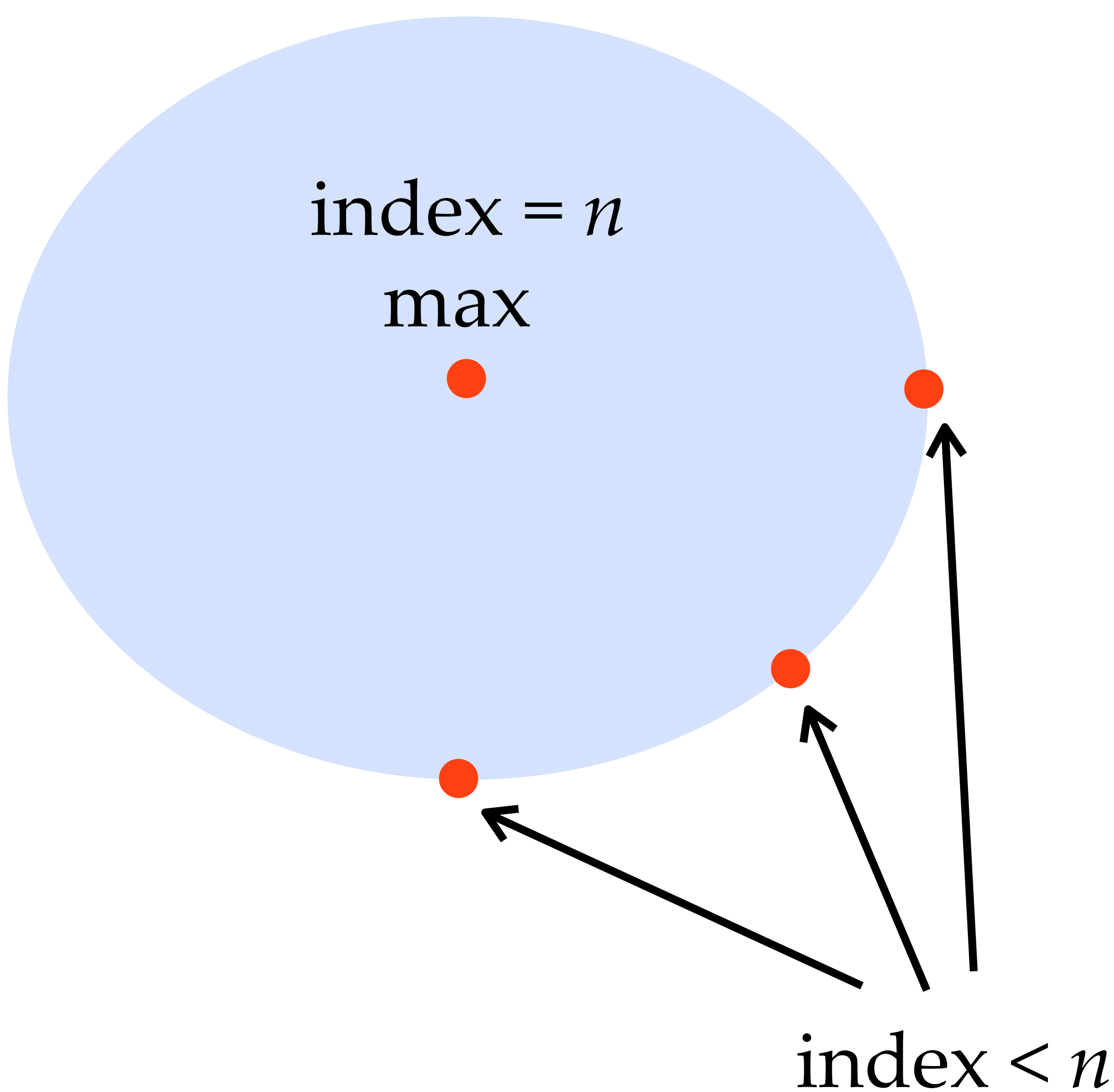

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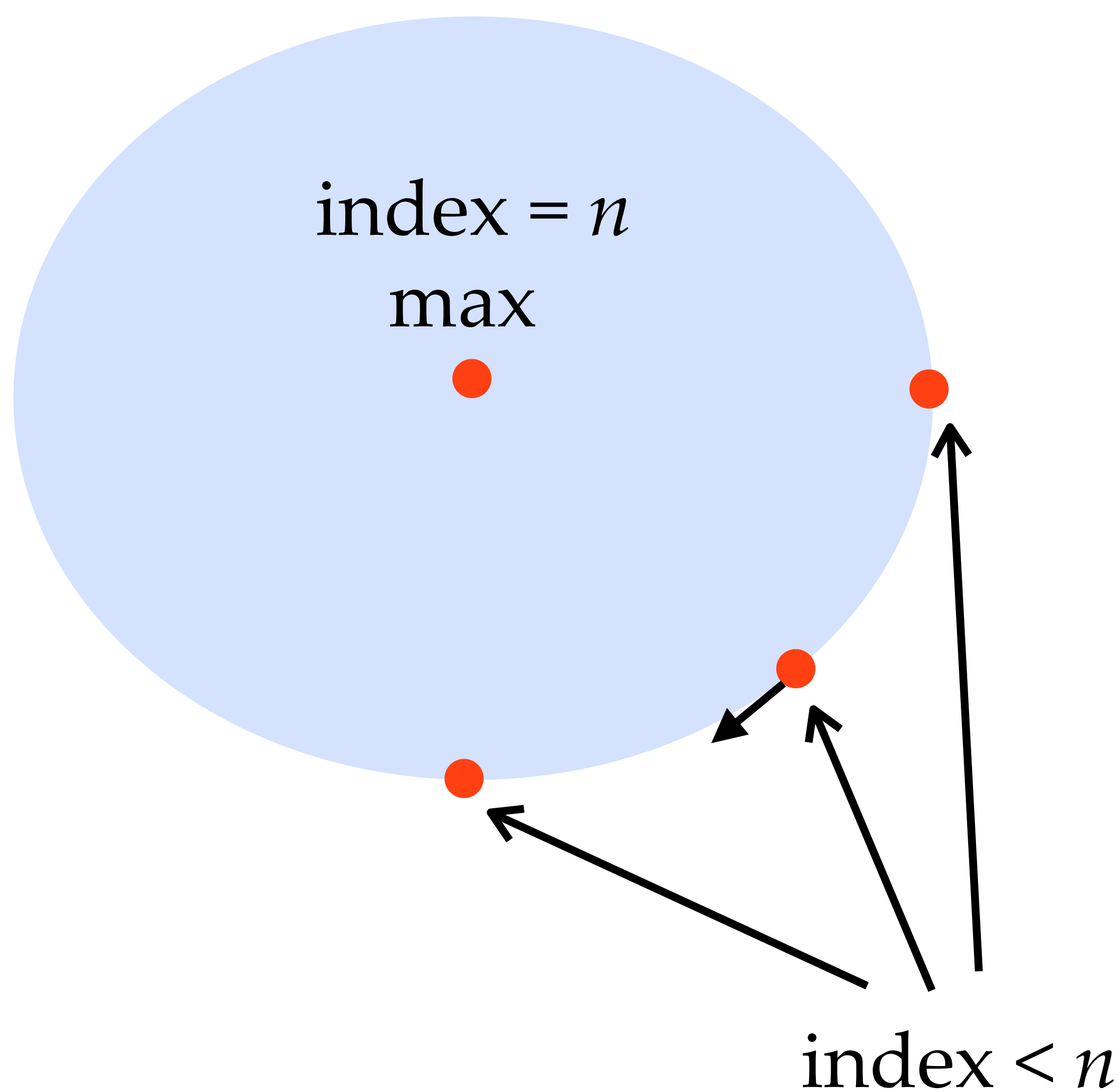

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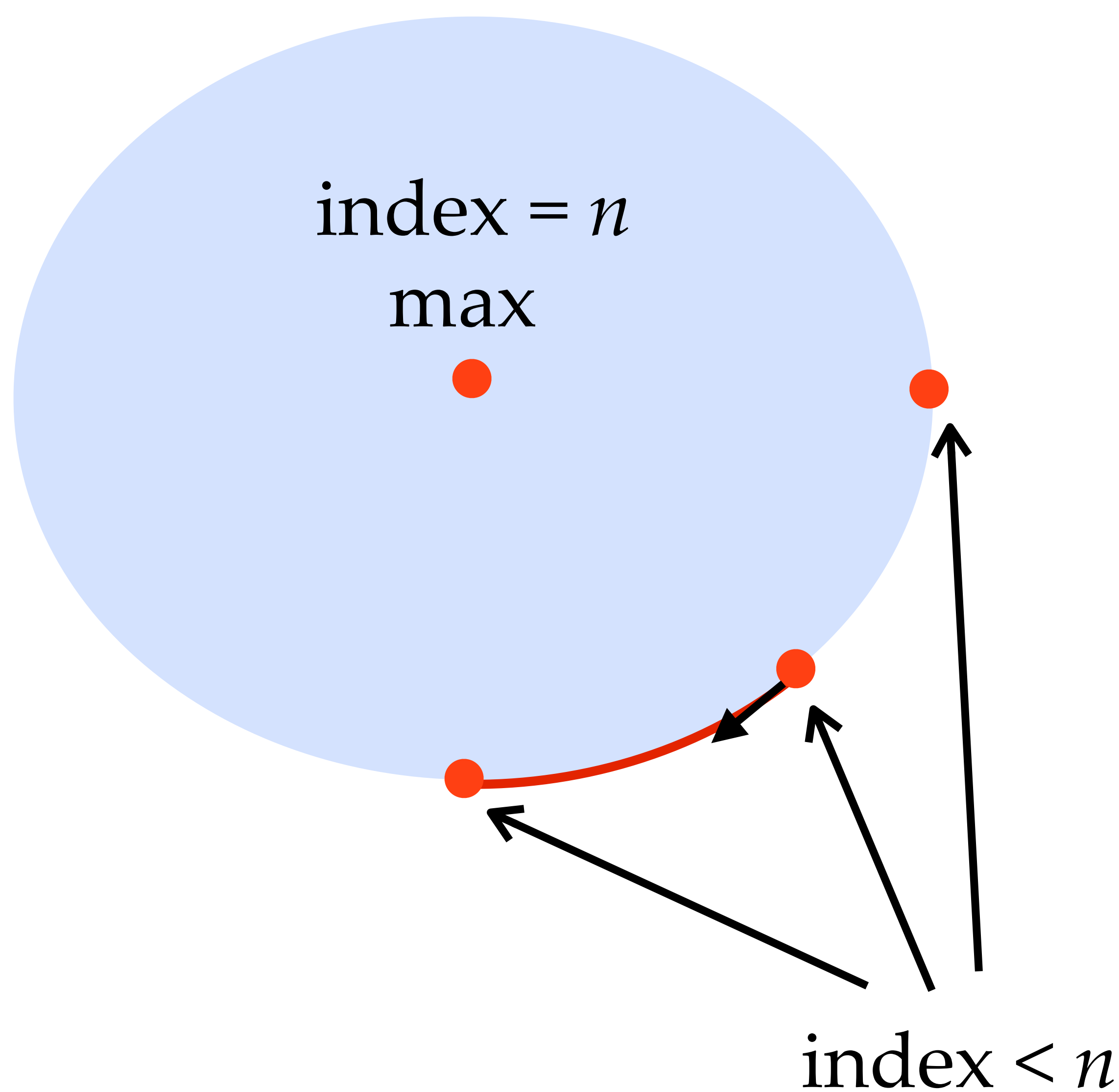

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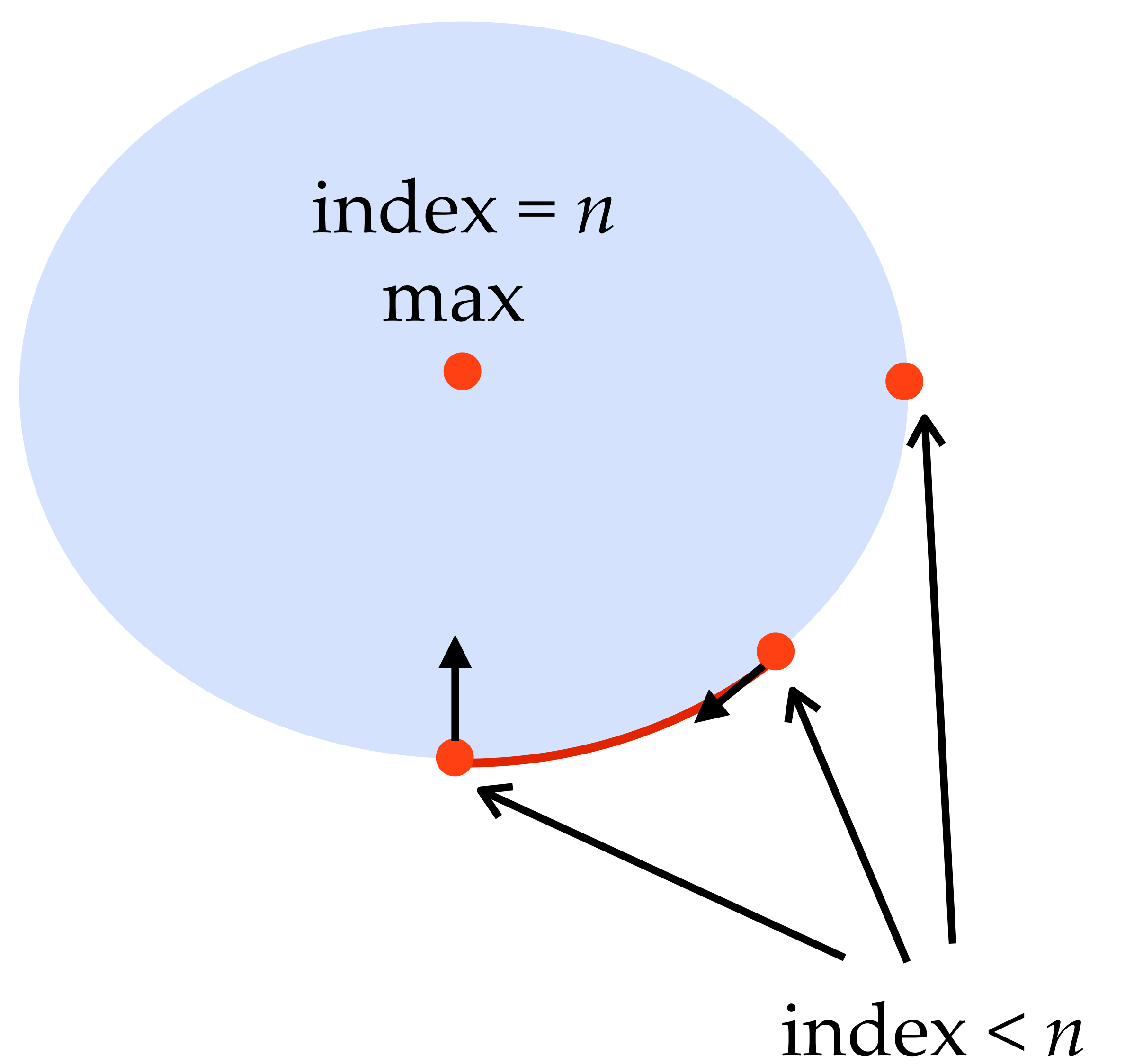

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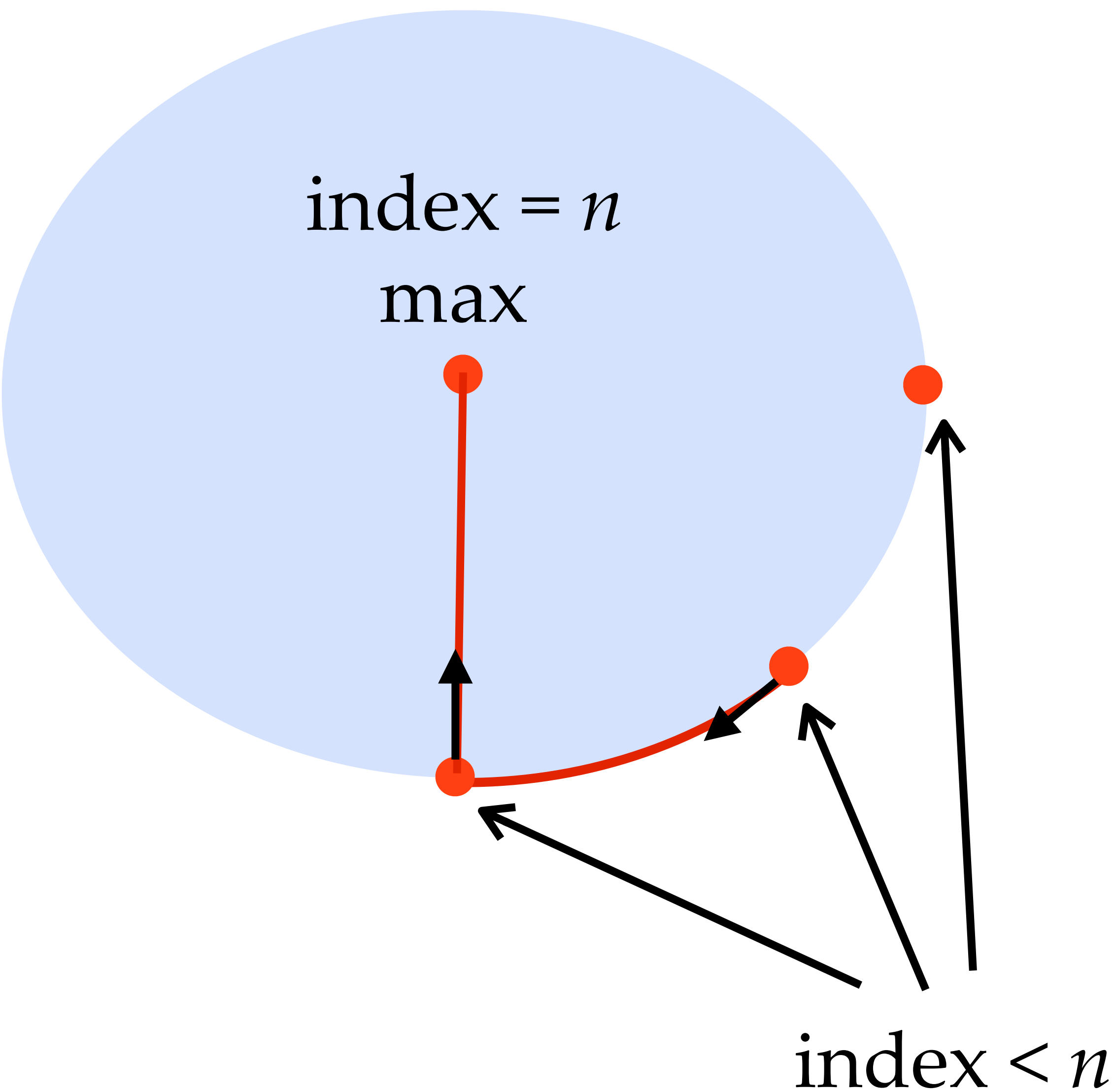

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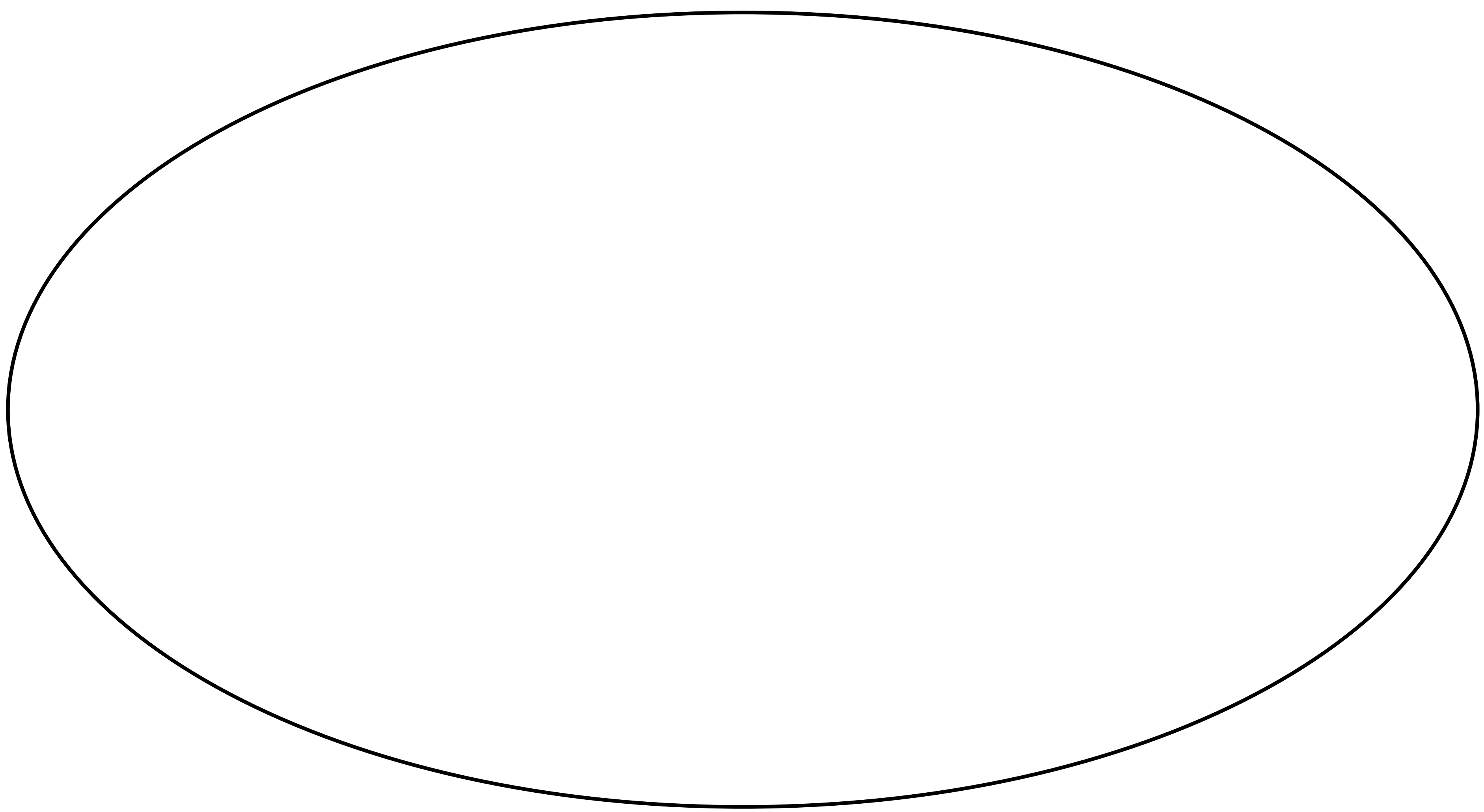

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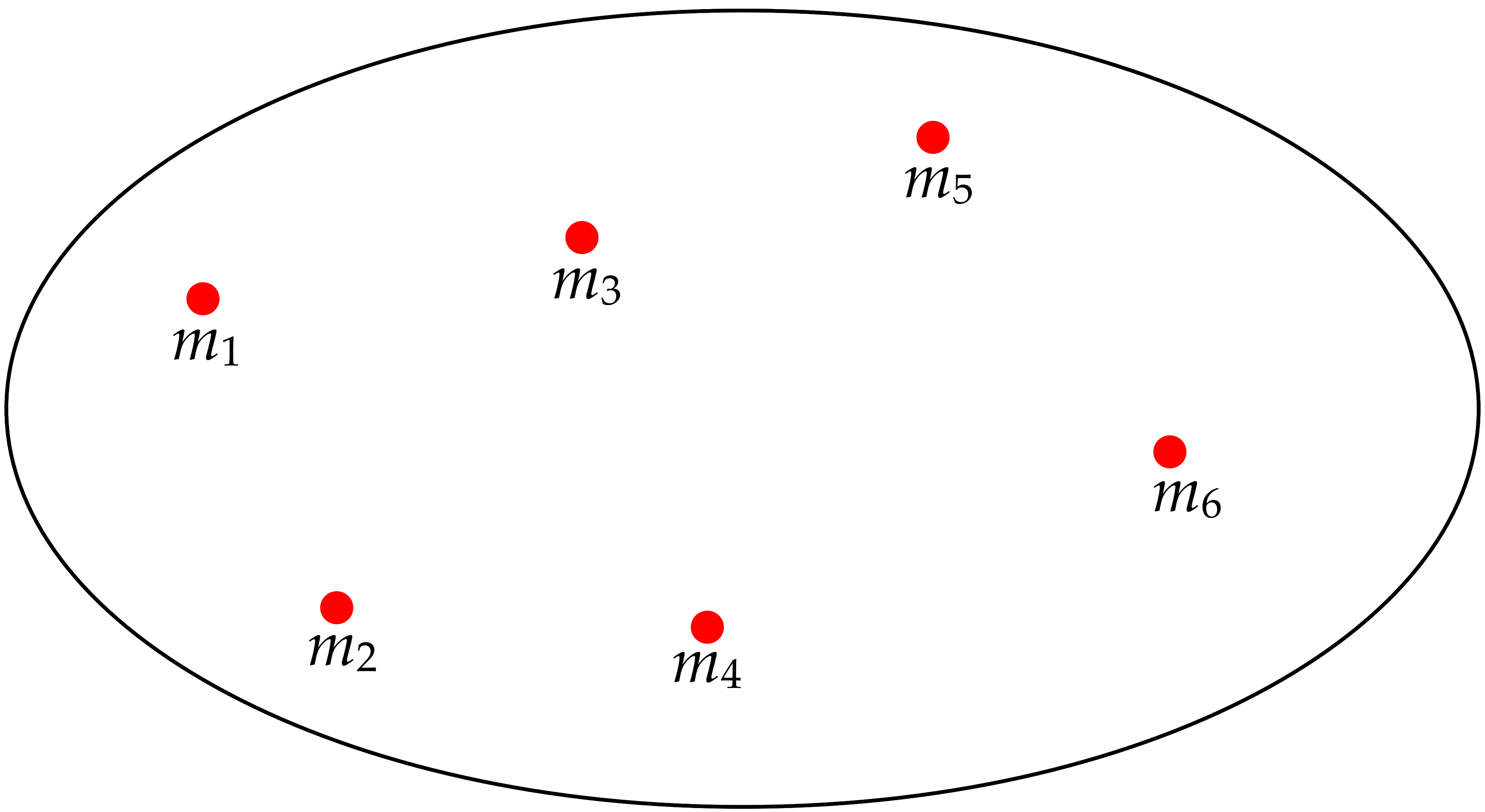

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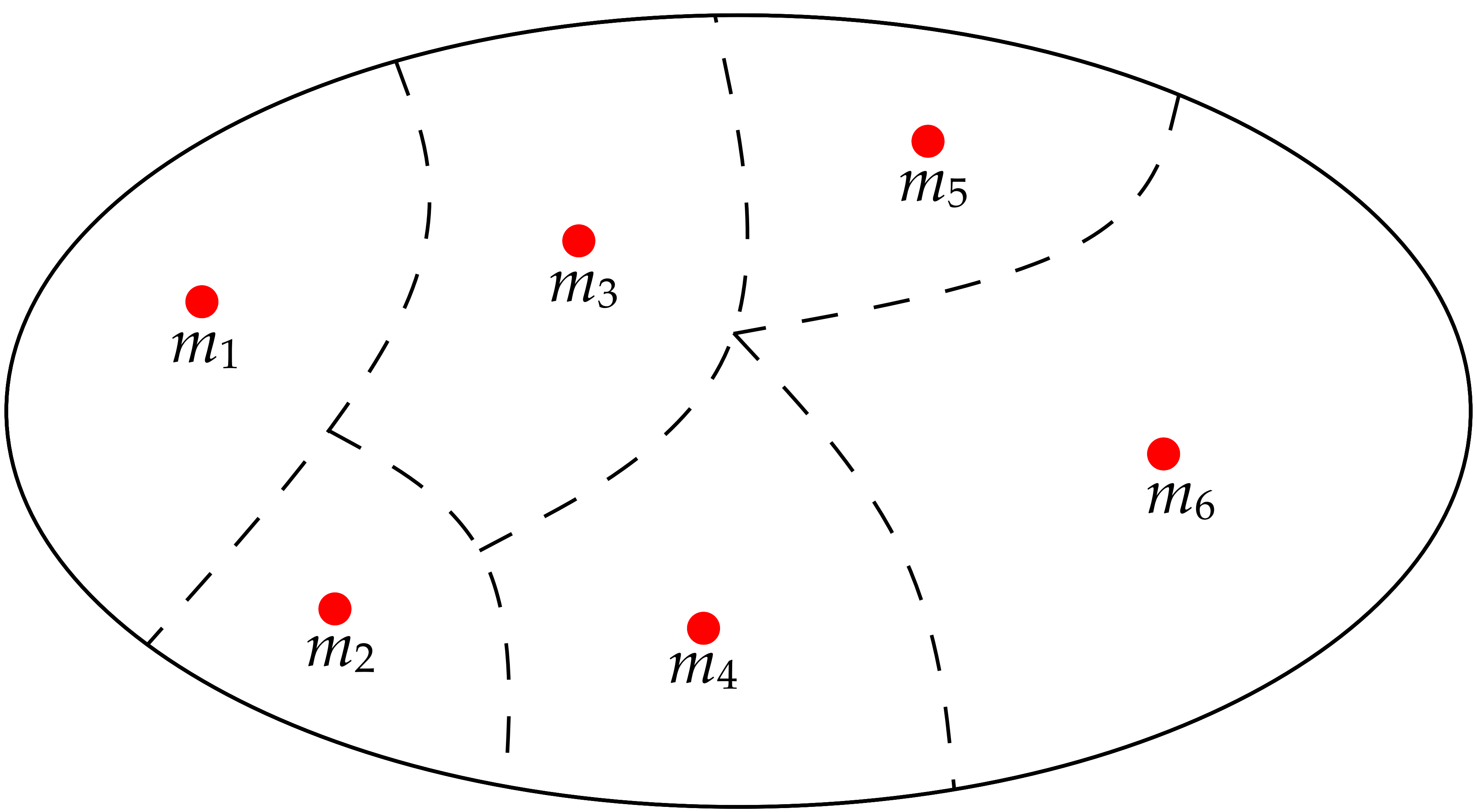

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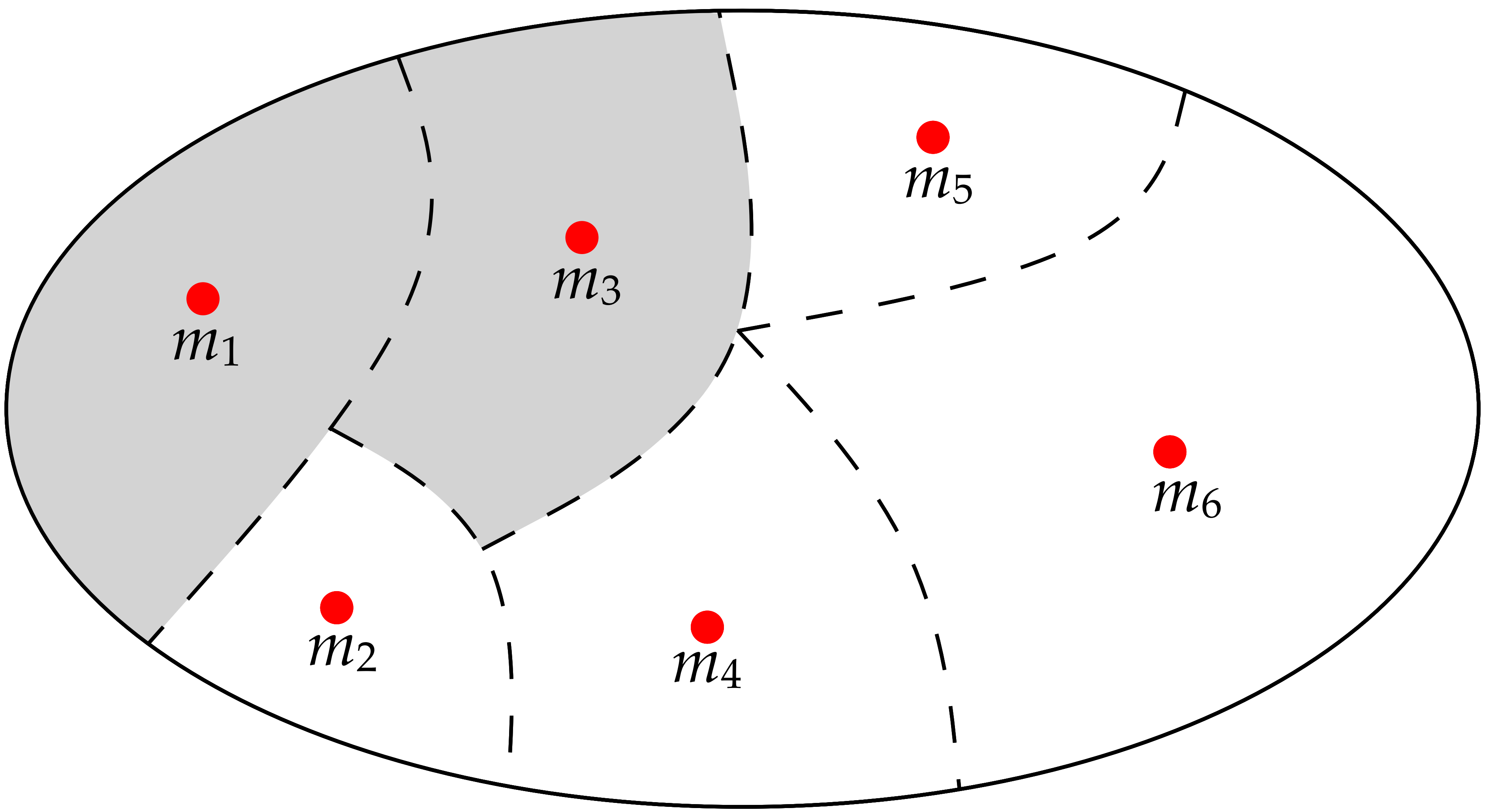

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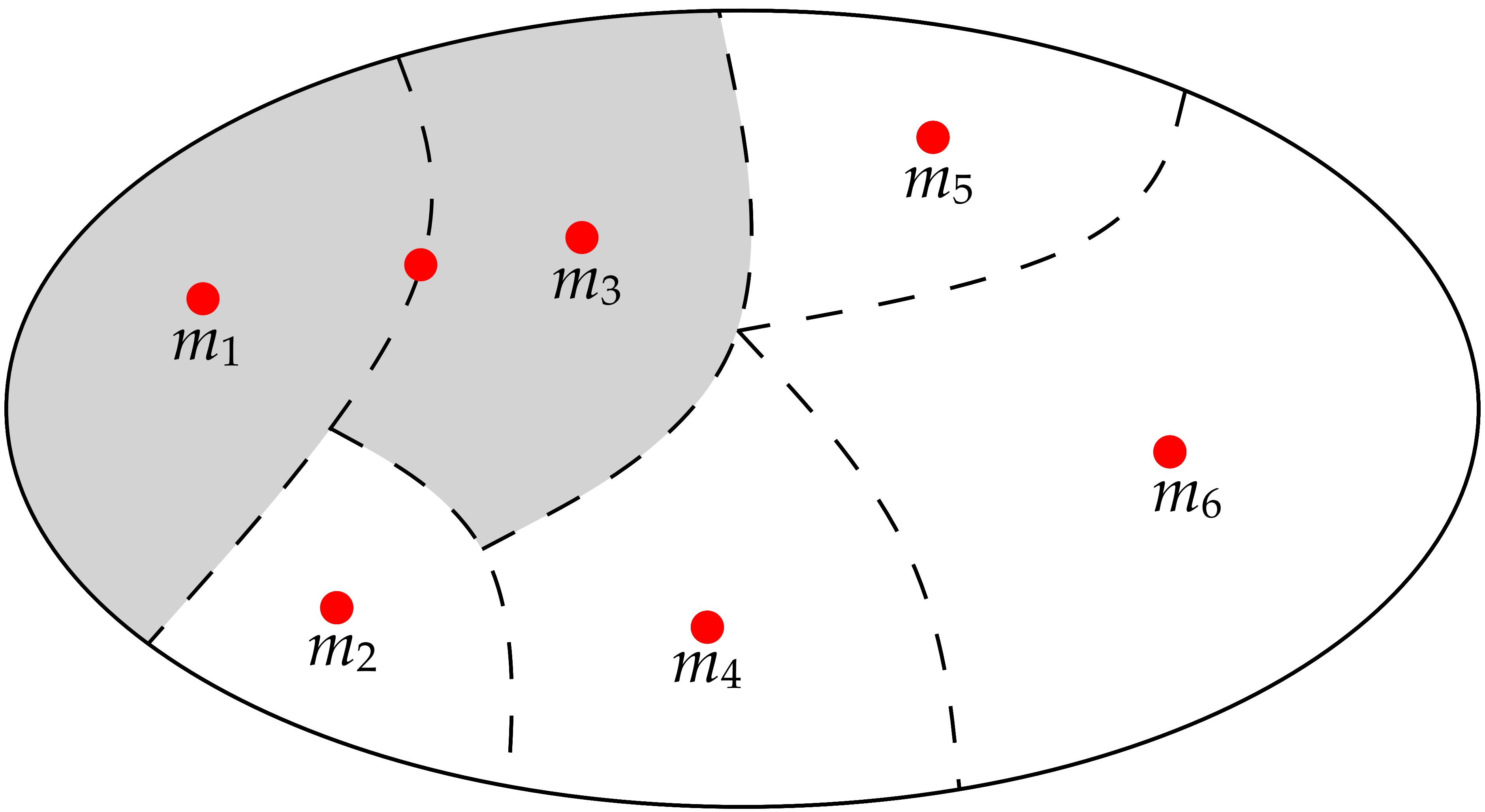

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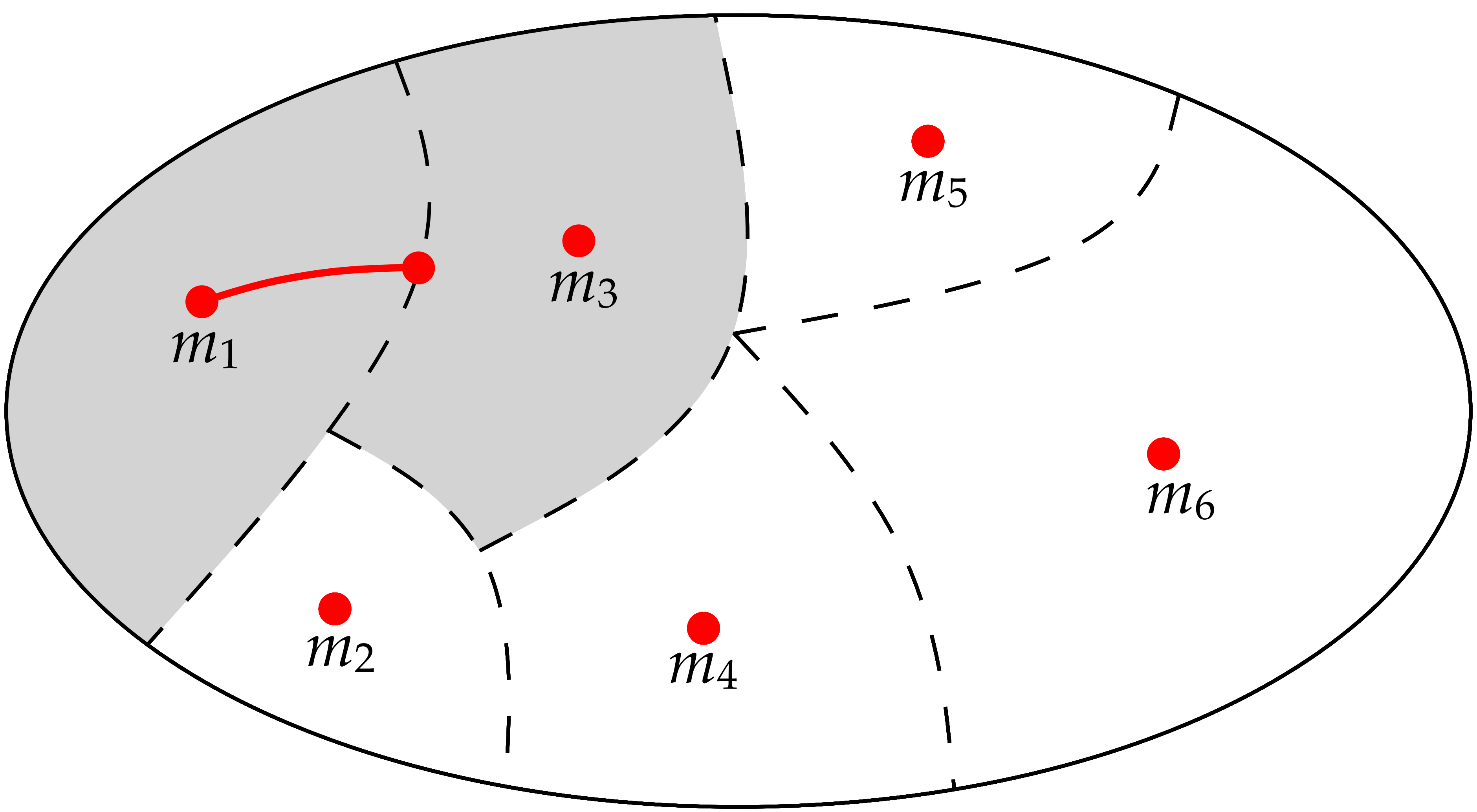

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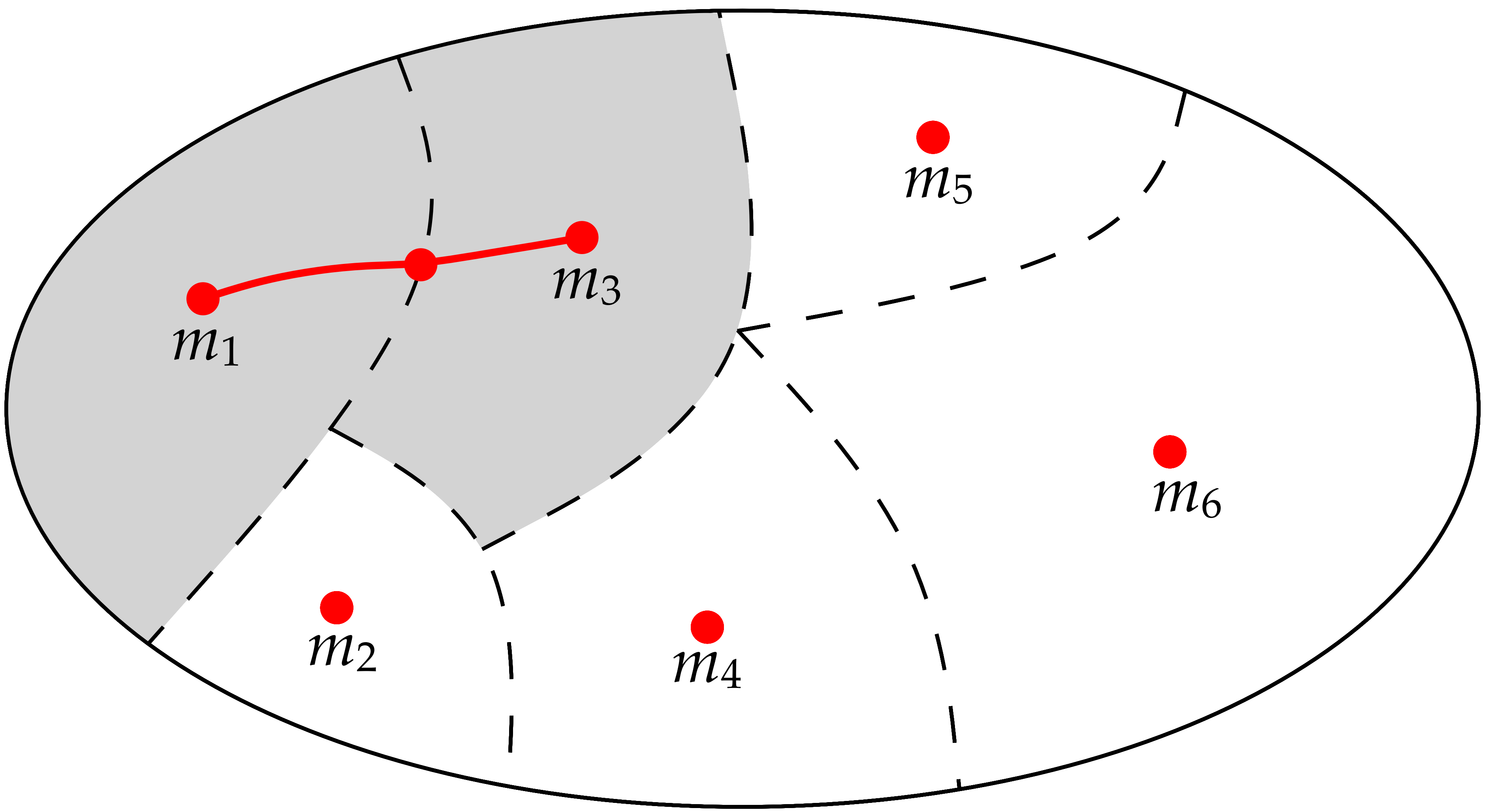

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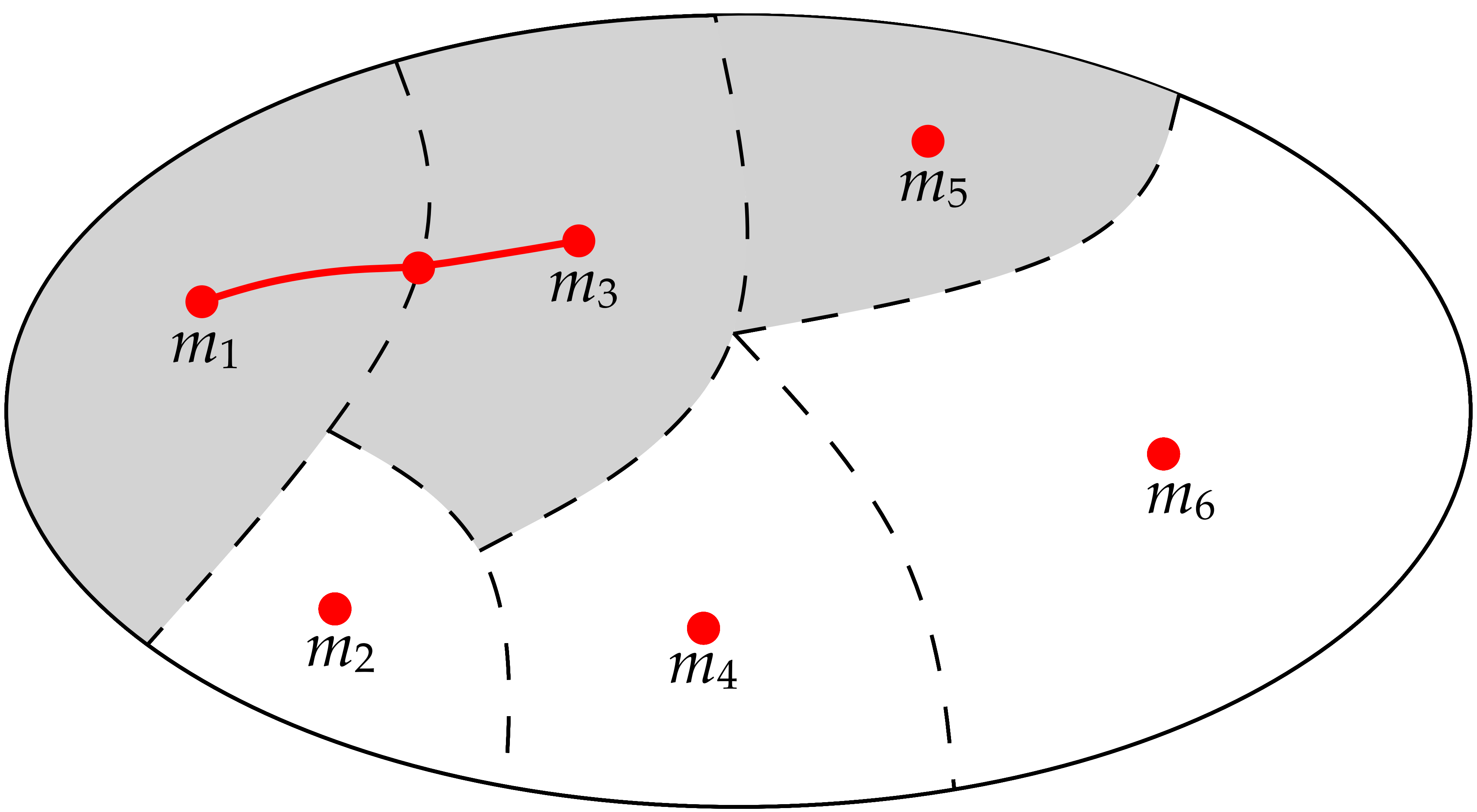

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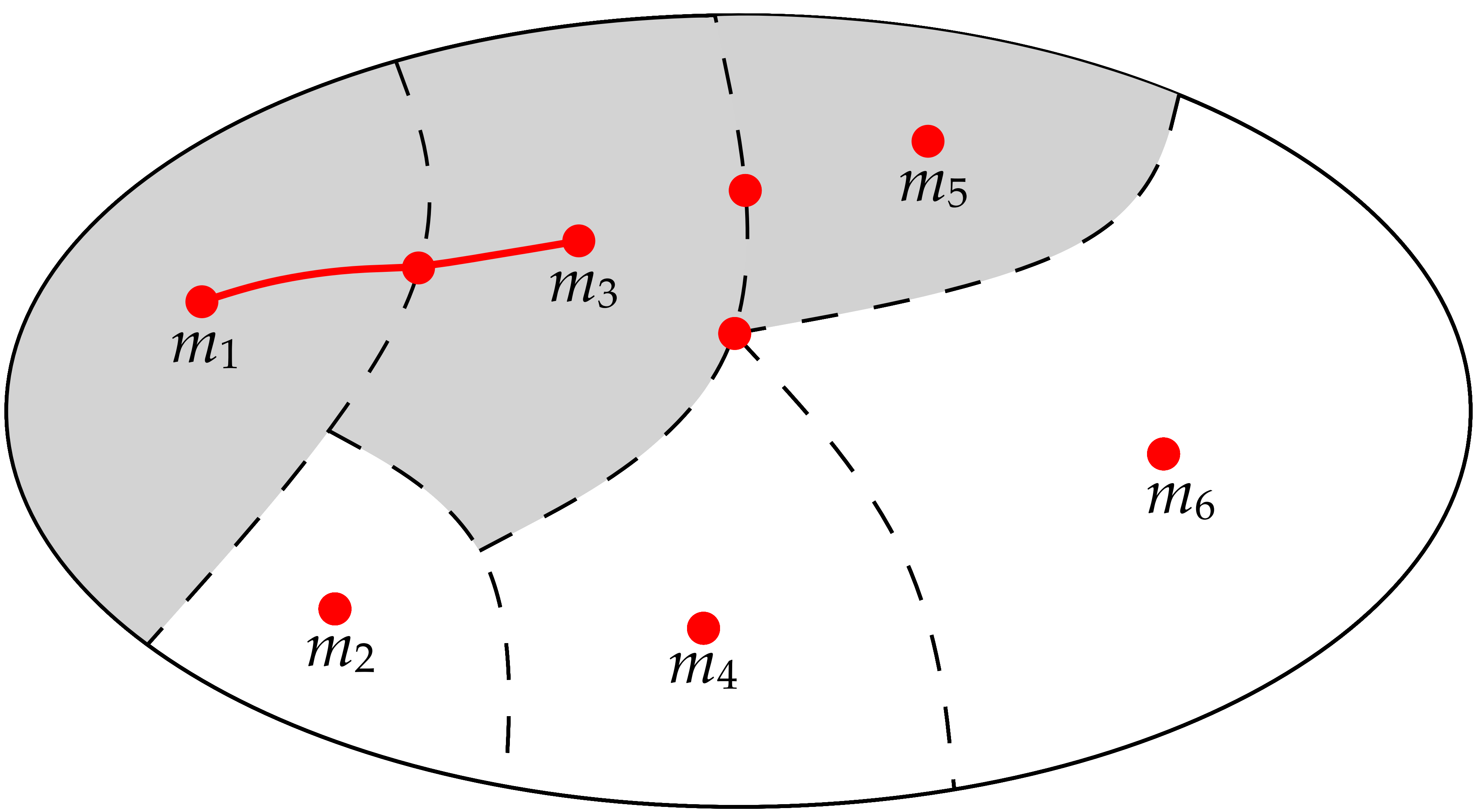

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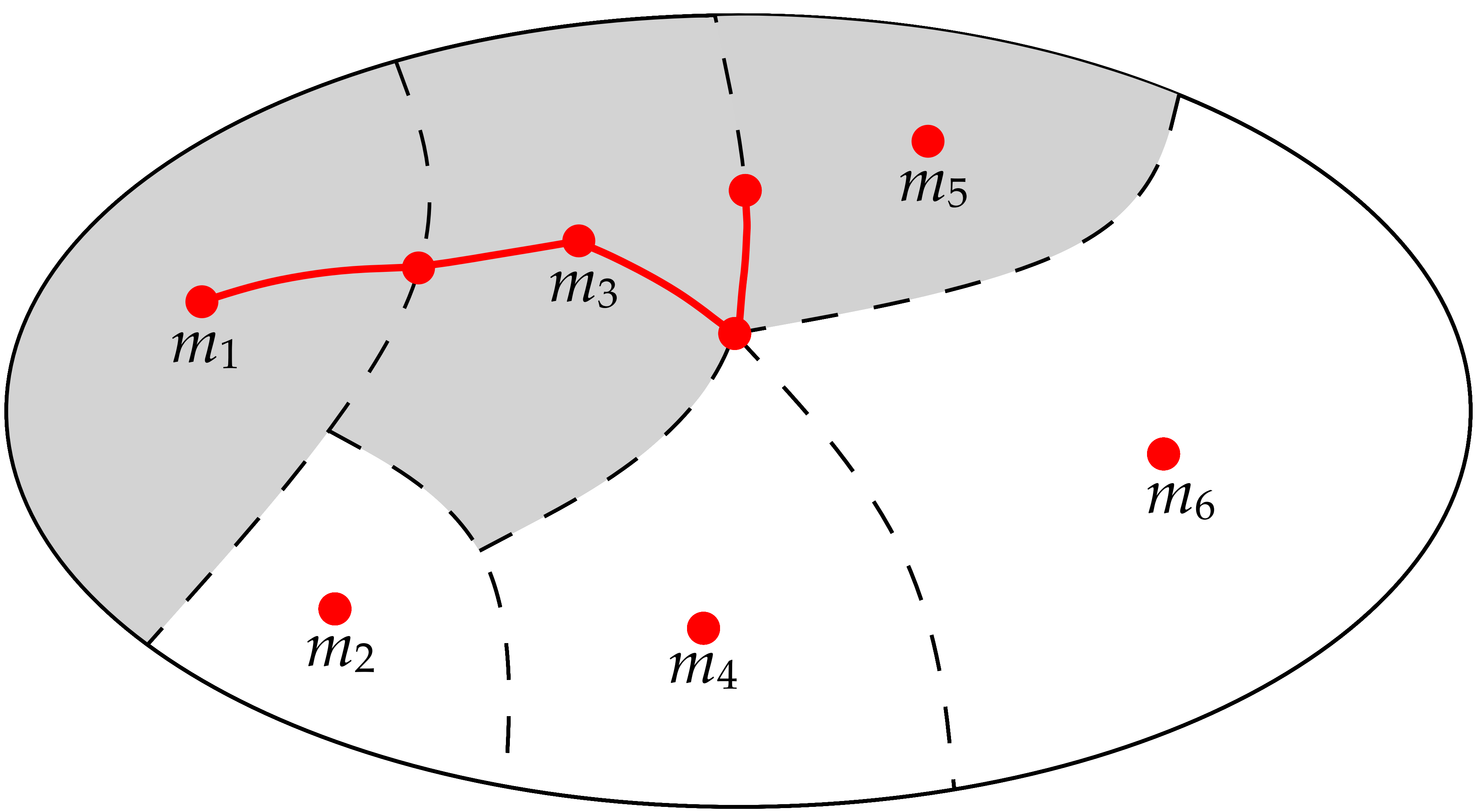

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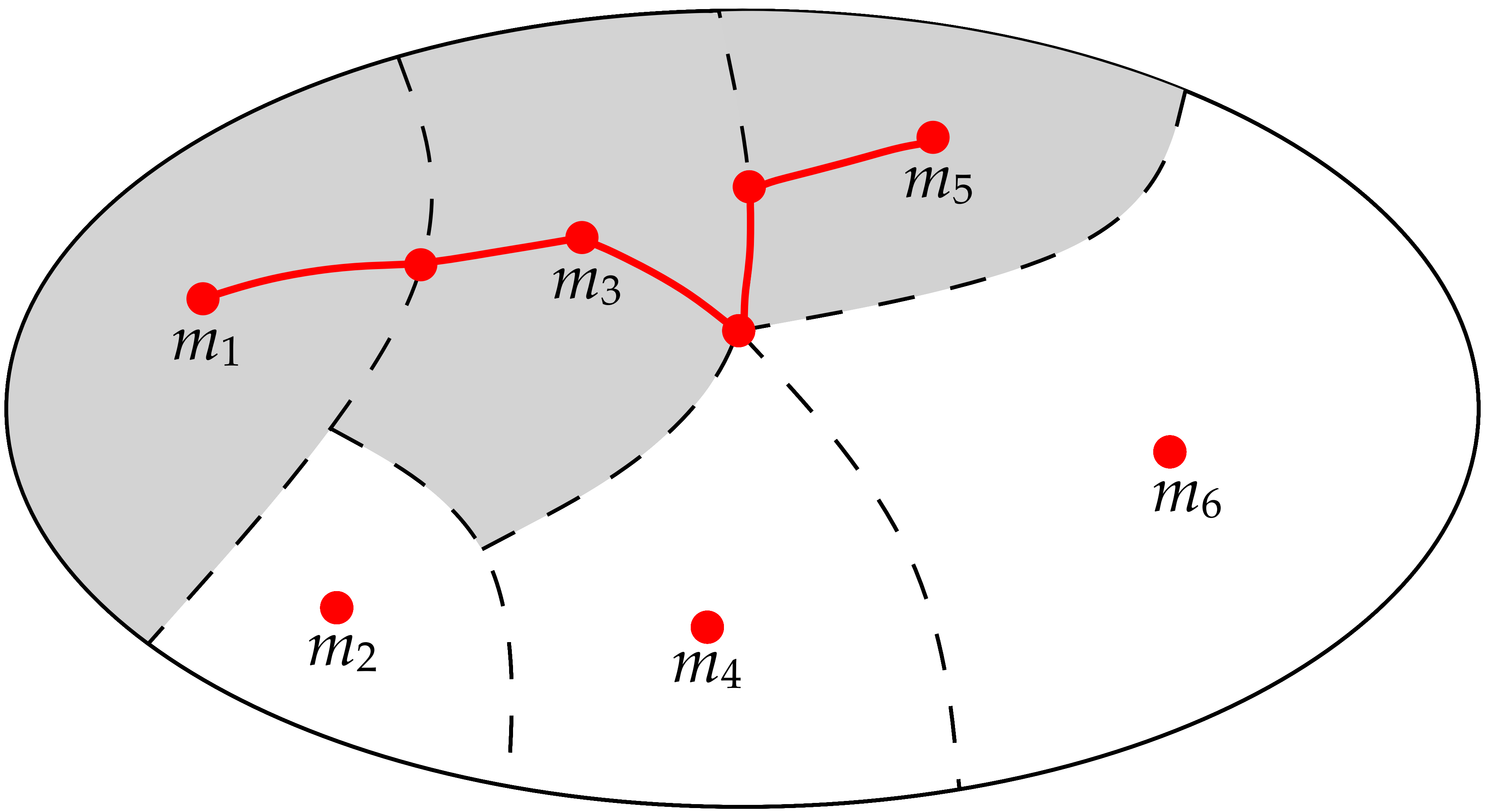

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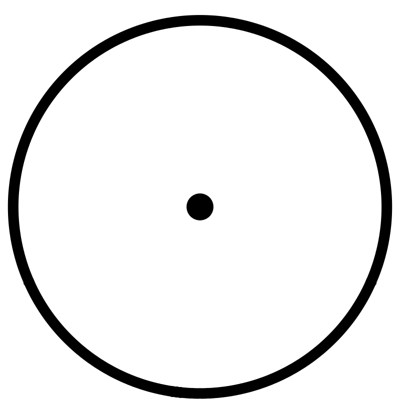

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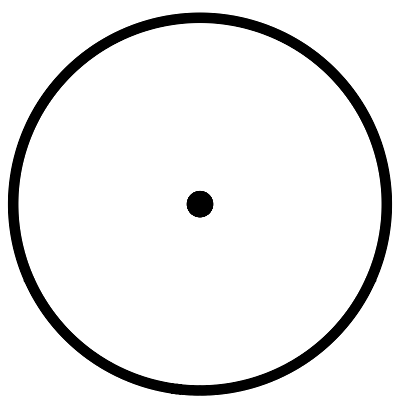

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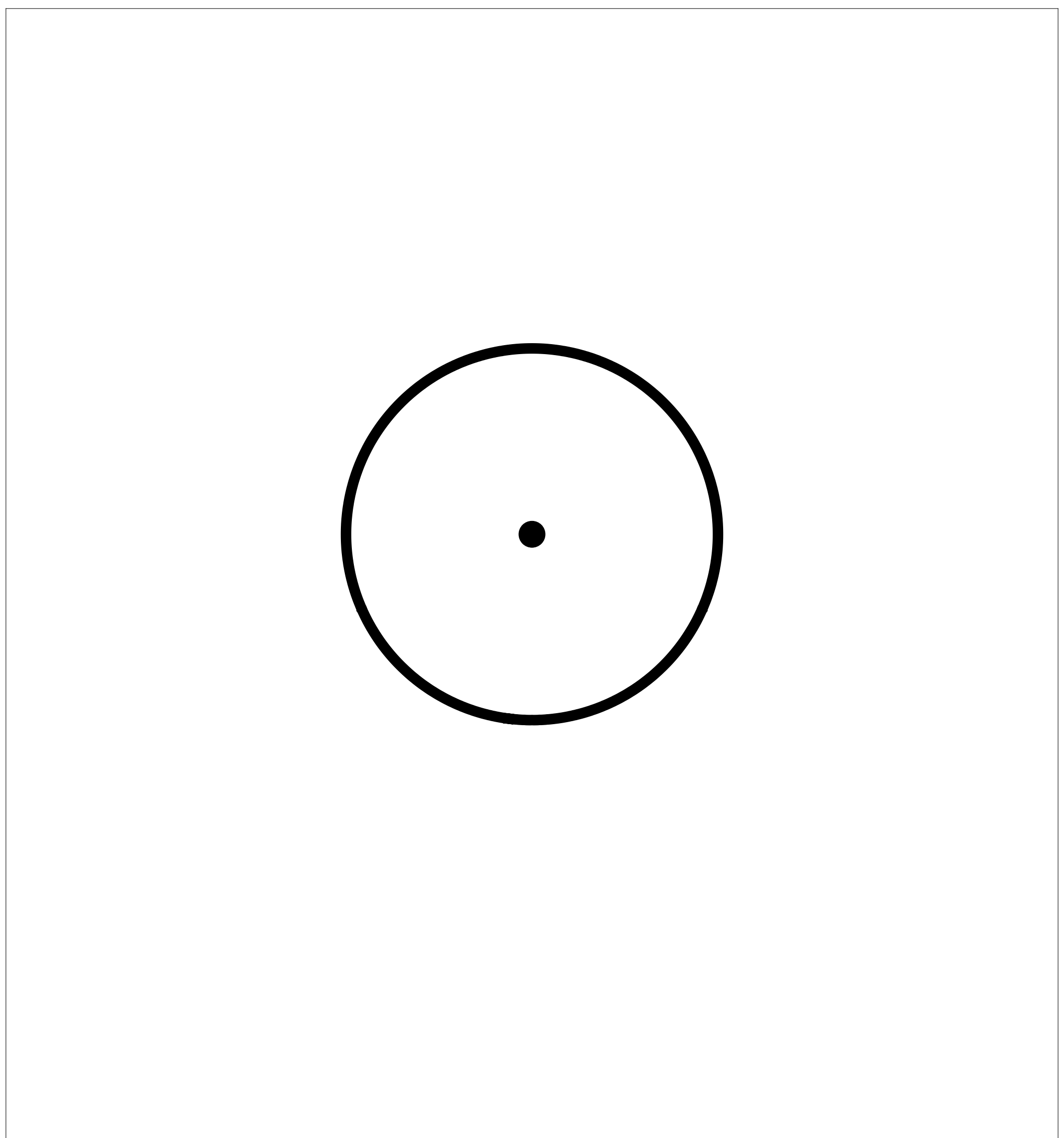

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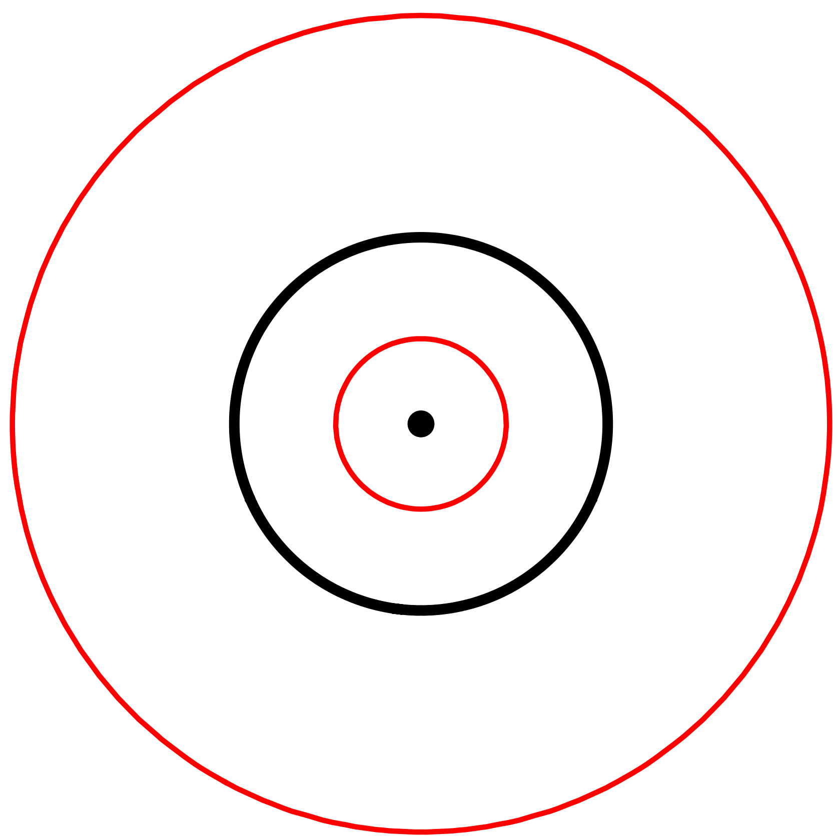

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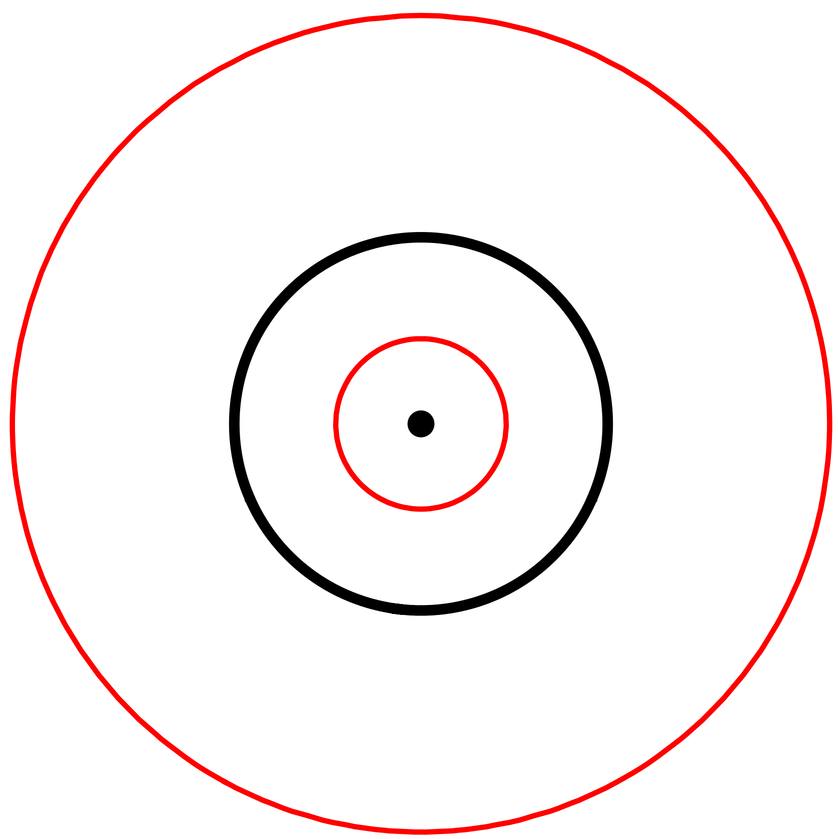

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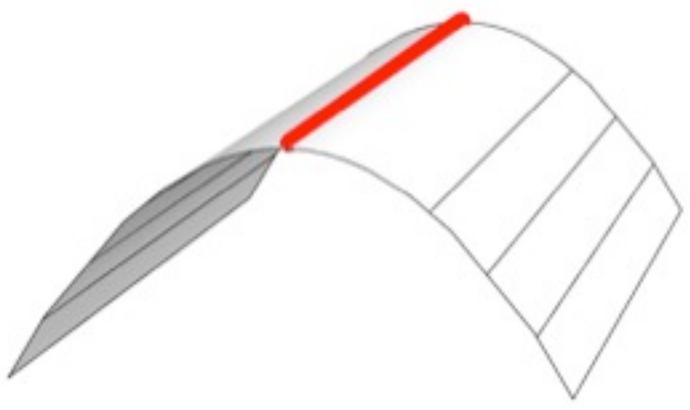

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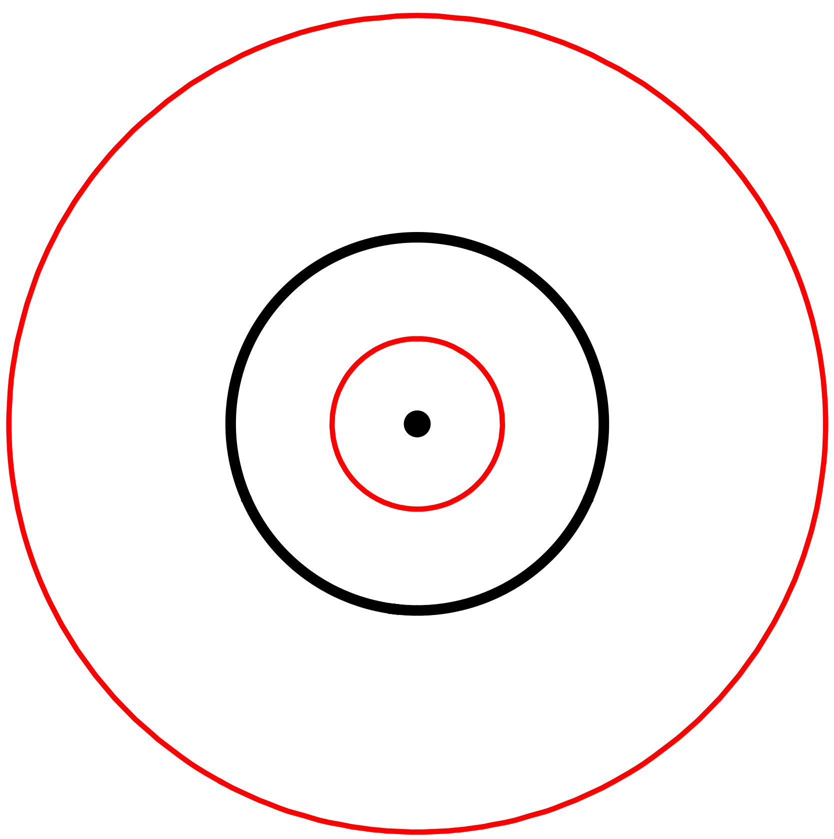

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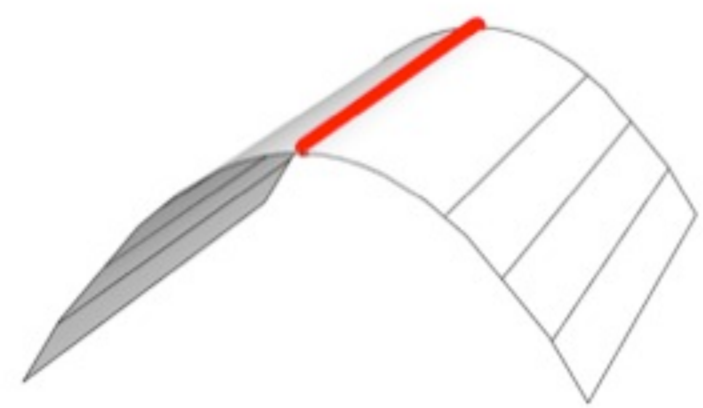

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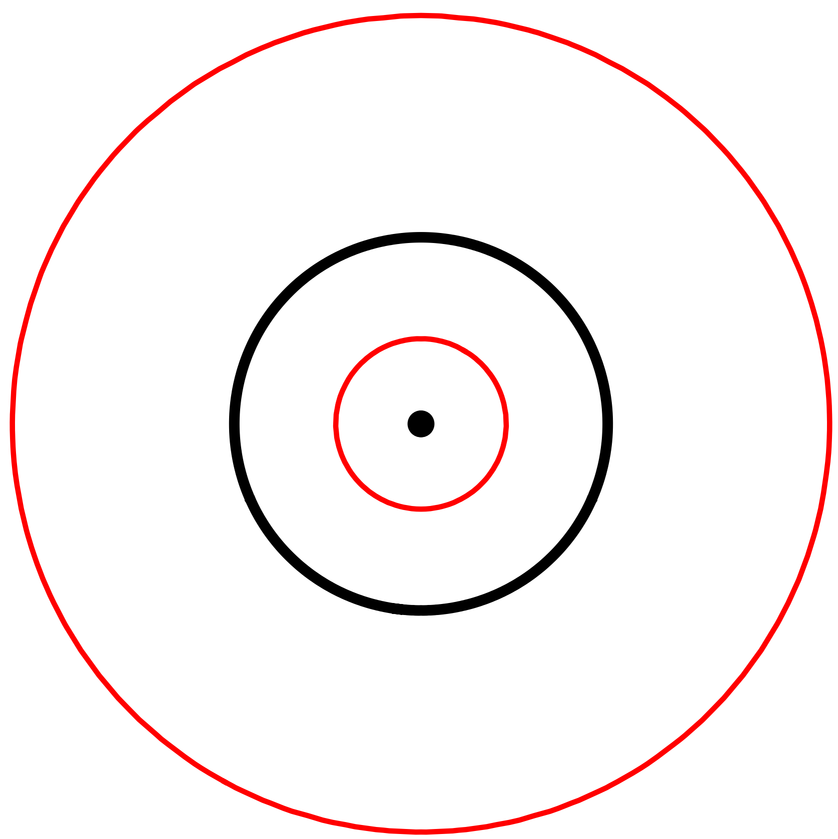

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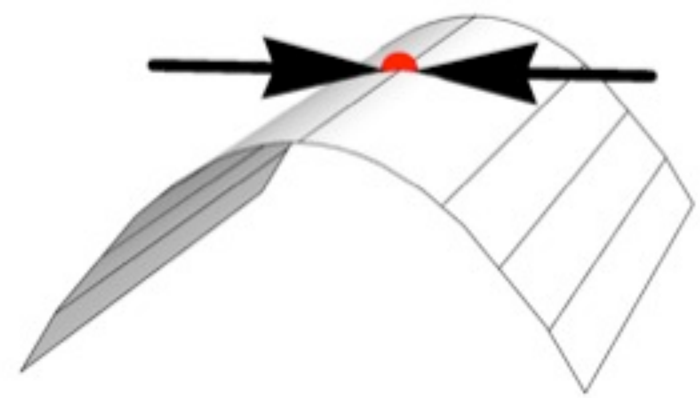

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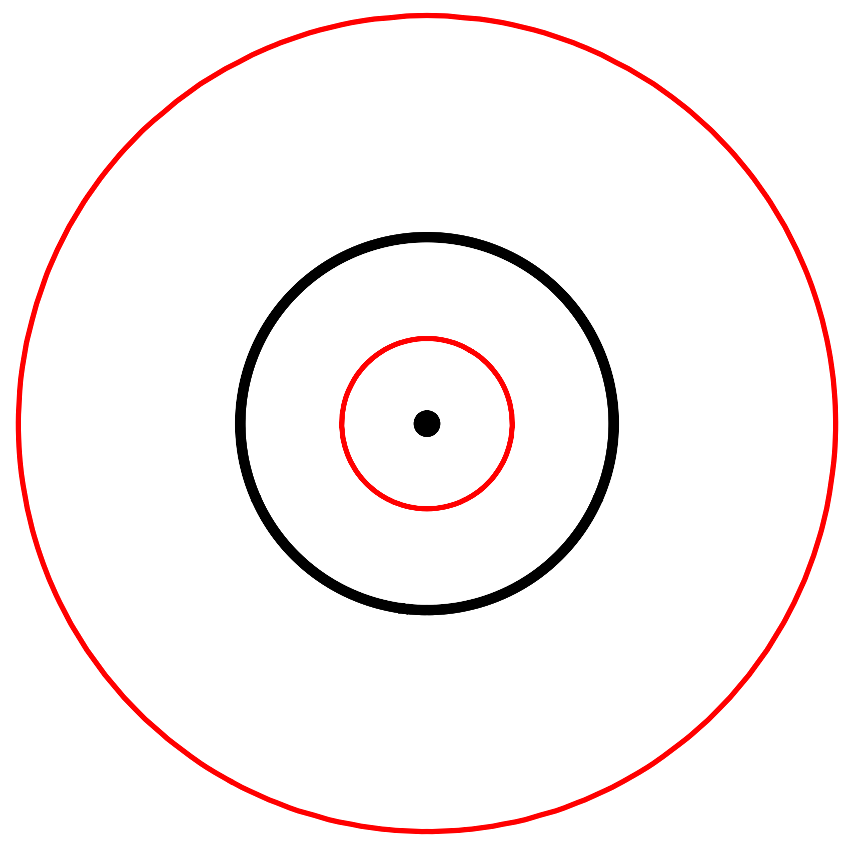

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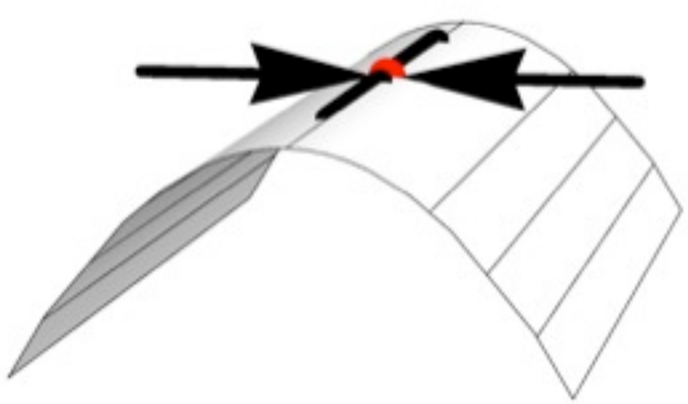

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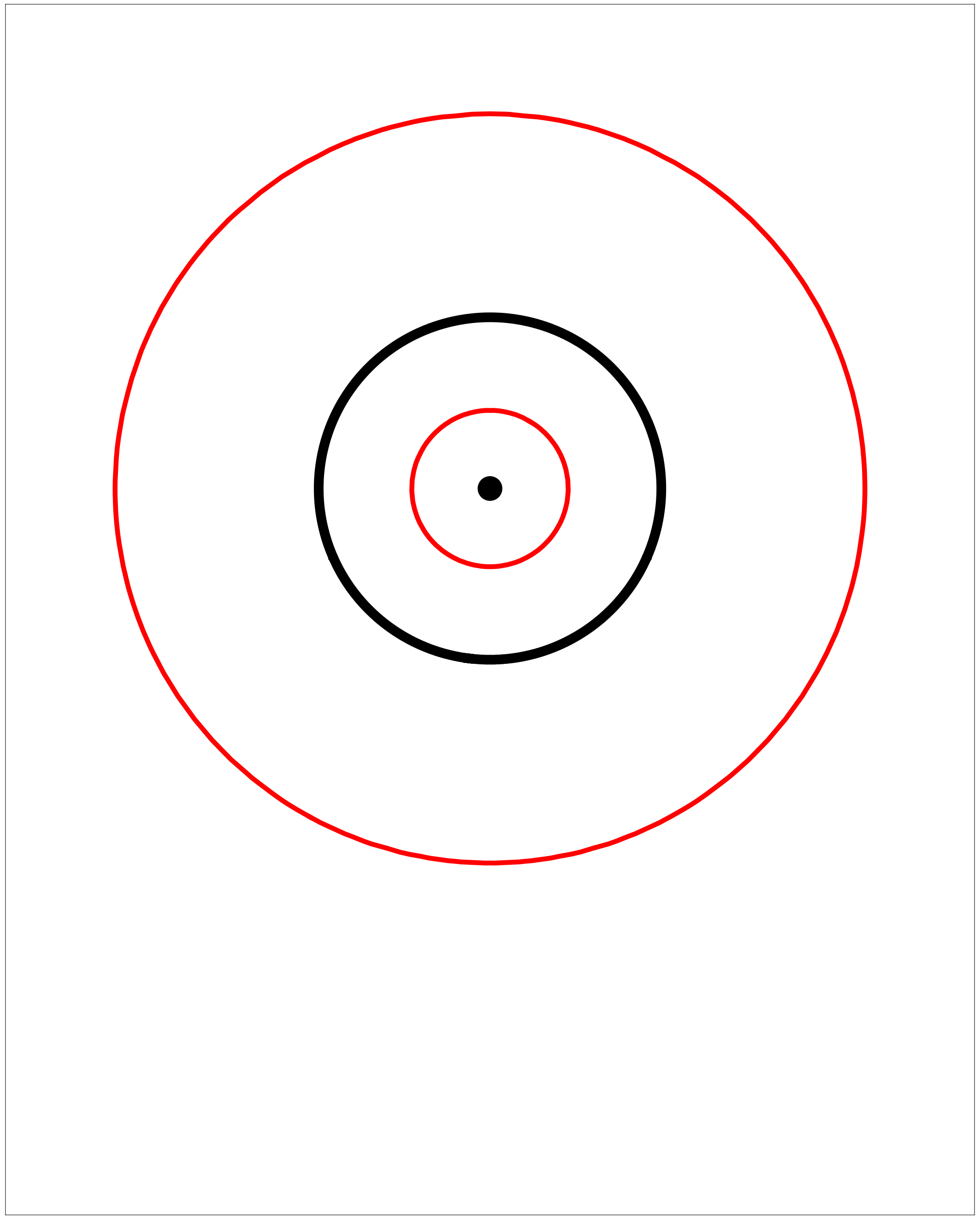

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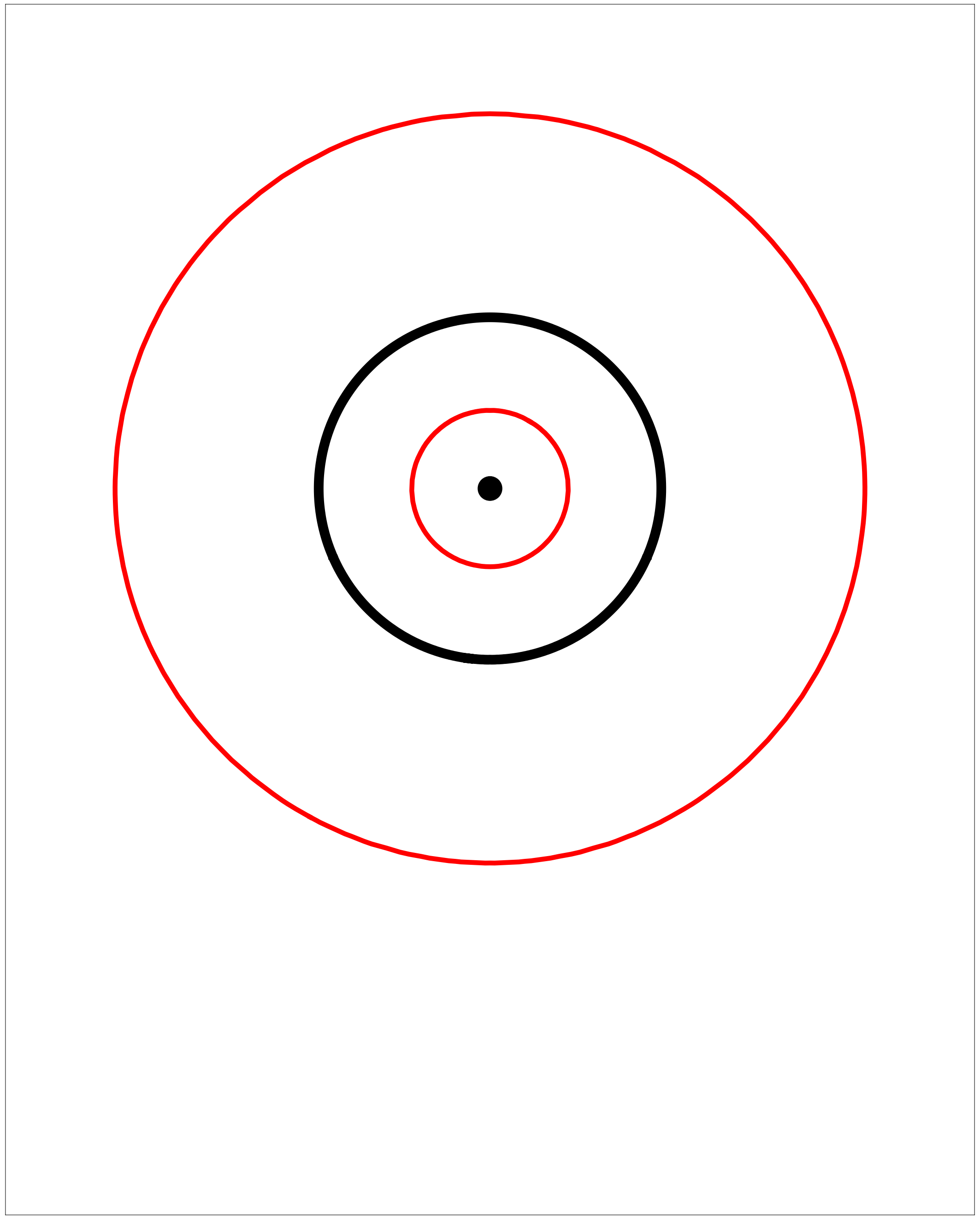
$$g = \frac{f^2}{\left(x_1^2 + x_2^2 + 1\right)^{\deg(f)+1}} \xrightarrow{\text{perturb}} g = \frac{f^2}{\left((x_1 - \color{red}{0}\color{black})^2 + (x_2 - \color{red}{1}\color{black})^2 + 1\right)^{\deg(f)+1}}$$



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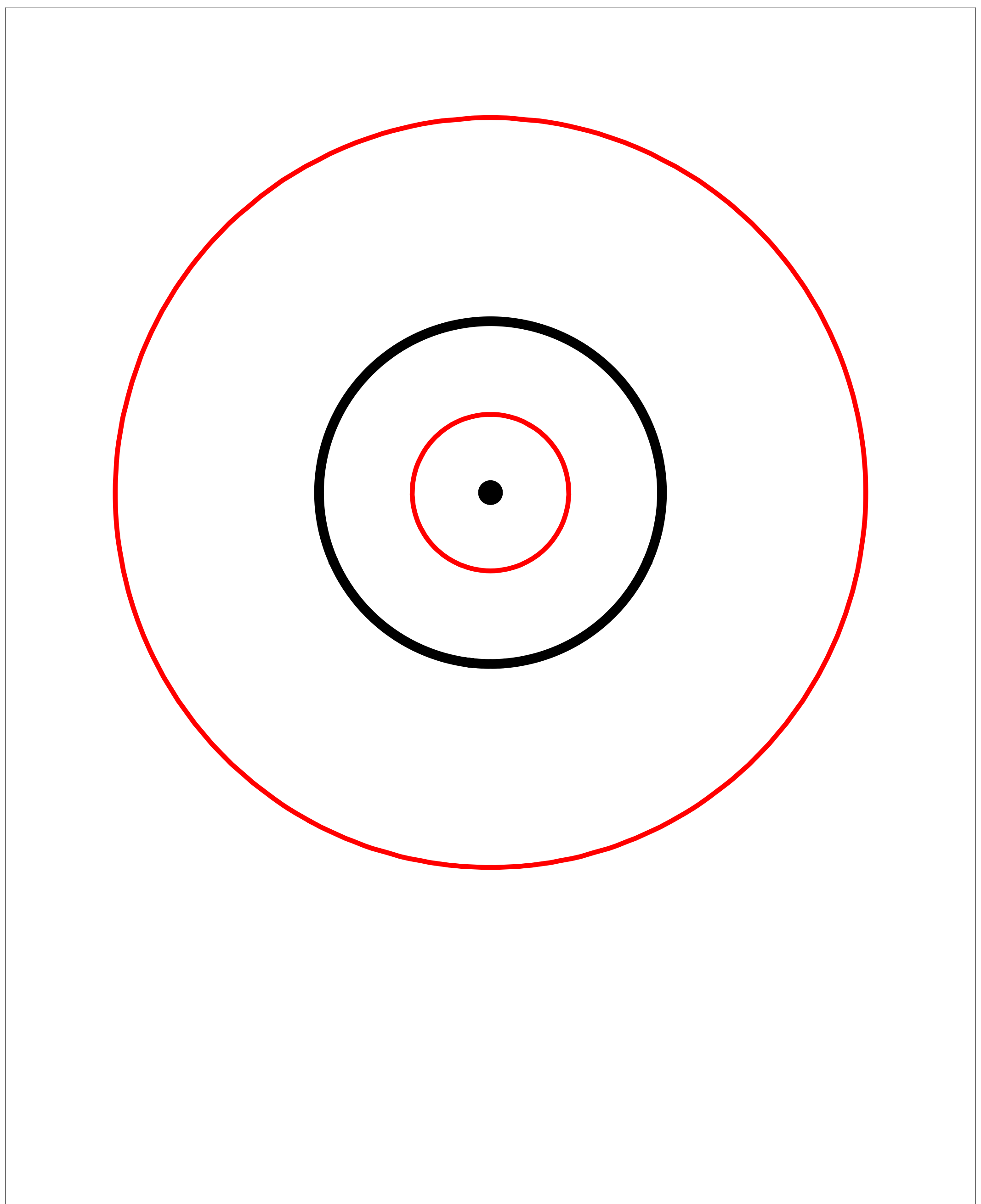

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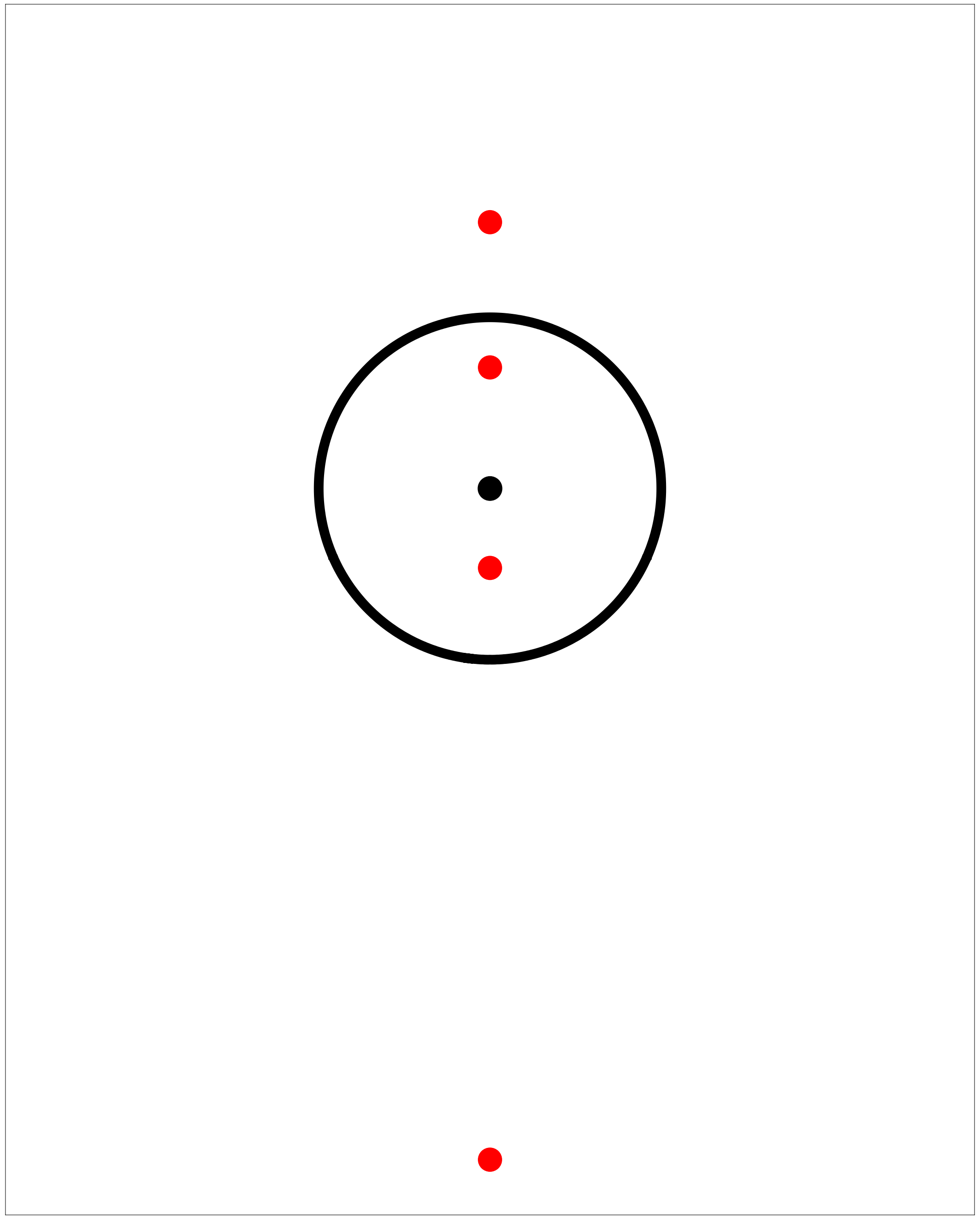

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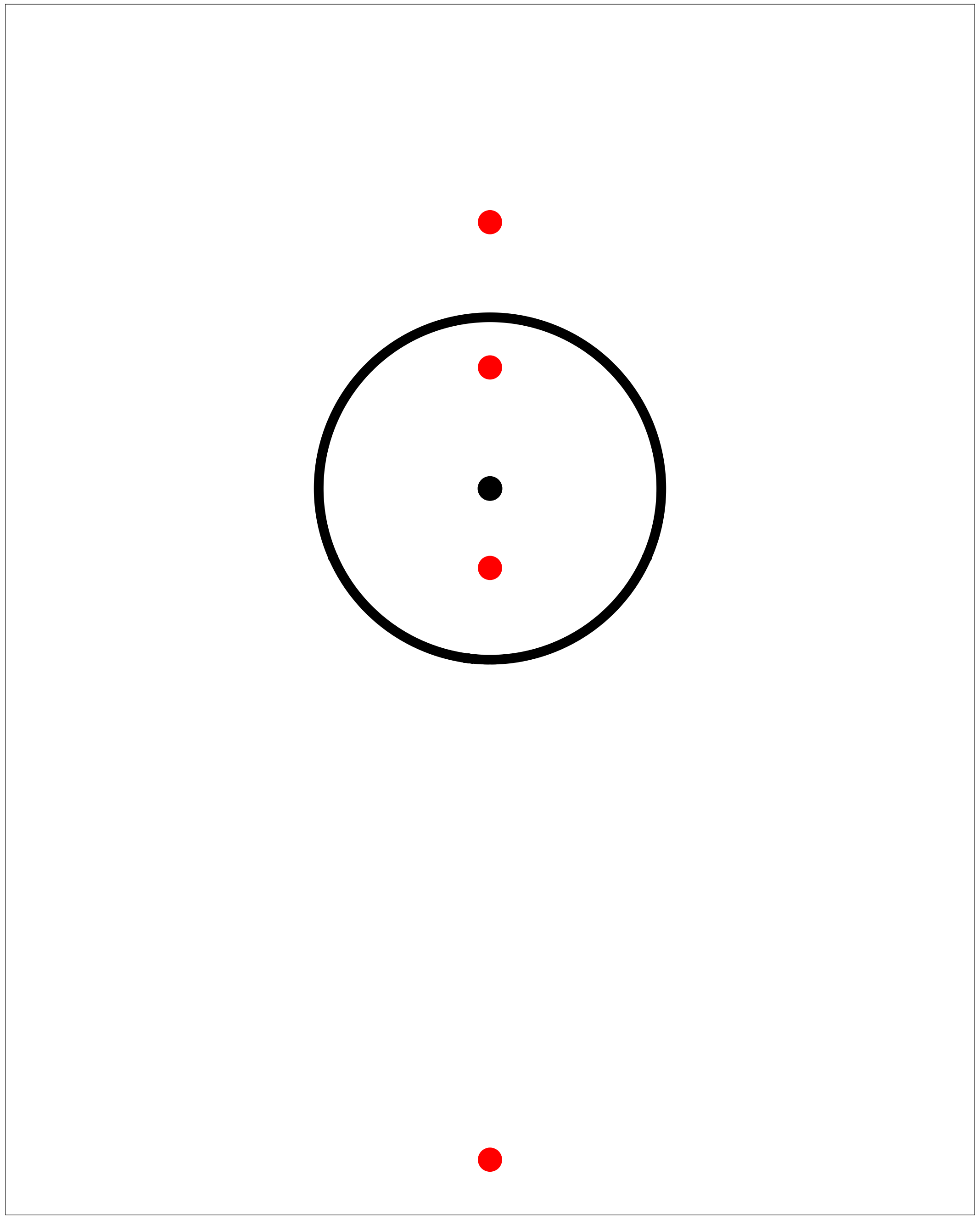

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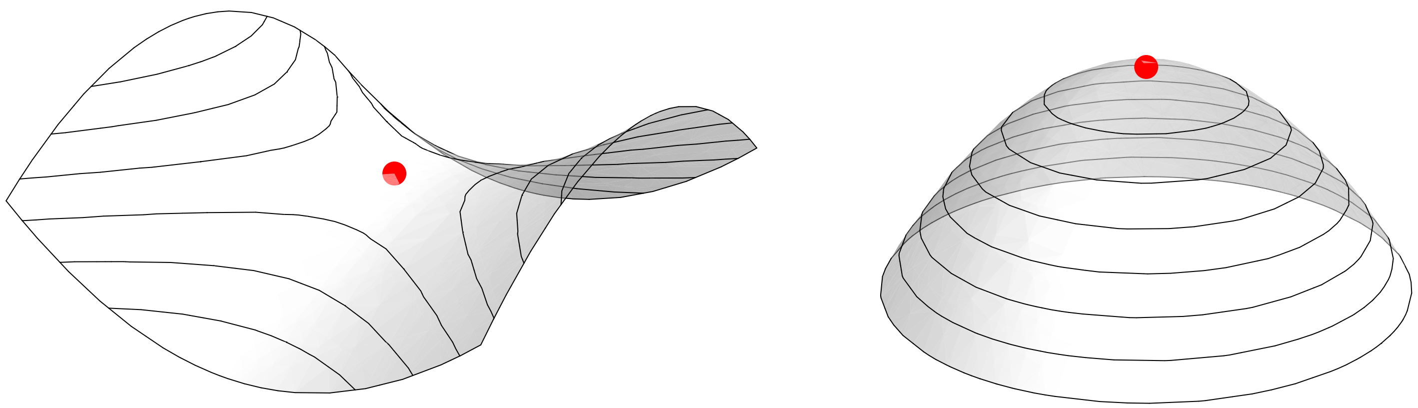
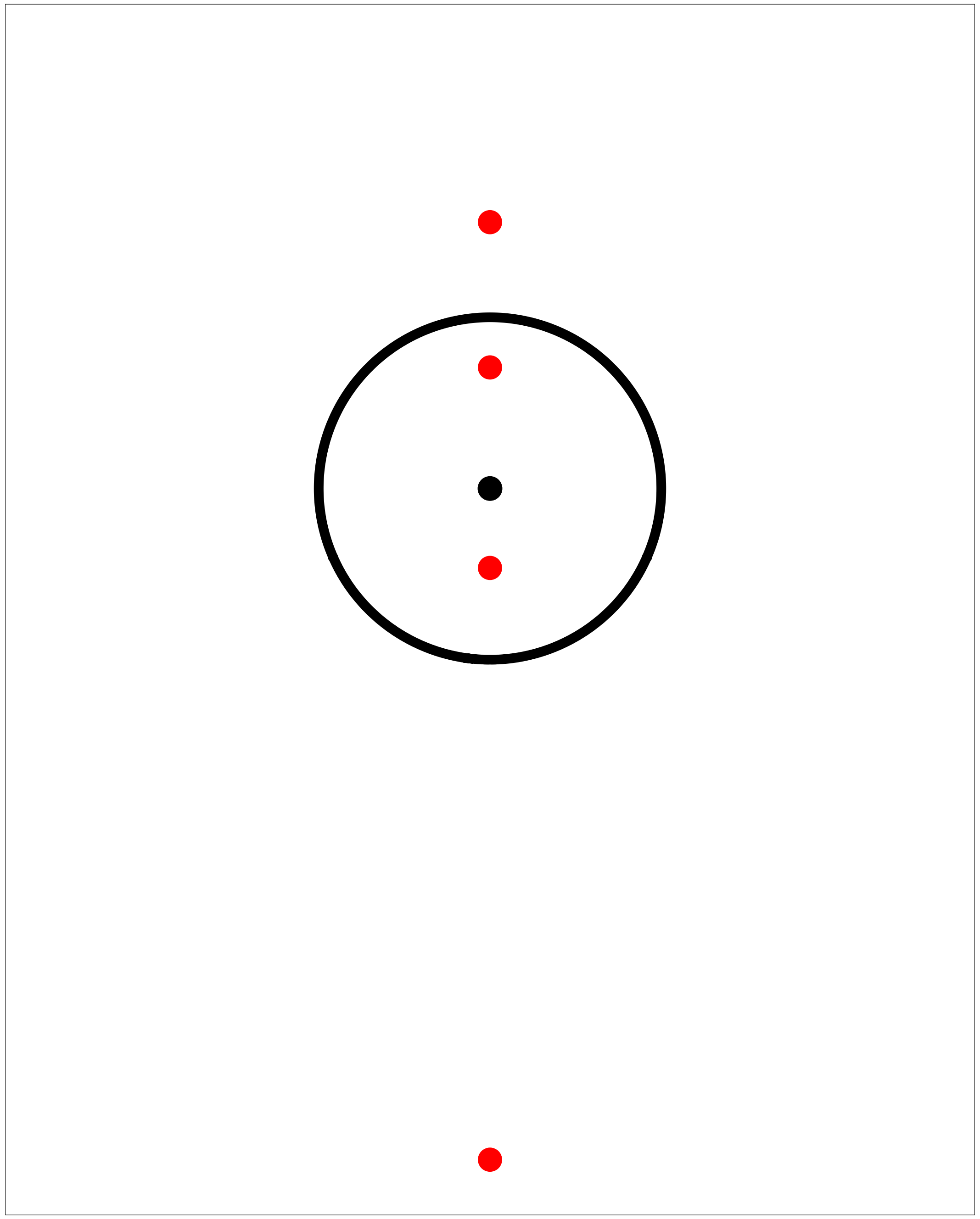


1. Correctness: Fixing by Perturbation

$$g = \frac{f^2}{\left(x_1^2 + x_2^2 + 1\right)^{\deg(f)+1}} \xrightarrow{\text{perturb}} g = \frac{f^2}{\left((x_1 - 0)^2 + (x_2 - 1)^2 + 1\right)^{\deg(f)+1}}$$



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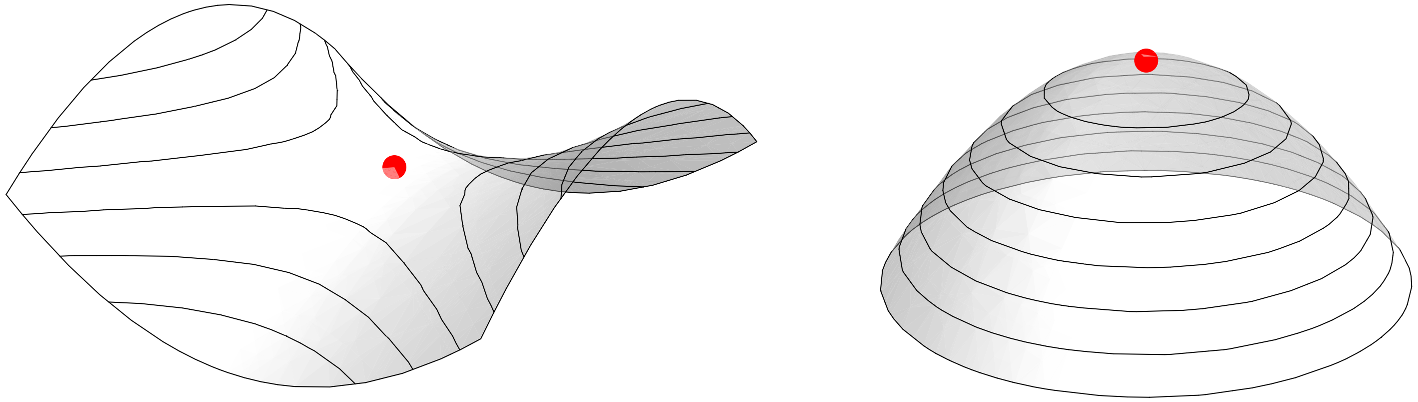

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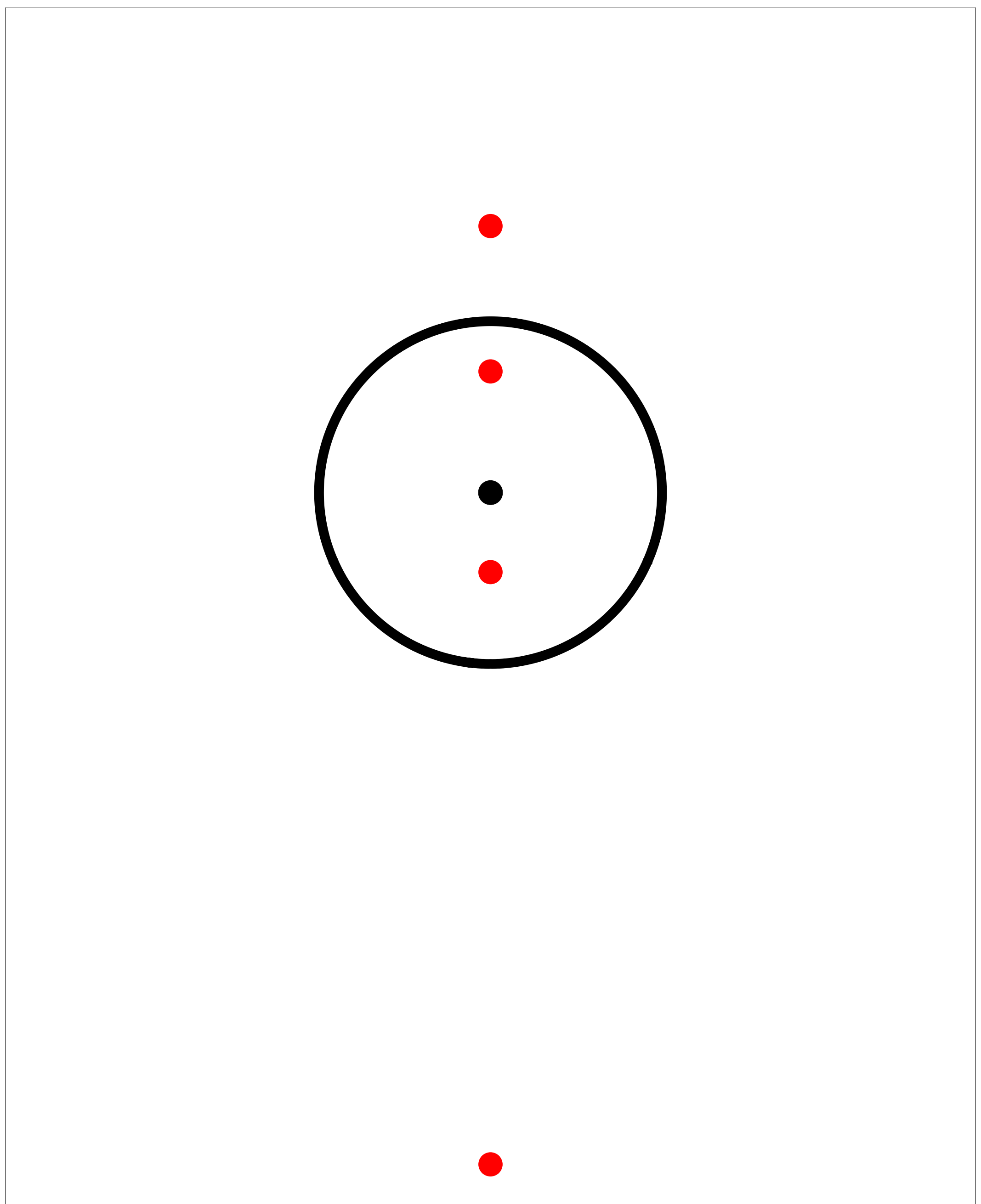

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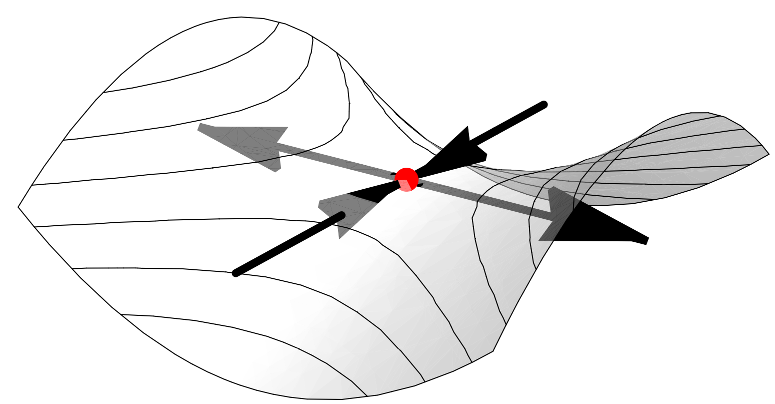

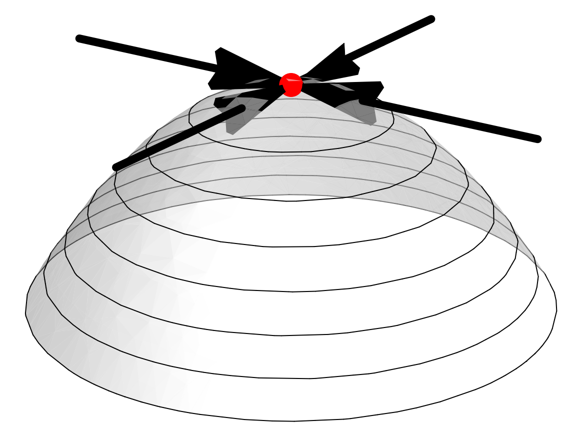

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2. Termination: Theorem

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$\forall f \in \mathbb{Z}[x_1, \ldots, x_n]$

$\exists$ semialgebraic set $S \subset \mathbb{R}^n$

$\dim (\mathbb{R}^n \setminus S) < n$

$\forall (c_1, \ldots, c_n) \in S$

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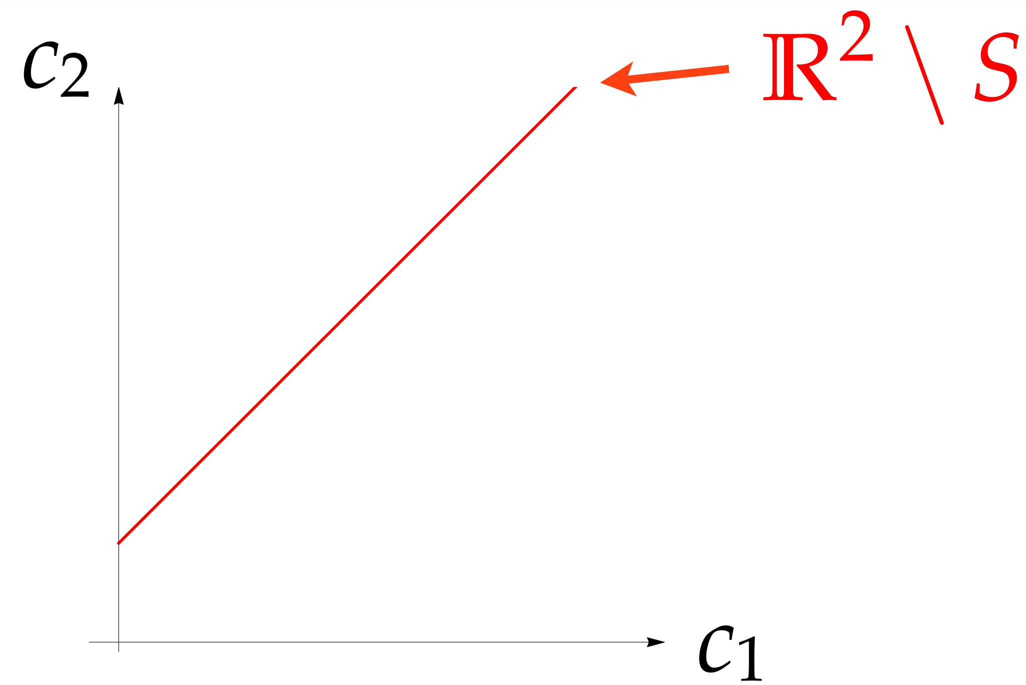

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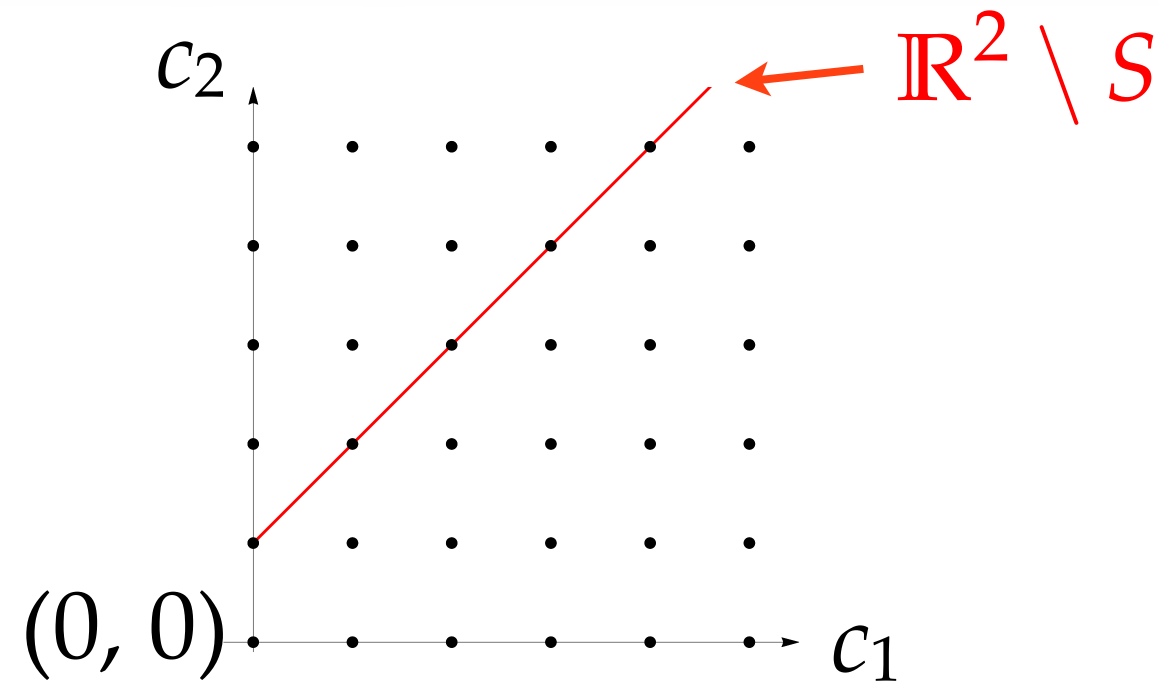

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$(0, 0)$

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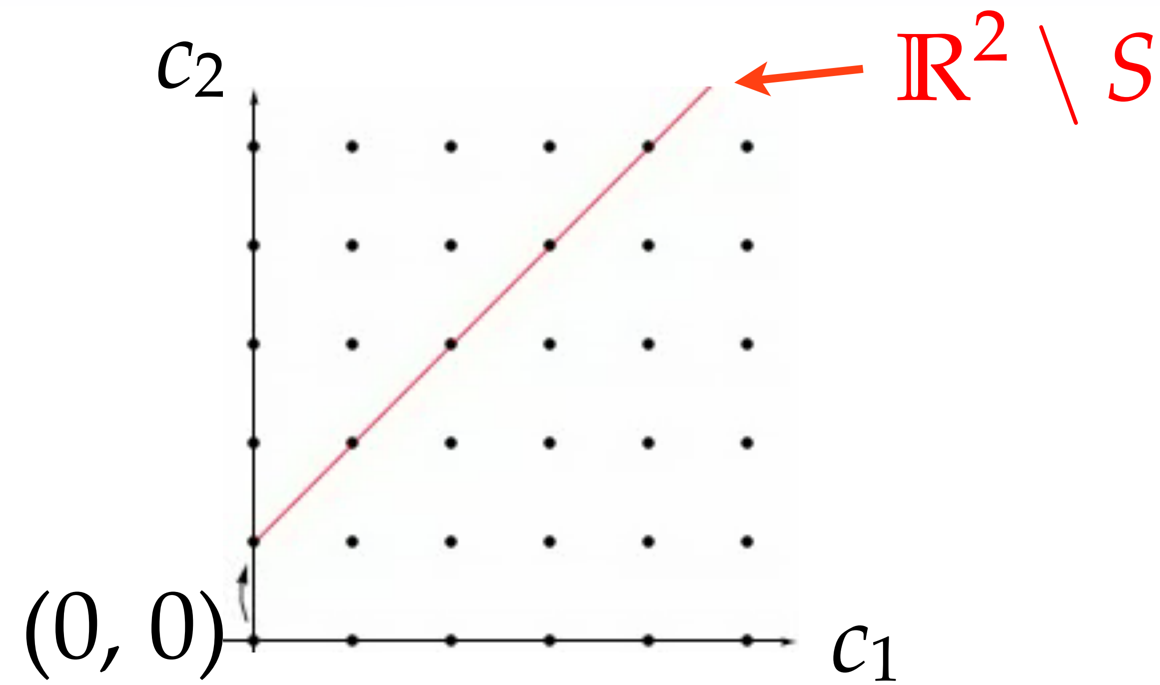

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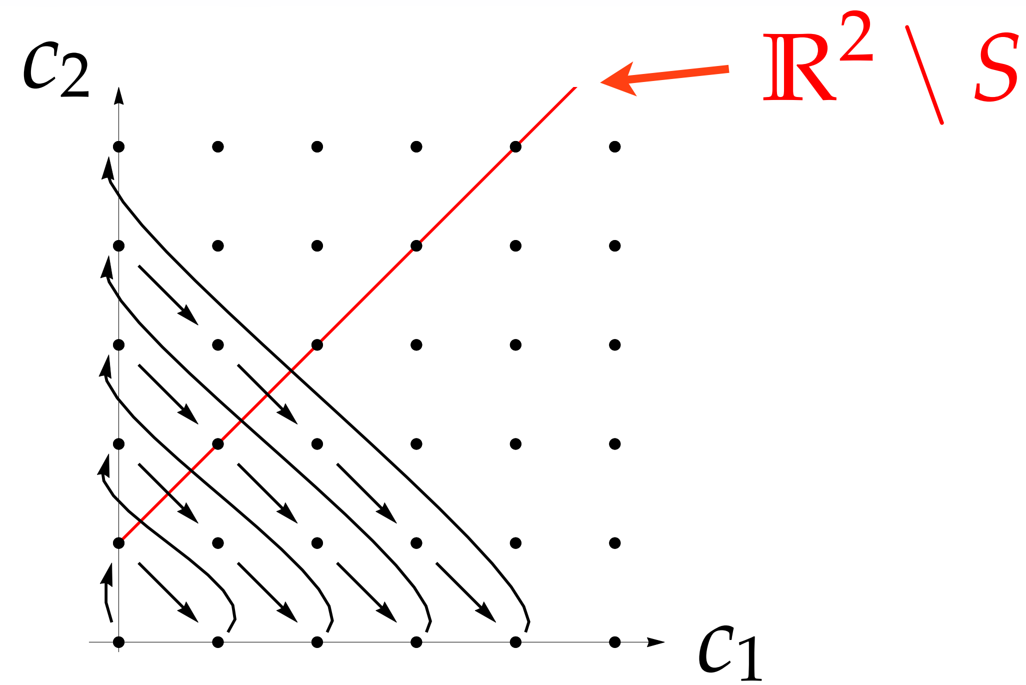

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Proof Idea: Sard's Theorem and Constant Rank Theorem

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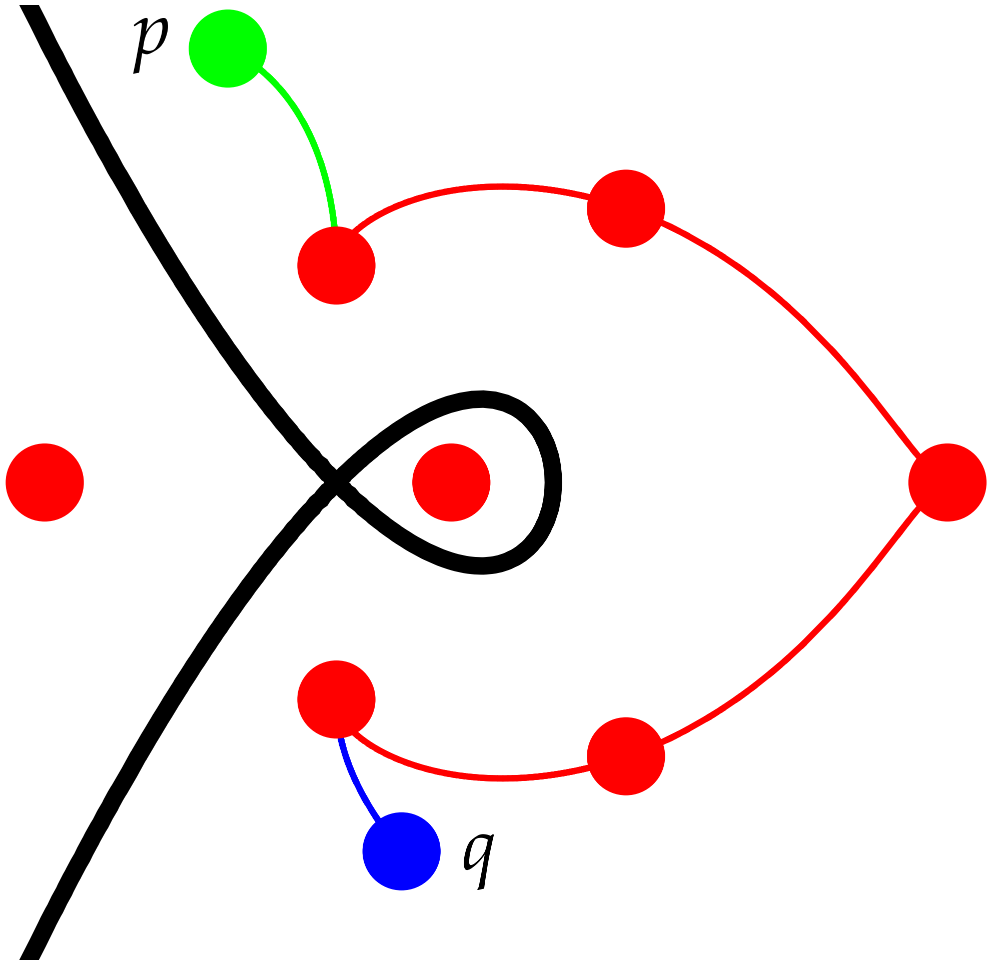

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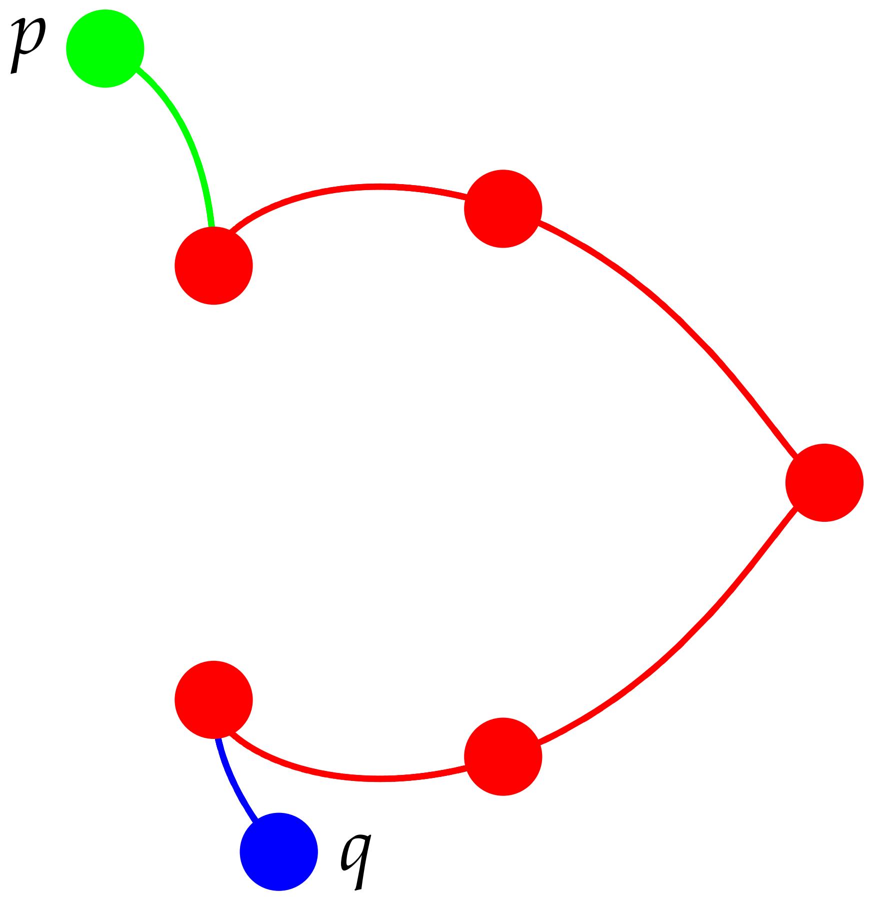

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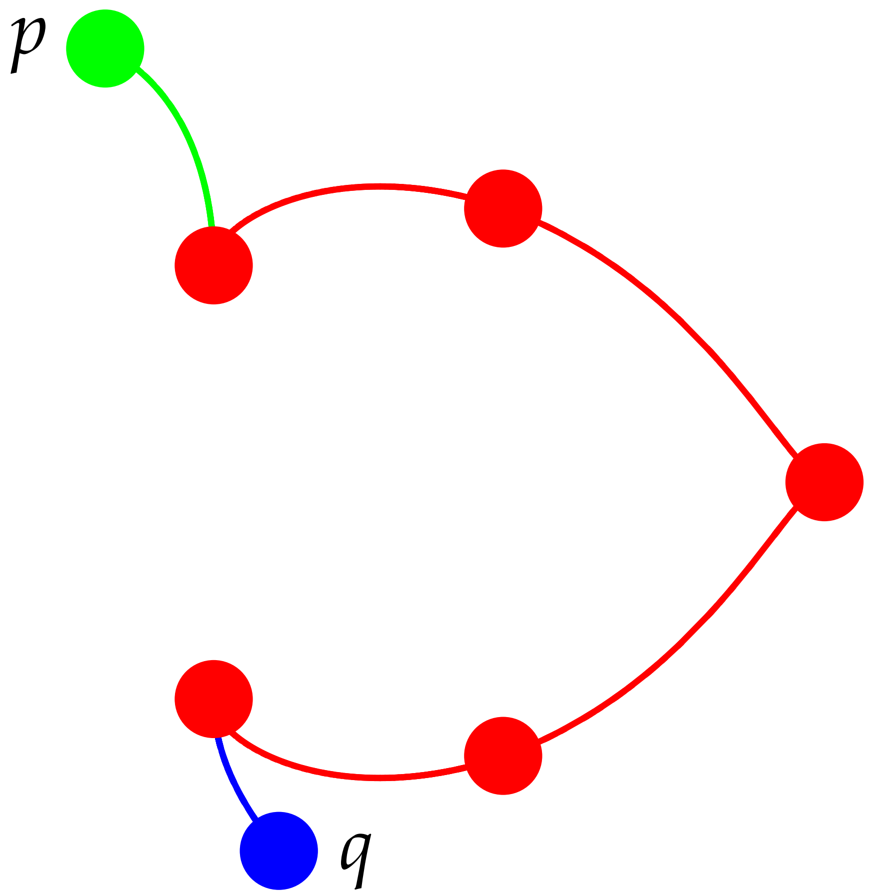

Length of
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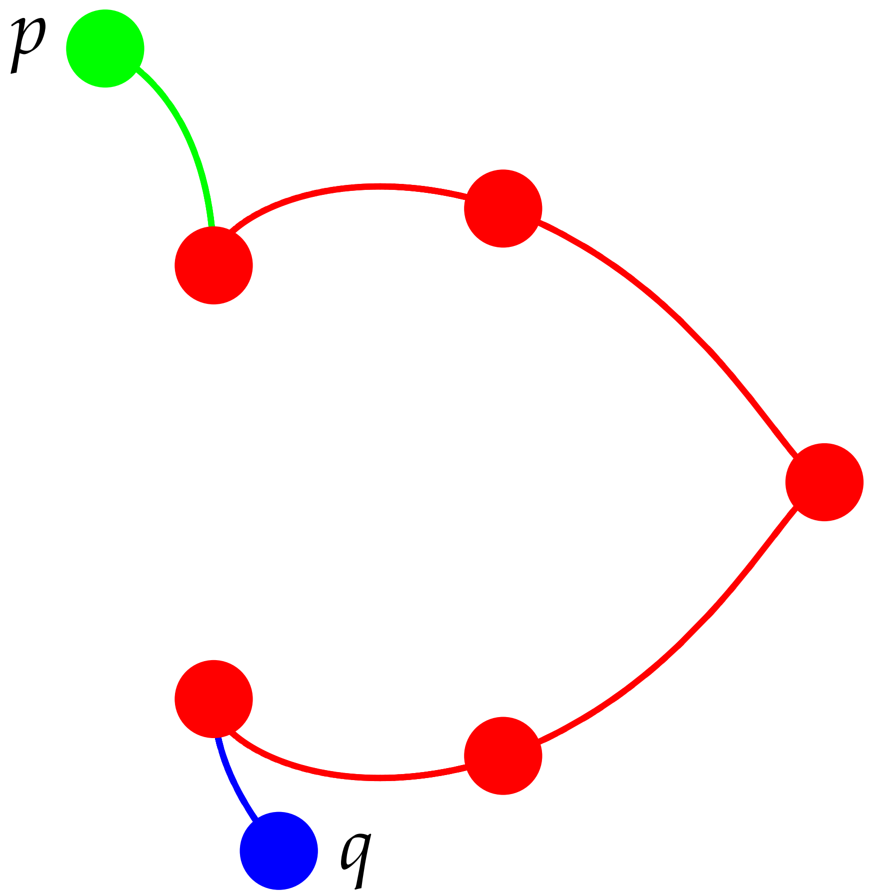

Given $f \in \mathbb{Z}[x_1, \ldots, x_n]$ $\quad d = \deg(f) \geq 2$

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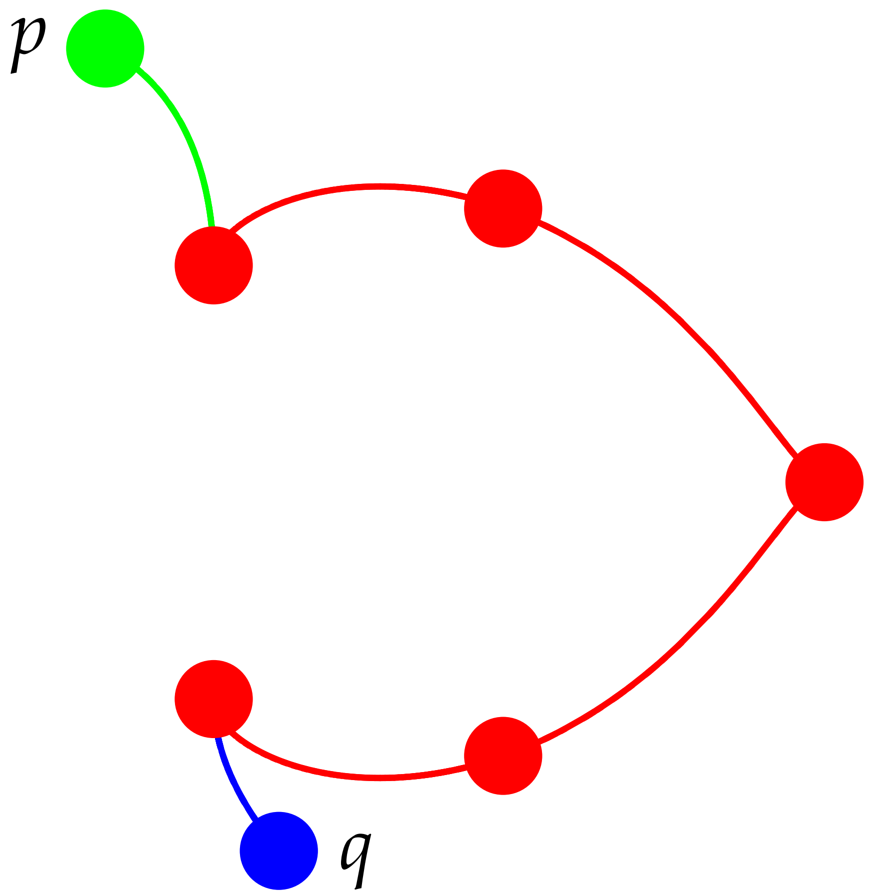

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Find A such that

Length $\leq A(n, d, H, c_1, \ldots, c_n, p, q)$

$H = \max\left|\text{coefficients of } f\right|$

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3. Length Bound: Proof Steps

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1. Radius Bound

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2. Trajectory Bound

3. Length Bound: Proof Steps

1. Radius Bound
2. Trajectory Bound
3. Proof Sketch

3. Length Bound: Radius Bound

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Connectivity path for p, q is contained in $\{x \in \mathbb{R}^n \mid g(x) \geq \varepsilon\}$

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$M = \min g(r)$, r is a routing point of g

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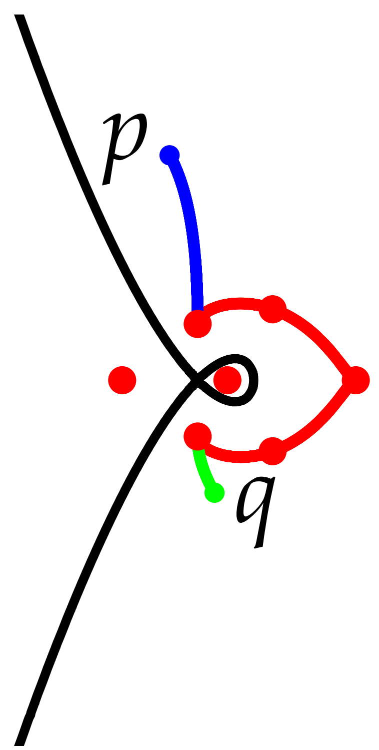

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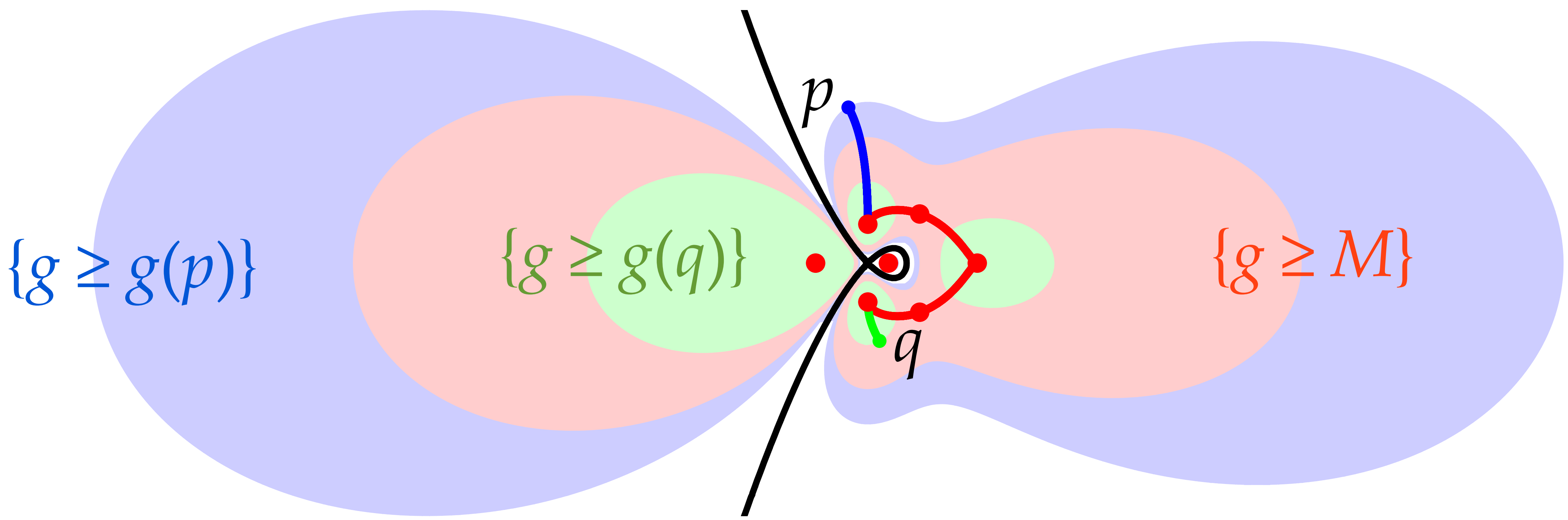

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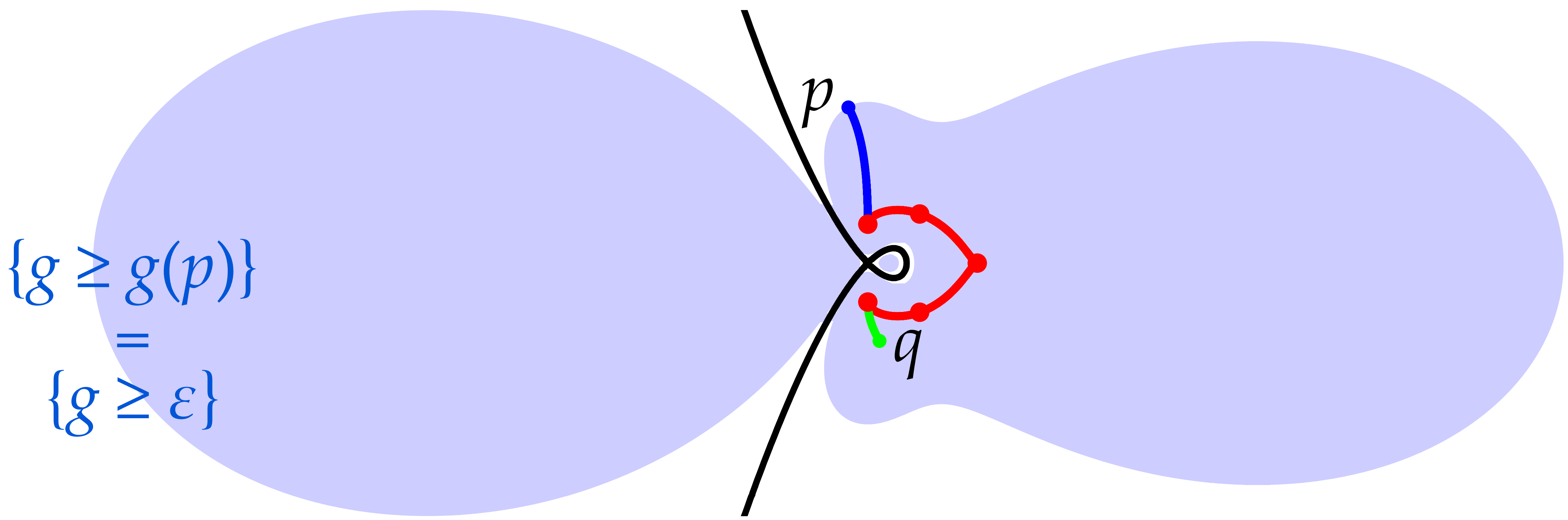

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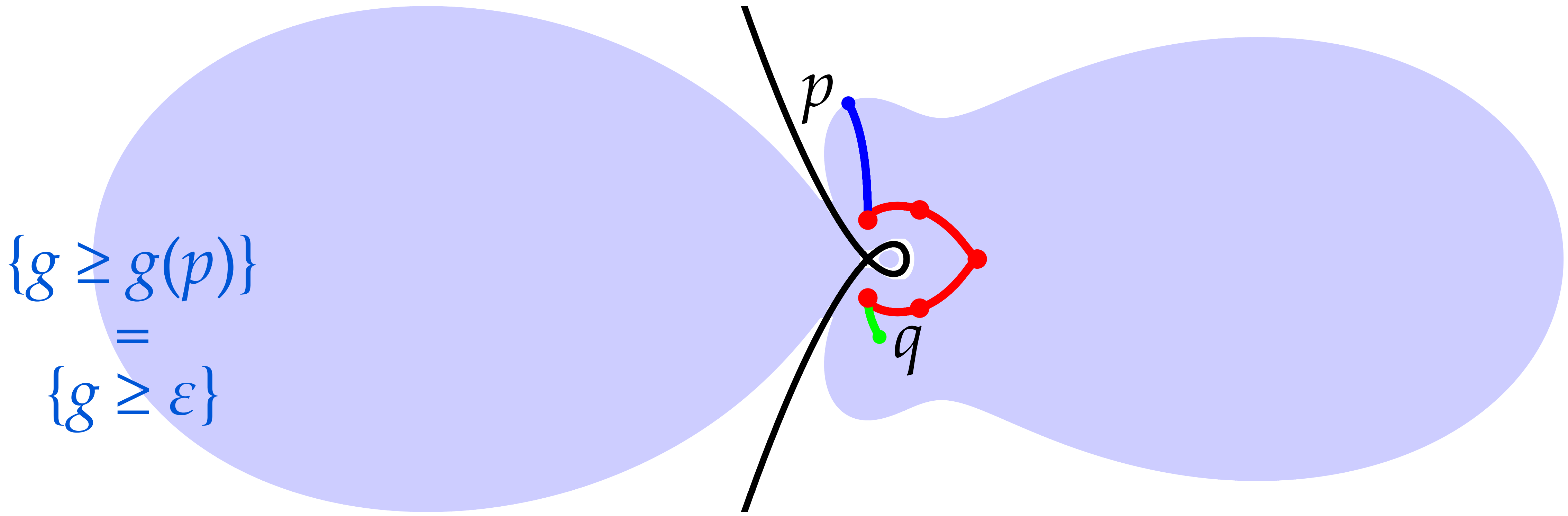

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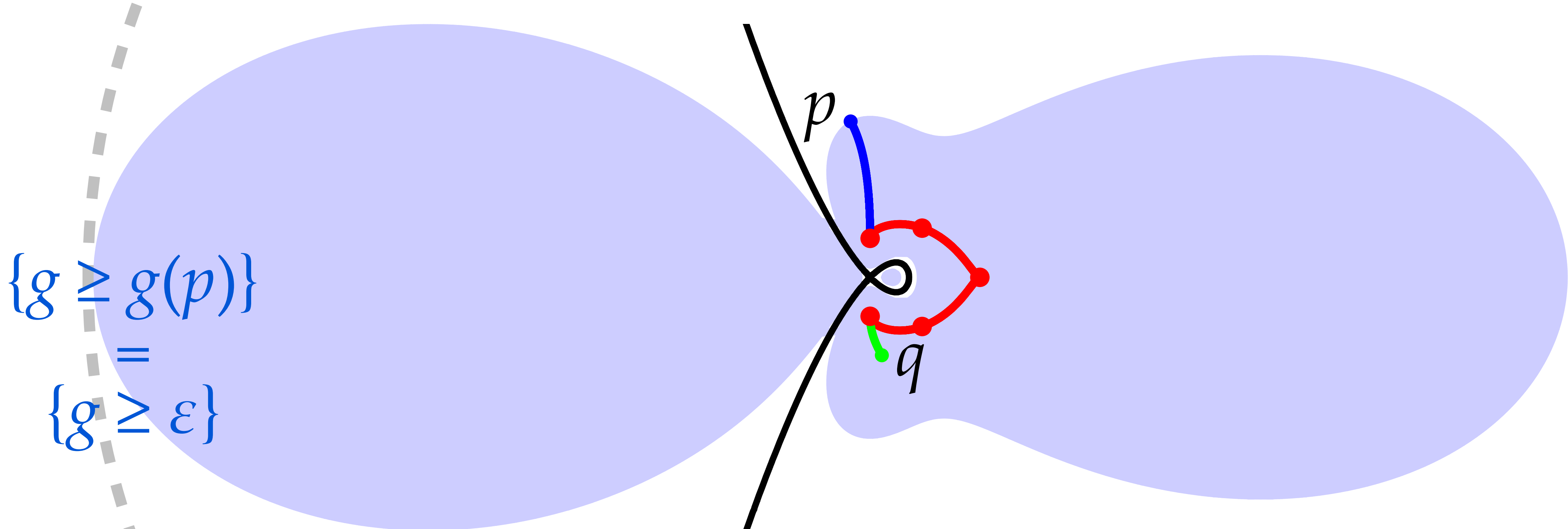

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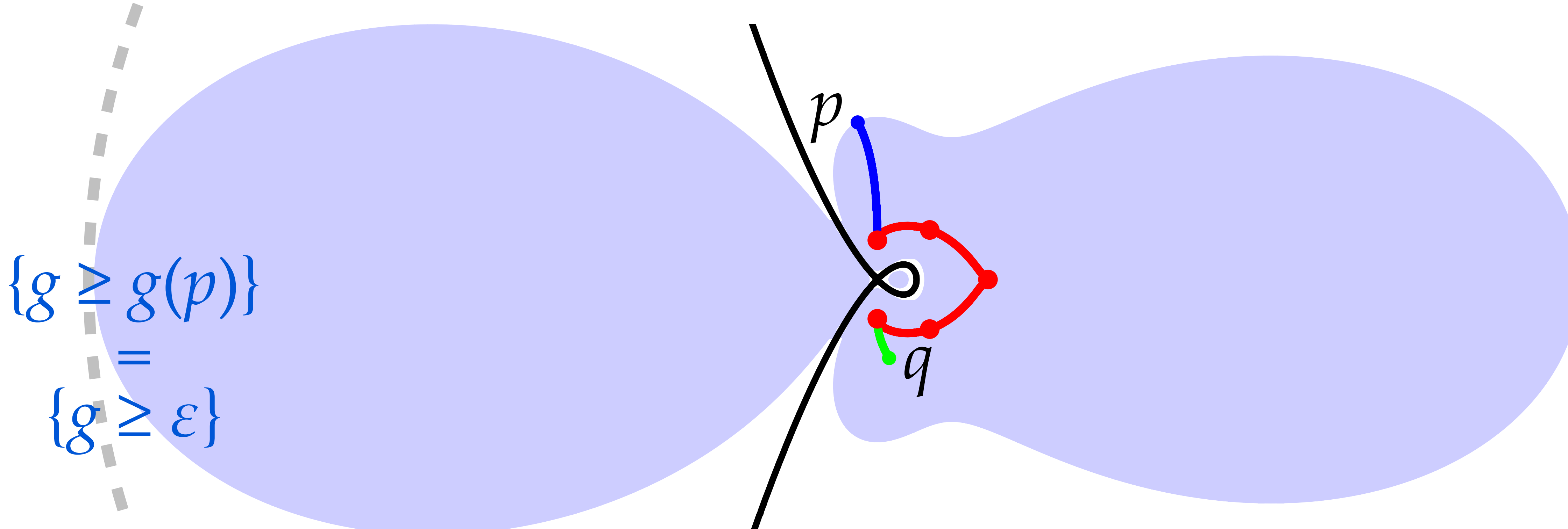

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We can contain $\left\{ g \geq \frac{A_1}{A_2} \right\}$ in a ball of radius r.

3. Length Bound: Trajectory Bound

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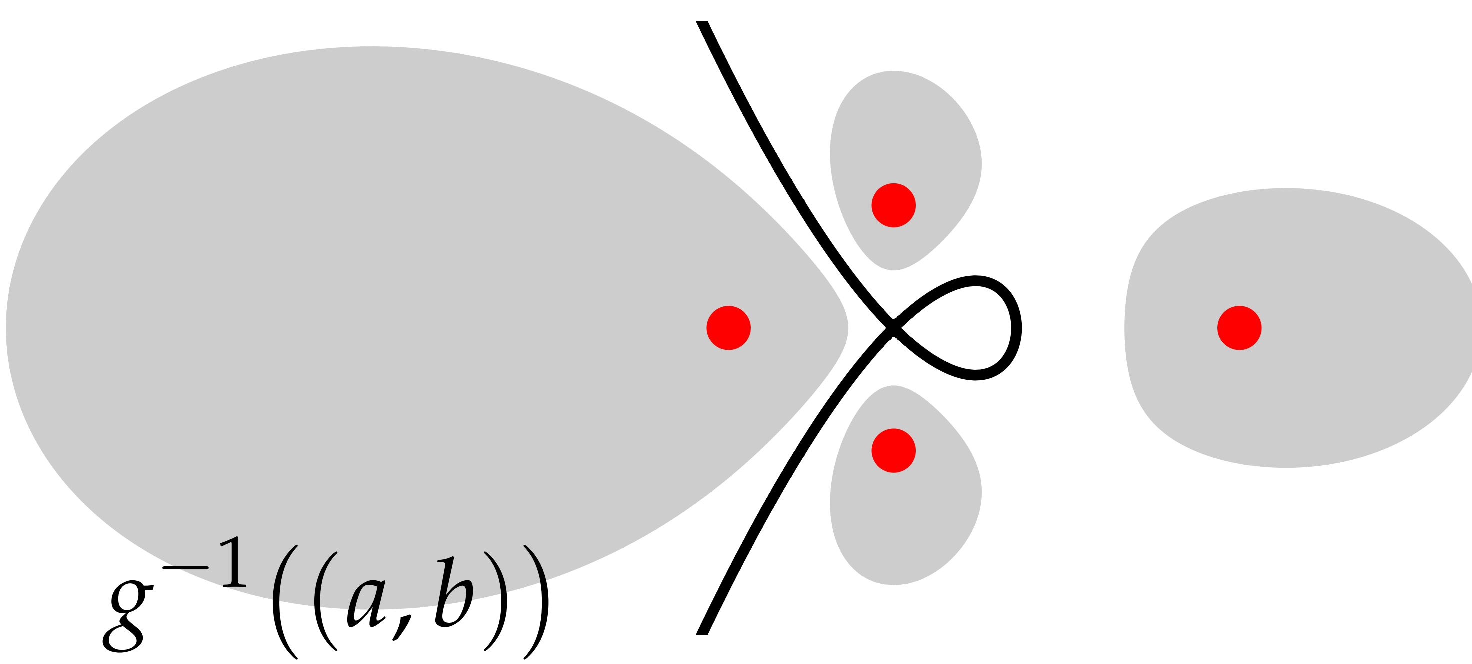

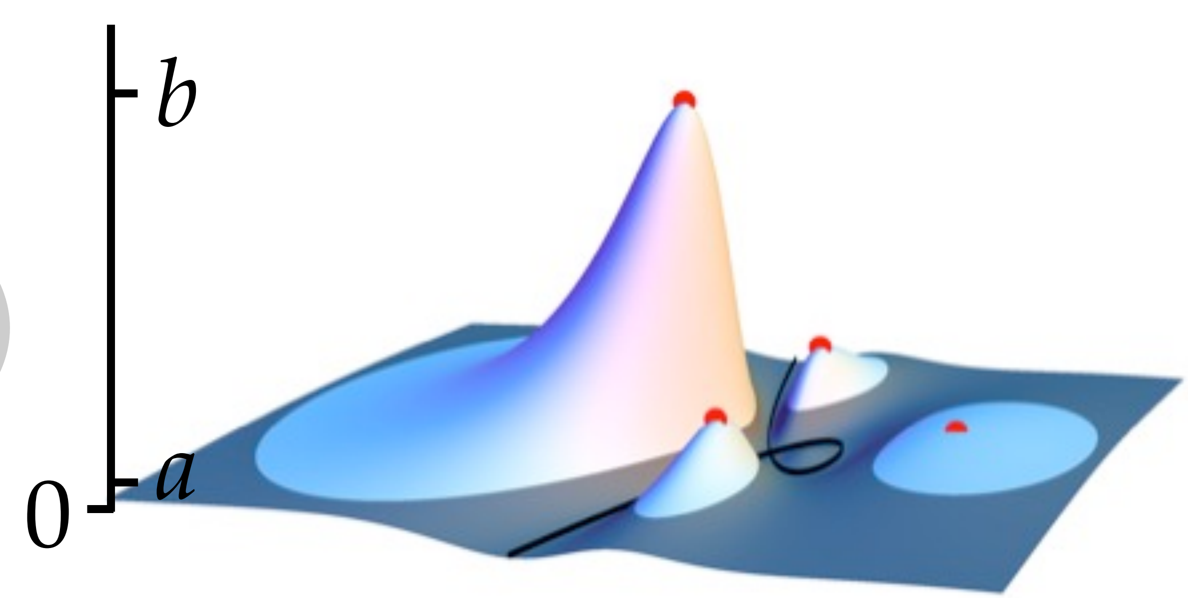

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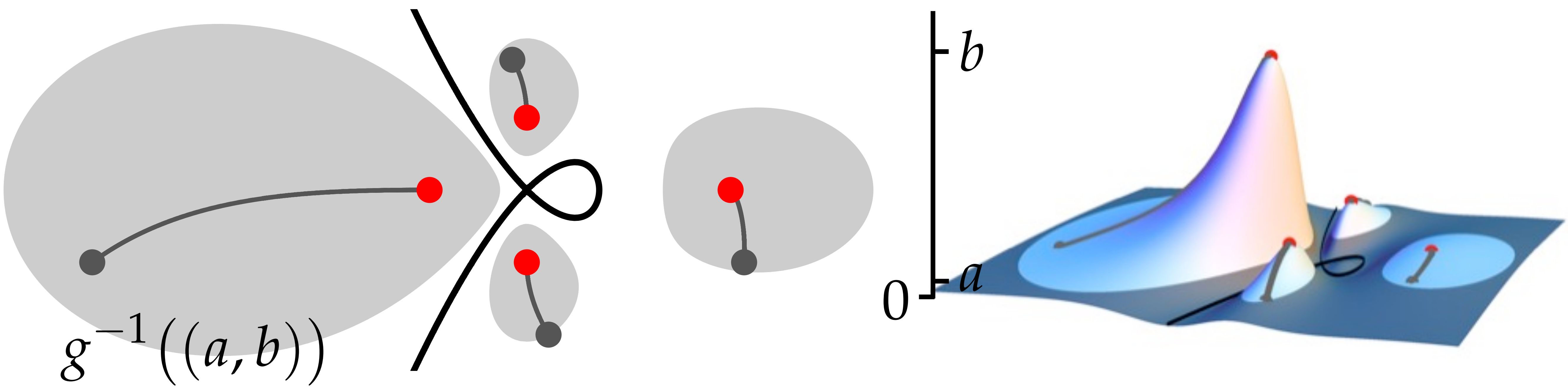

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Proof Idea

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Proof Idea

$\Omega : C^1$ curve

$\forall x \in \Omega$

$\forall y \in g^{-1}(g(x)) \quad \|\nabla g(x)\| \leq \|\nabla g(y)\|$

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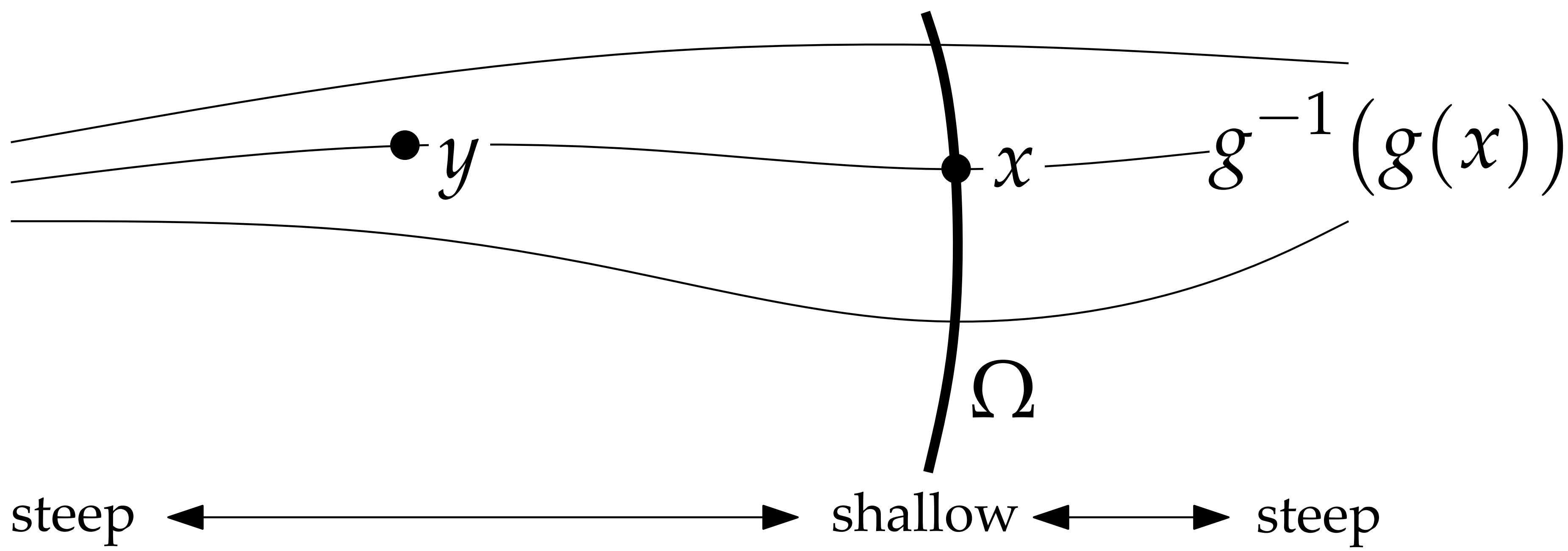

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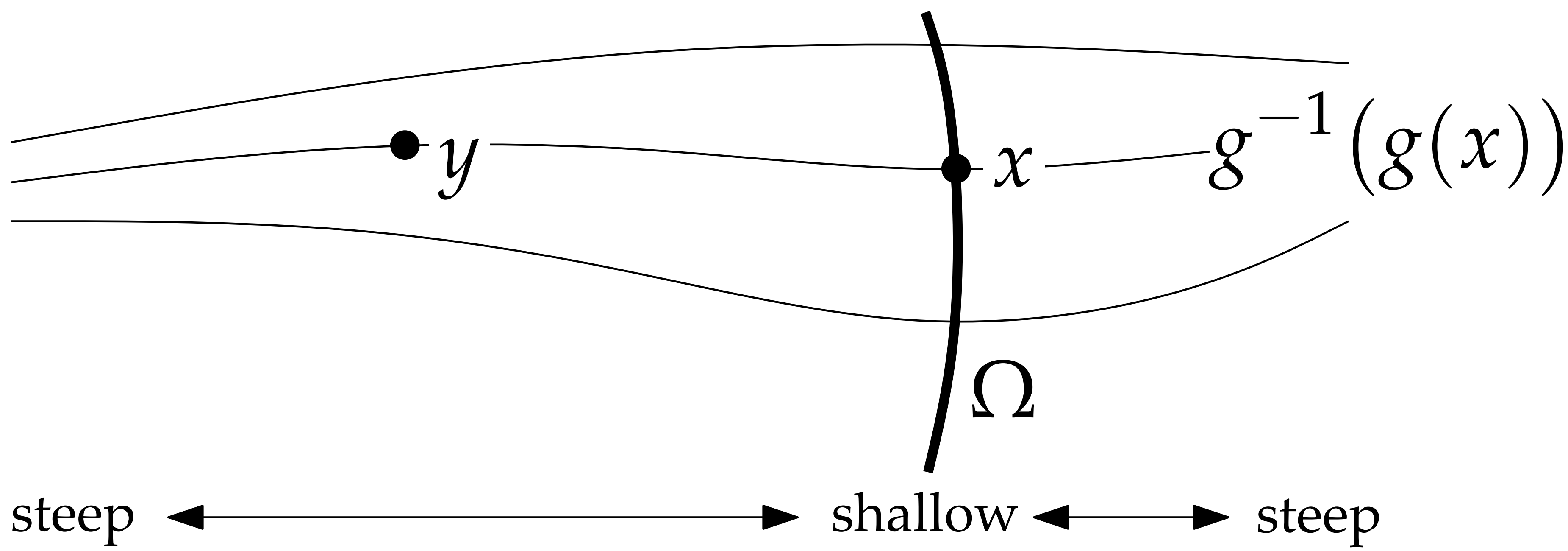

Length(trajectory) ≤ Length(Ω)

3. Length Bound: Trajectory Bound

Proof Idea

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3. Length Bound: Trajectory Bound

Proof Idea

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Length(trajectory) ≤ Length(Ω) ≤ Length(Θ)

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$$\text{Length(trajectory)} \le \text{Length}(\Omega) \le \text{Length}(\Theta) \le 2nr(6d+4)^{n-1}$$

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$$\frac{A_1}{A_2} = \min\left\{g(p), g(q), \frac{1}{\left(2dH\left(c_1^2 + \cdots + c_n^2 + 2\right)\right)^{104n^3(5d)^{5n}}}\right\}$$

3. Length Bound: Proof Sketch

p and q in a same component of $\{f \neq 0\}$ can be connected by a connectivity path of length bounded by

$$4nr(6d+4)^{n-1}$$

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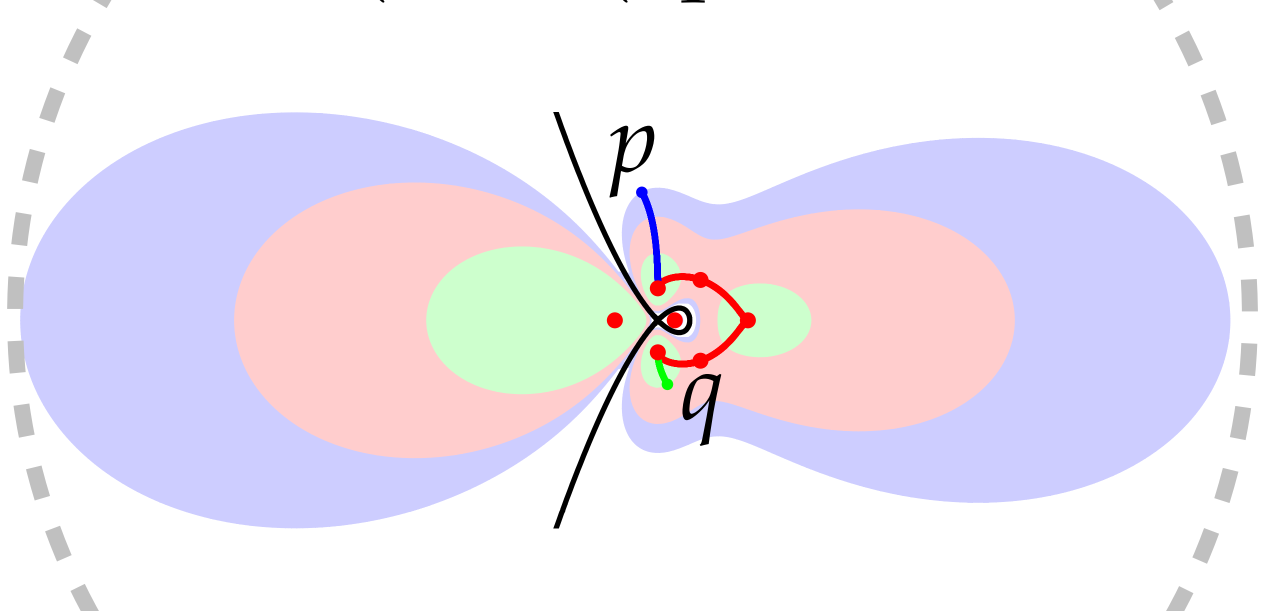

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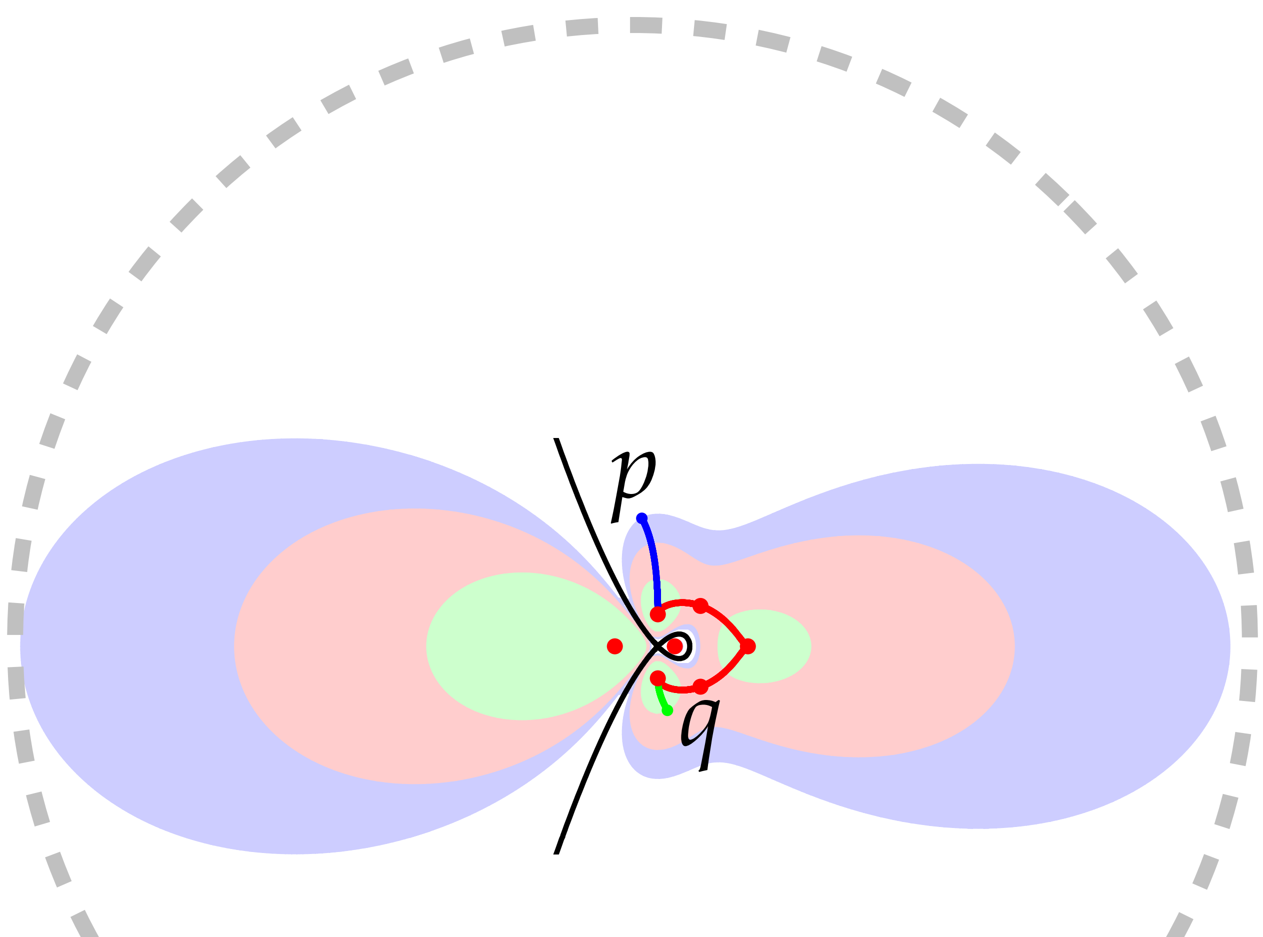

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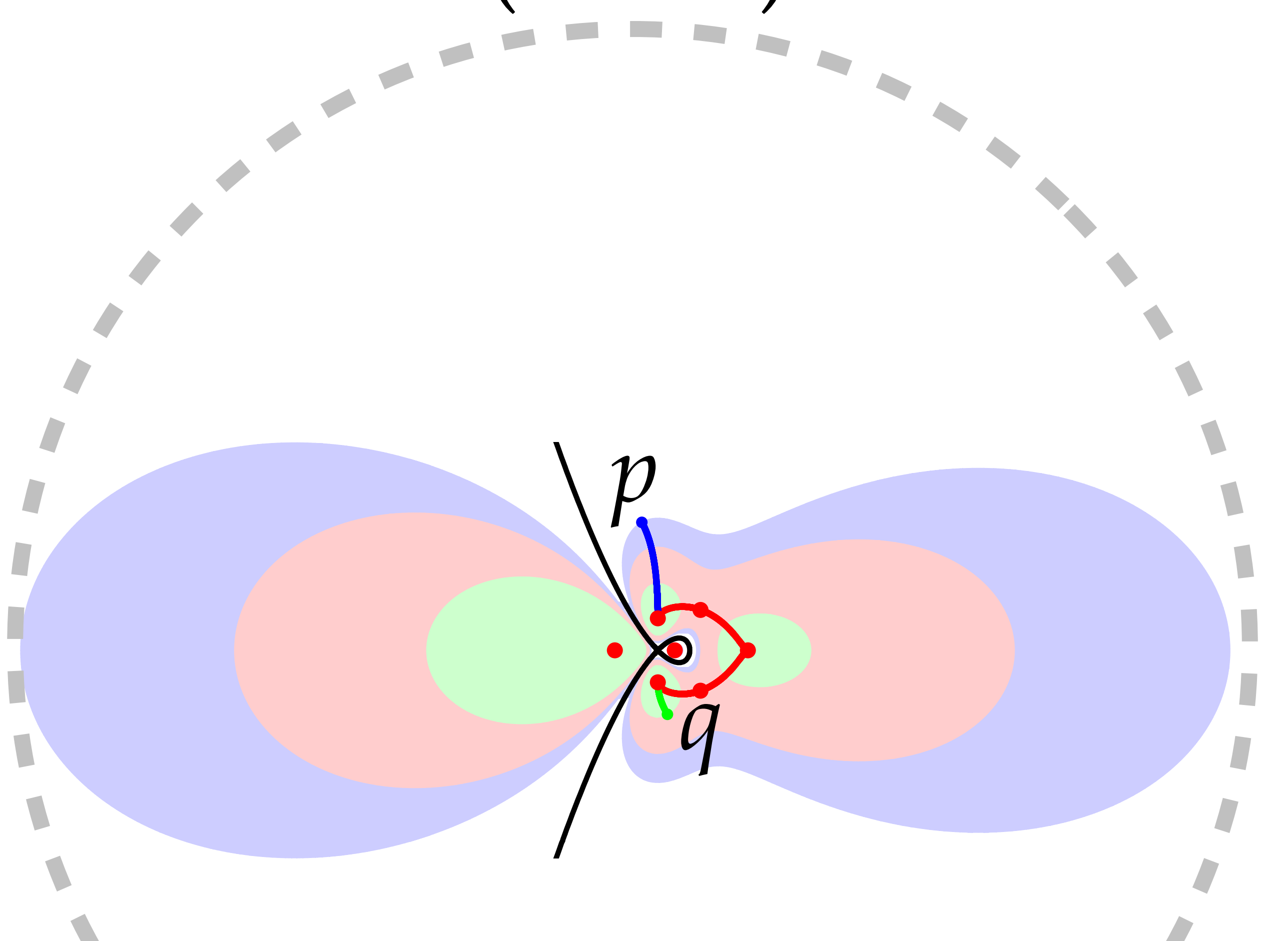

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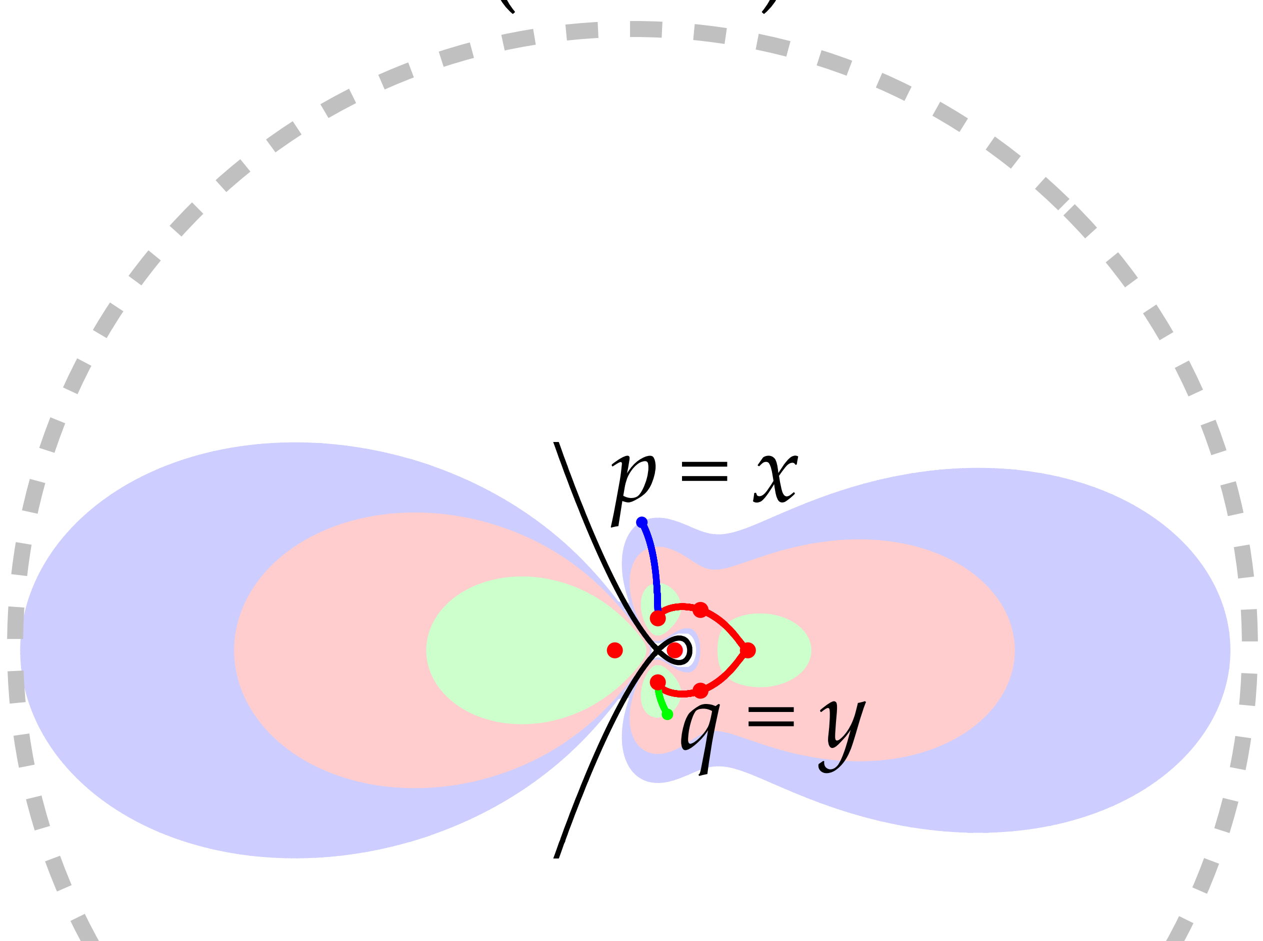

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Case 1: $k = 1$ routing point in component

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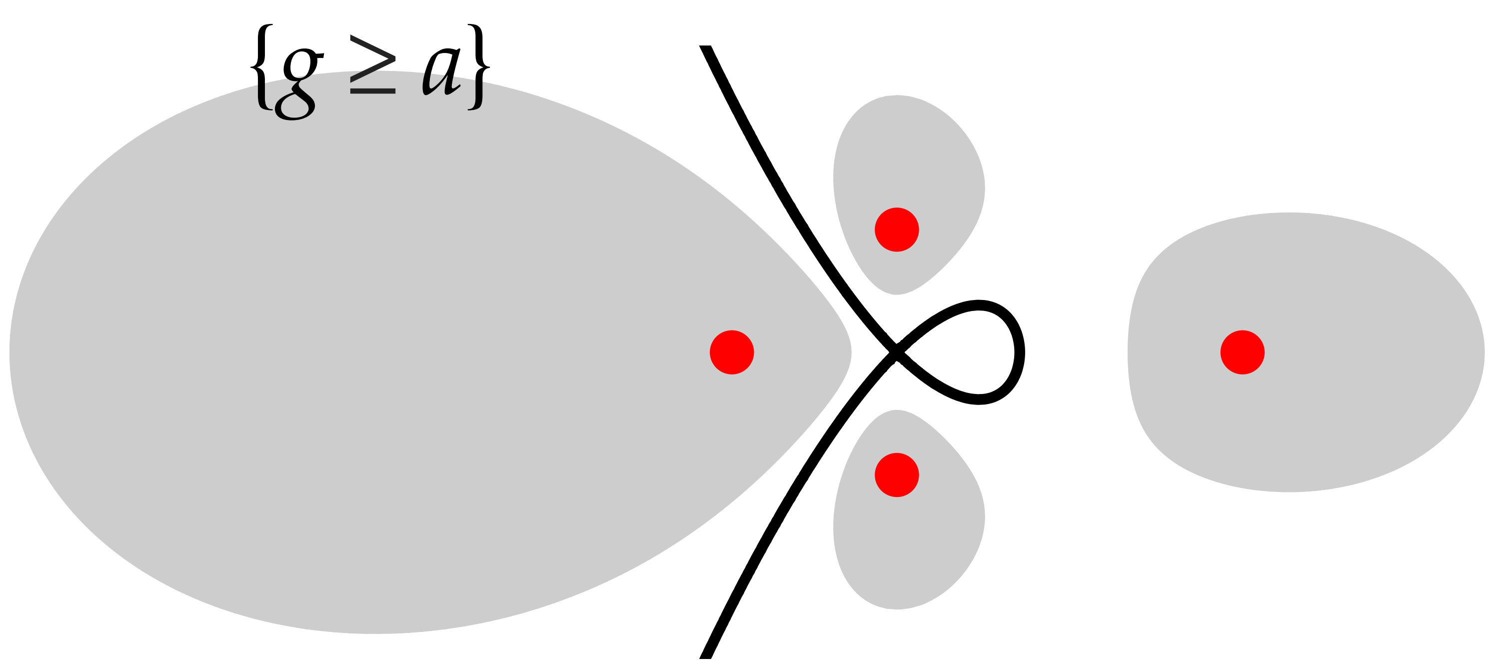

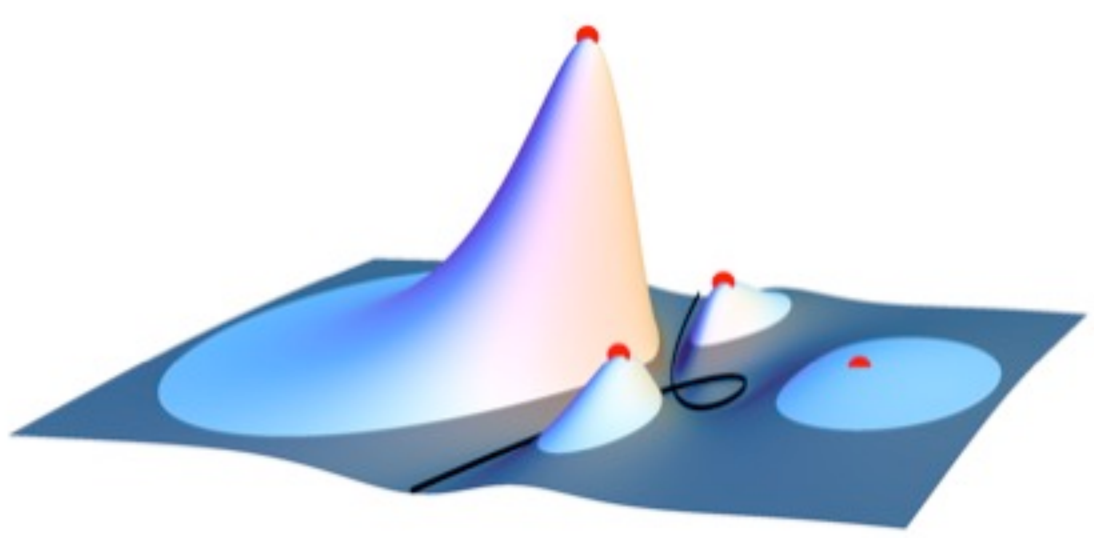

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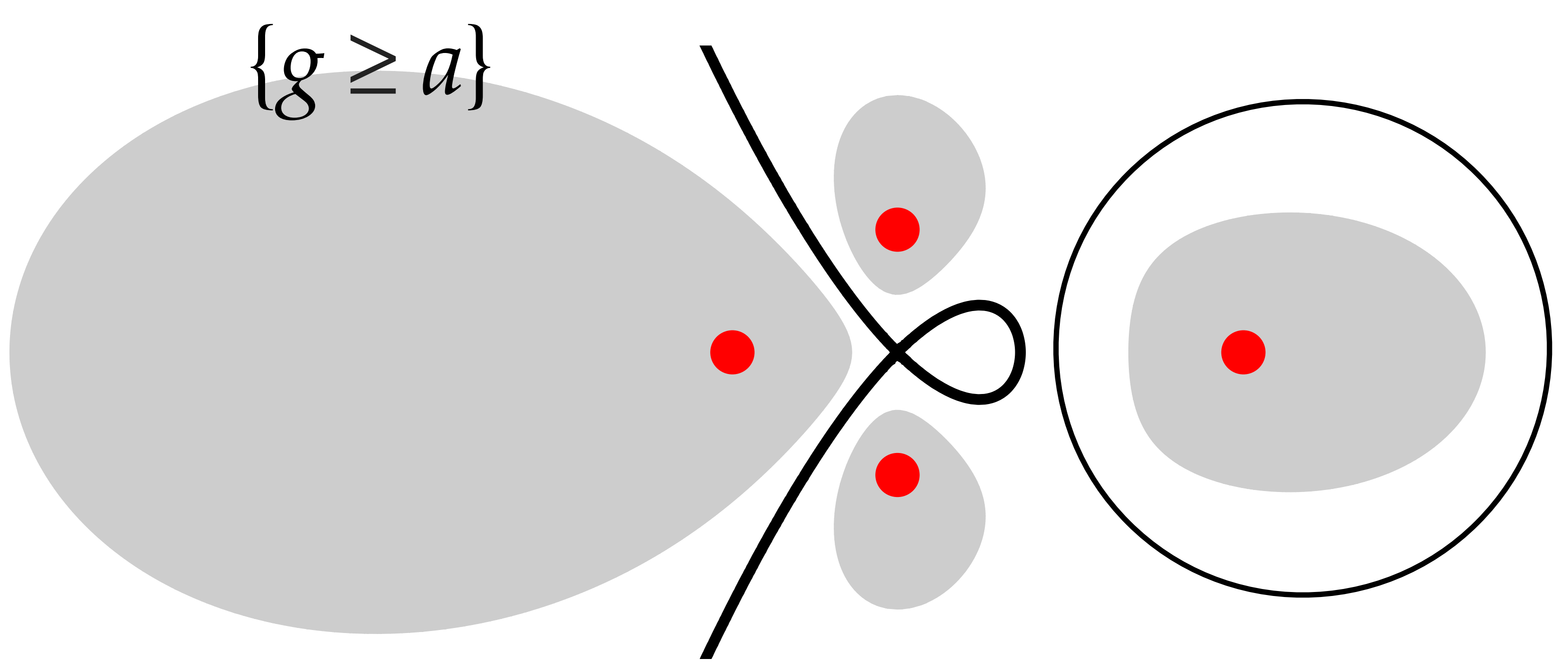

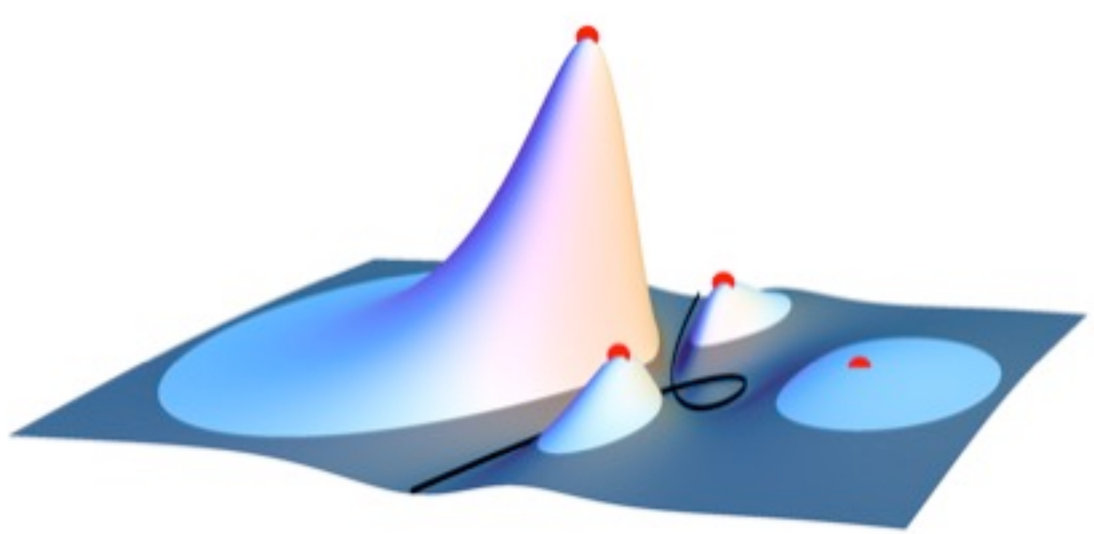

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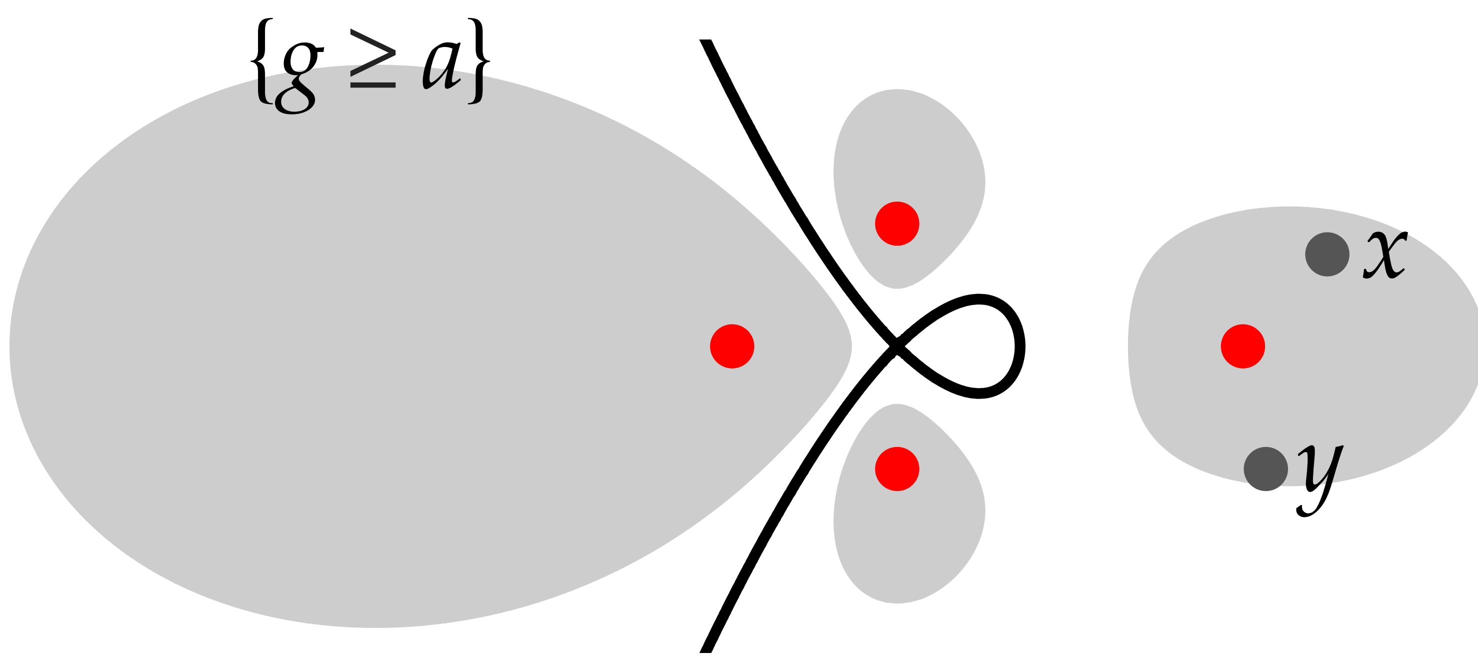

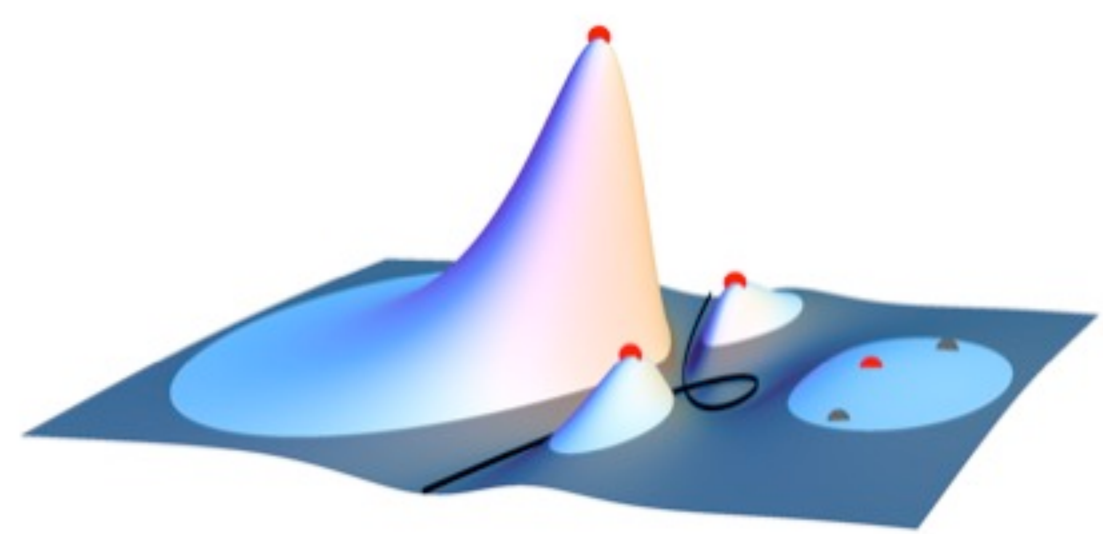

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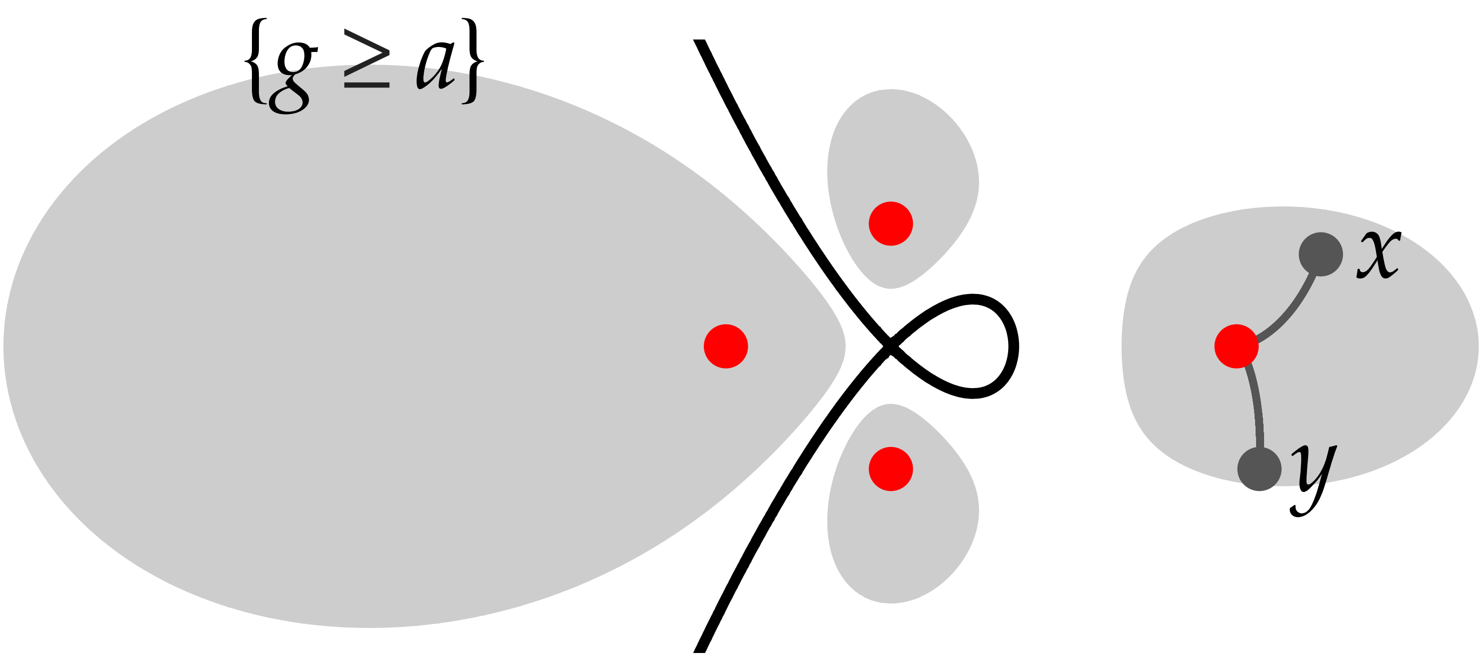

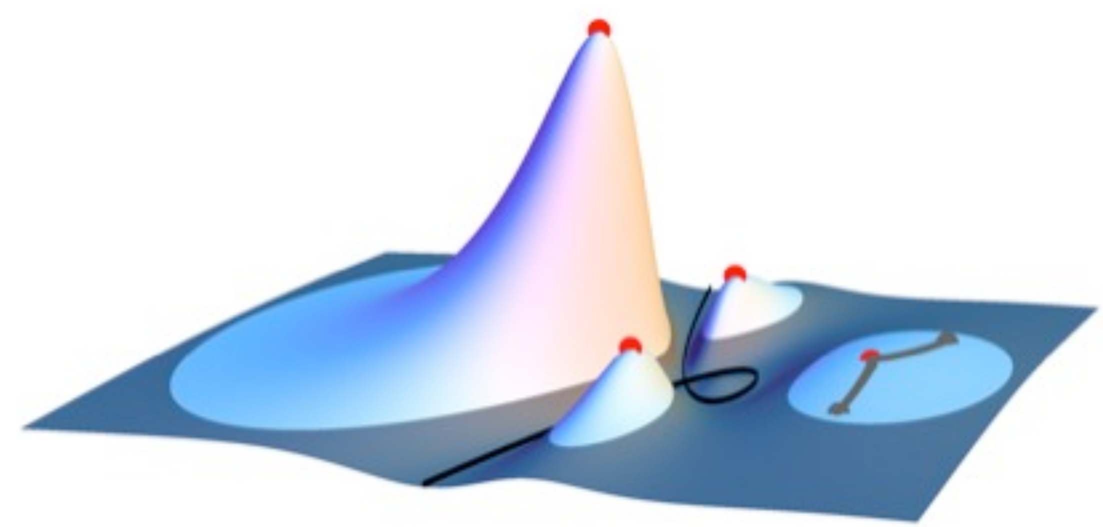

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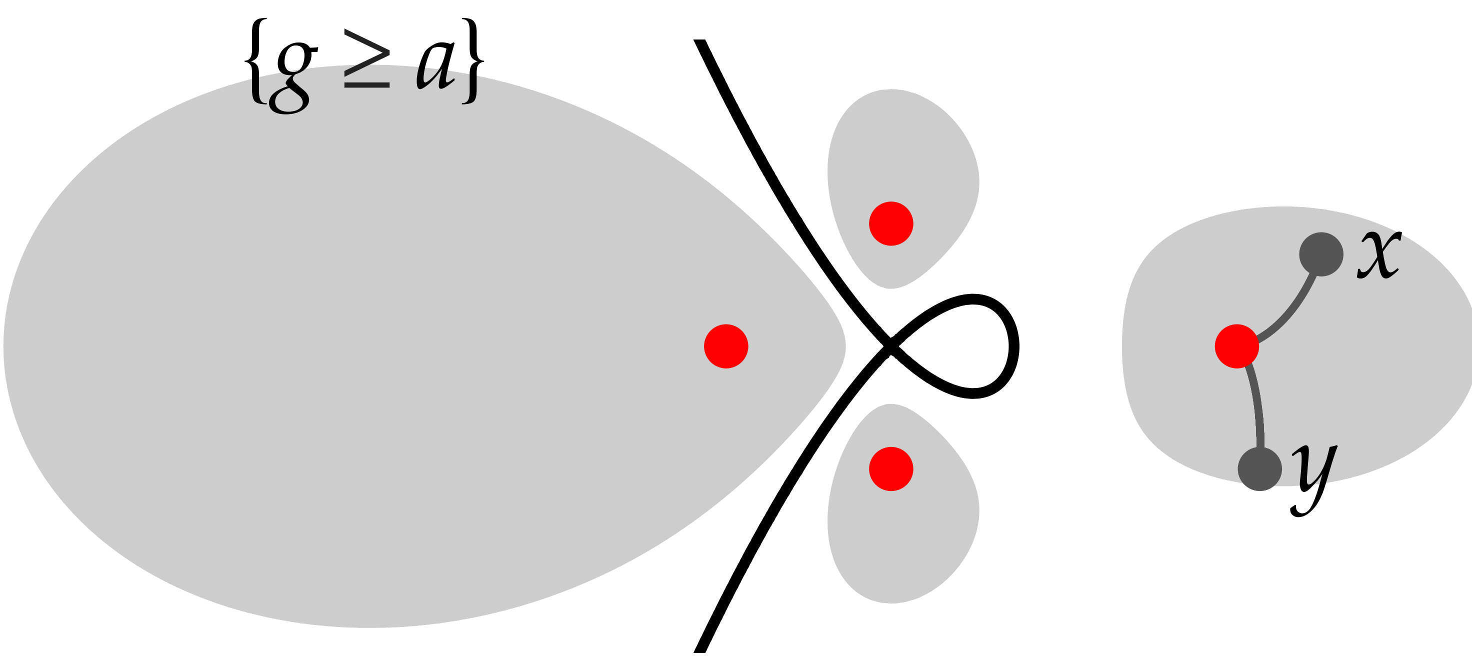
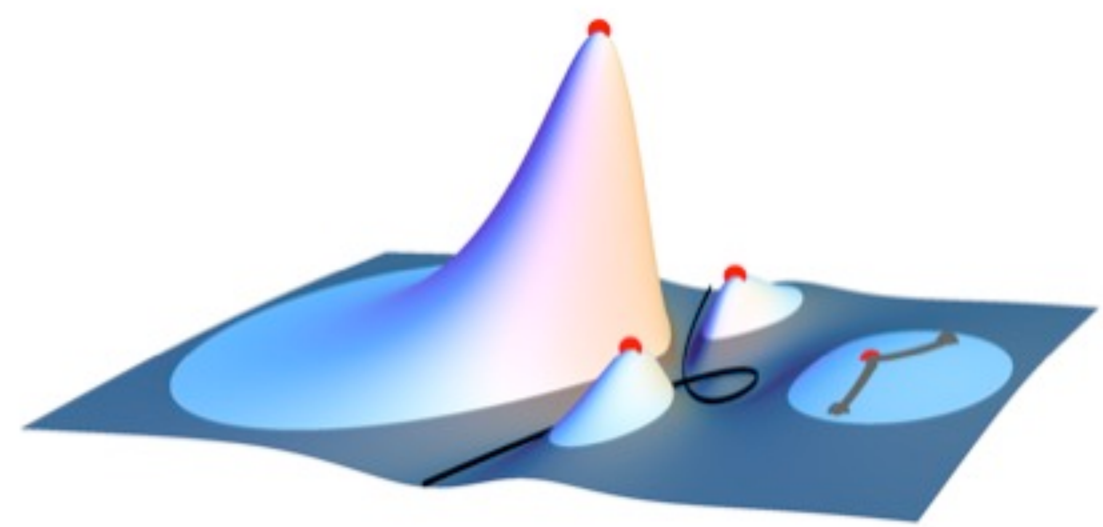
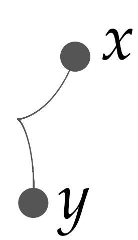




Total Length $\leq$ Length 

3. Length Bound: Proof Sketch

x and y in a same component of $\left\{ g \geq a \right\}$ can be connected by a connectivity path of length bounded by

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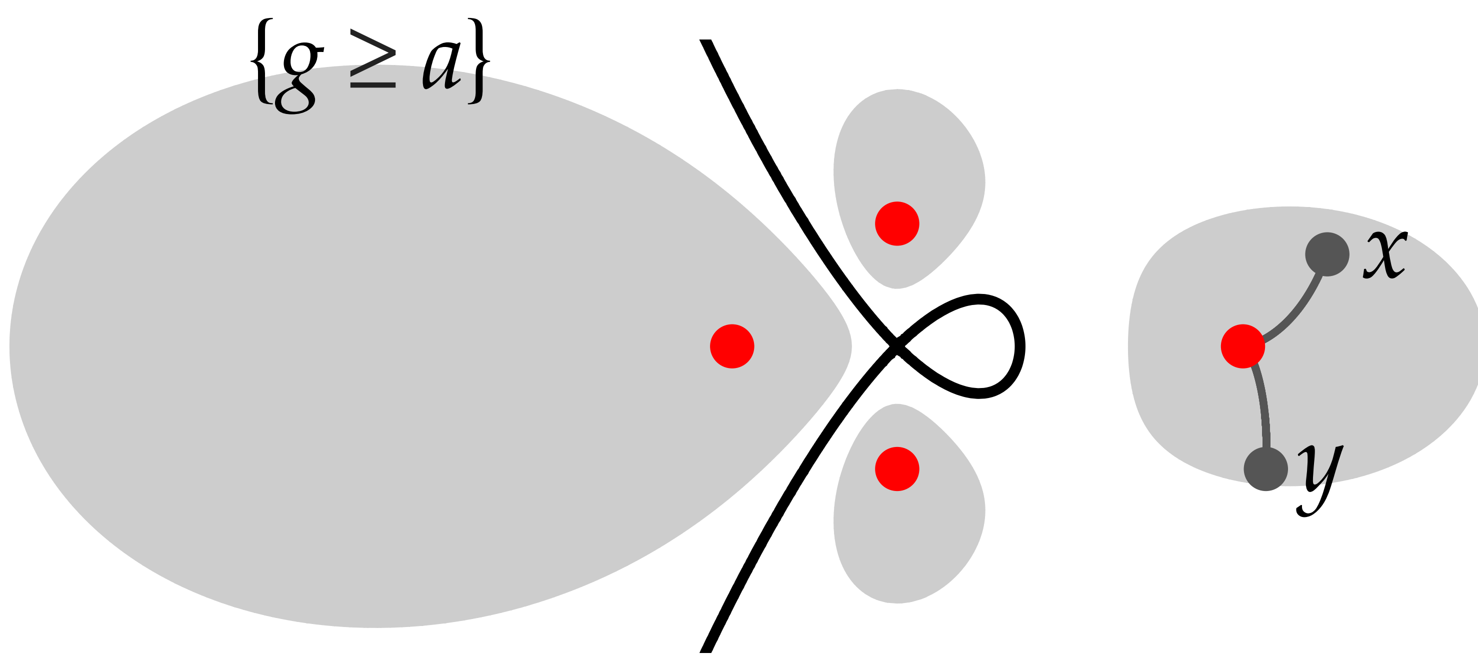
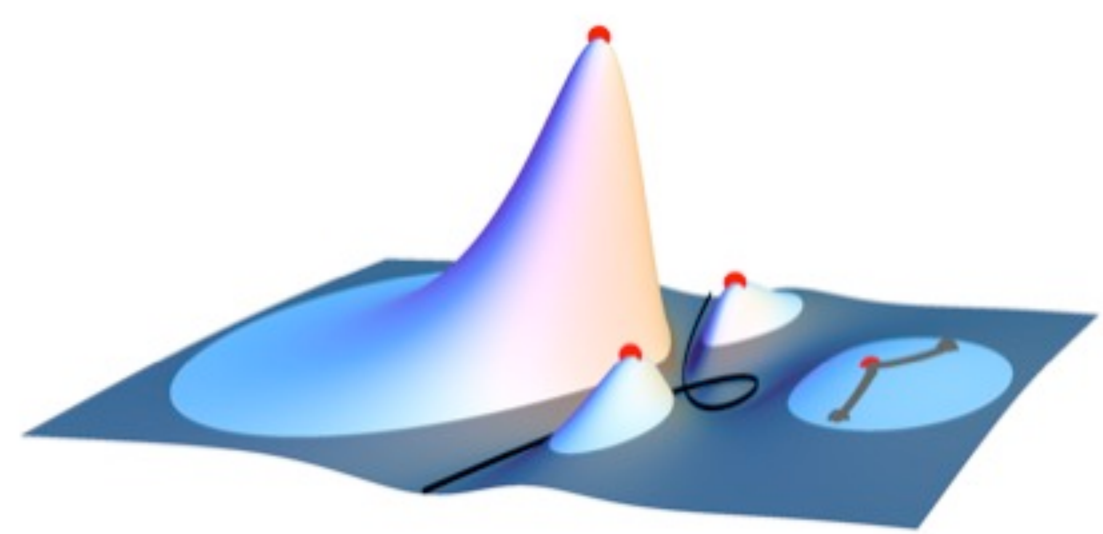
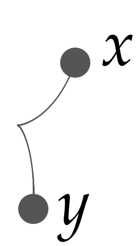




Total Length $\leq$ Length  $\leq 2 \cdot 2nr(6d + 4)^{n-1}$

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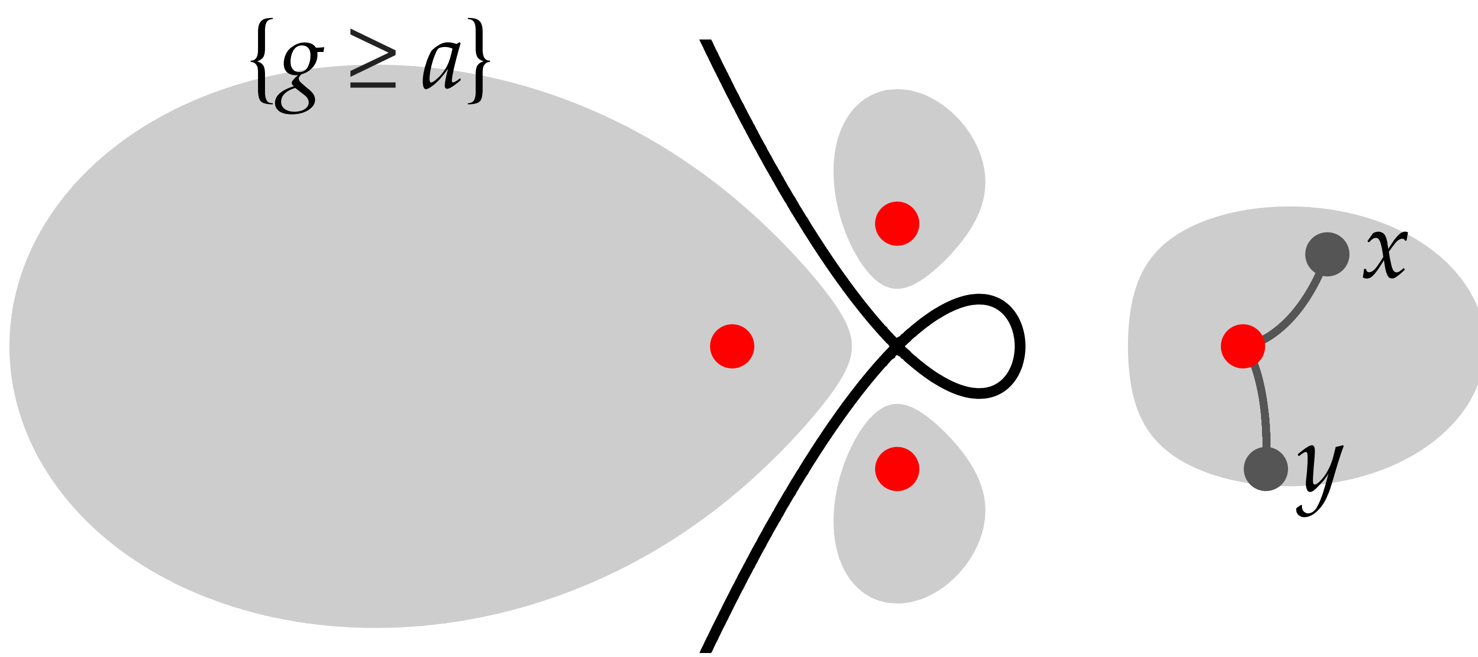
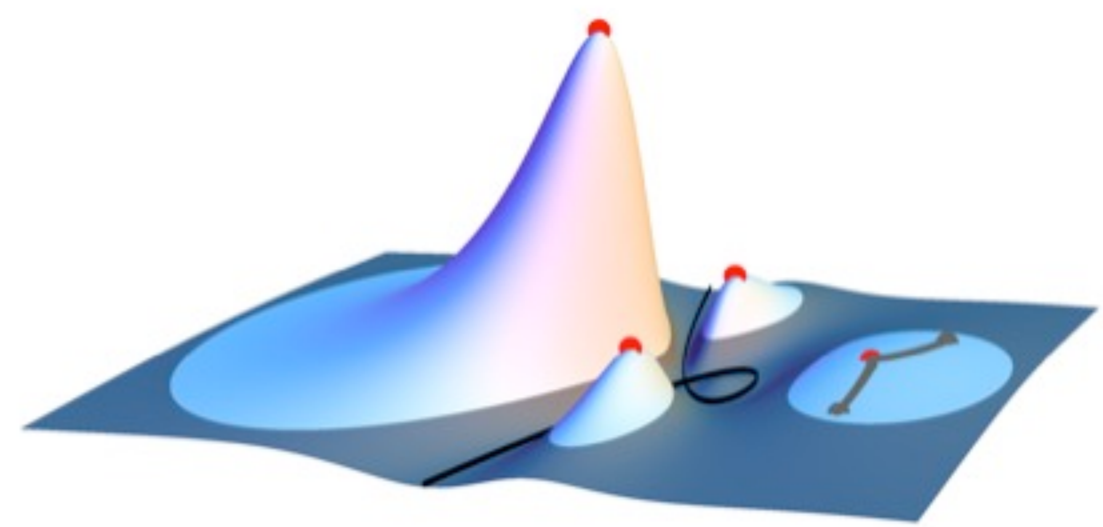
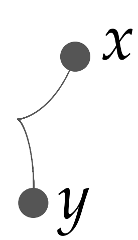




Total Length $\leq$ Length 

$\leq 2 \cdot 2nr(6d+4)^{n-1} = 4nr(6d+4)^{n-1}$

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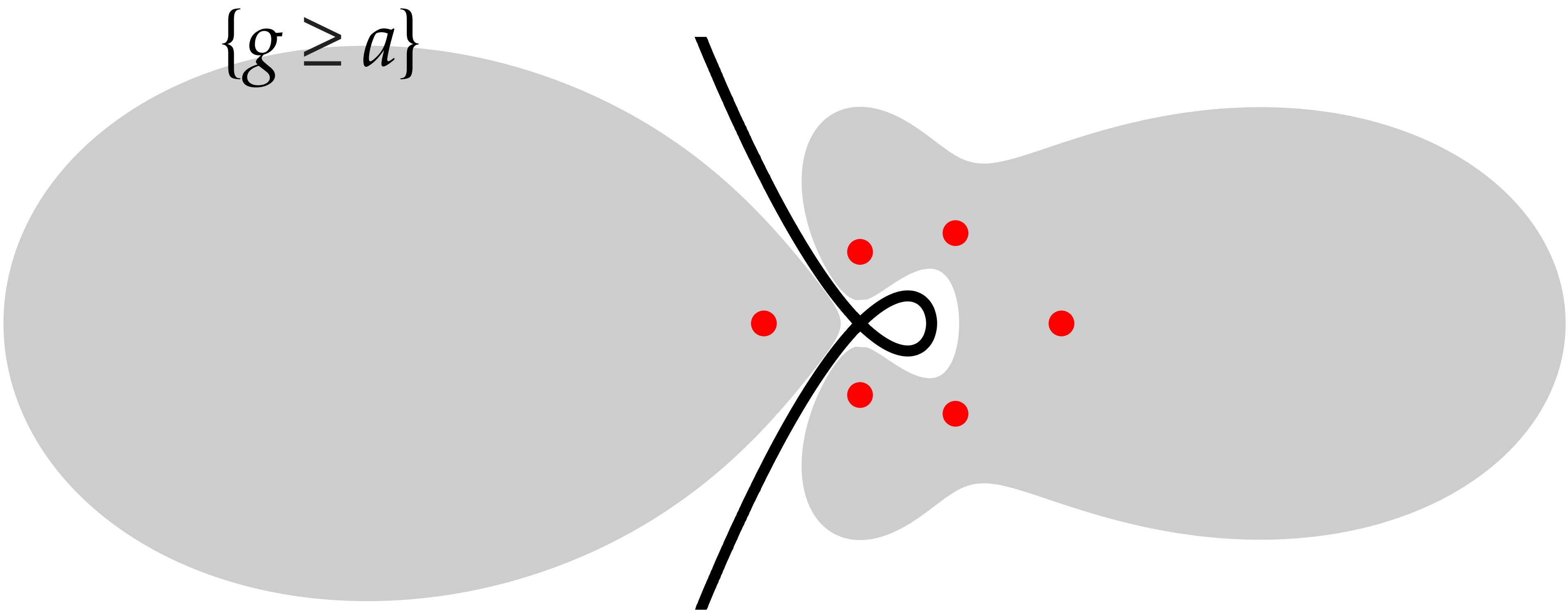

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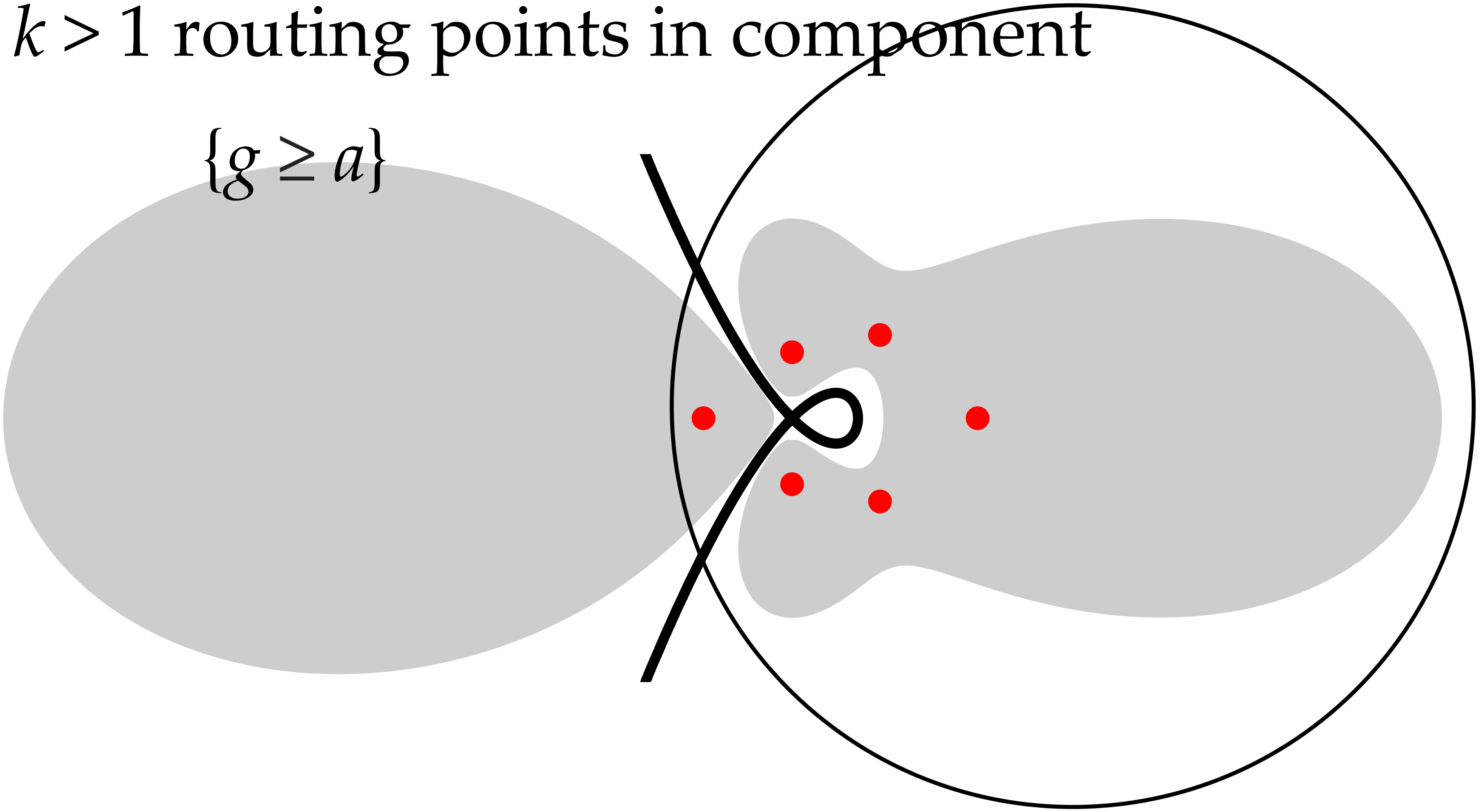

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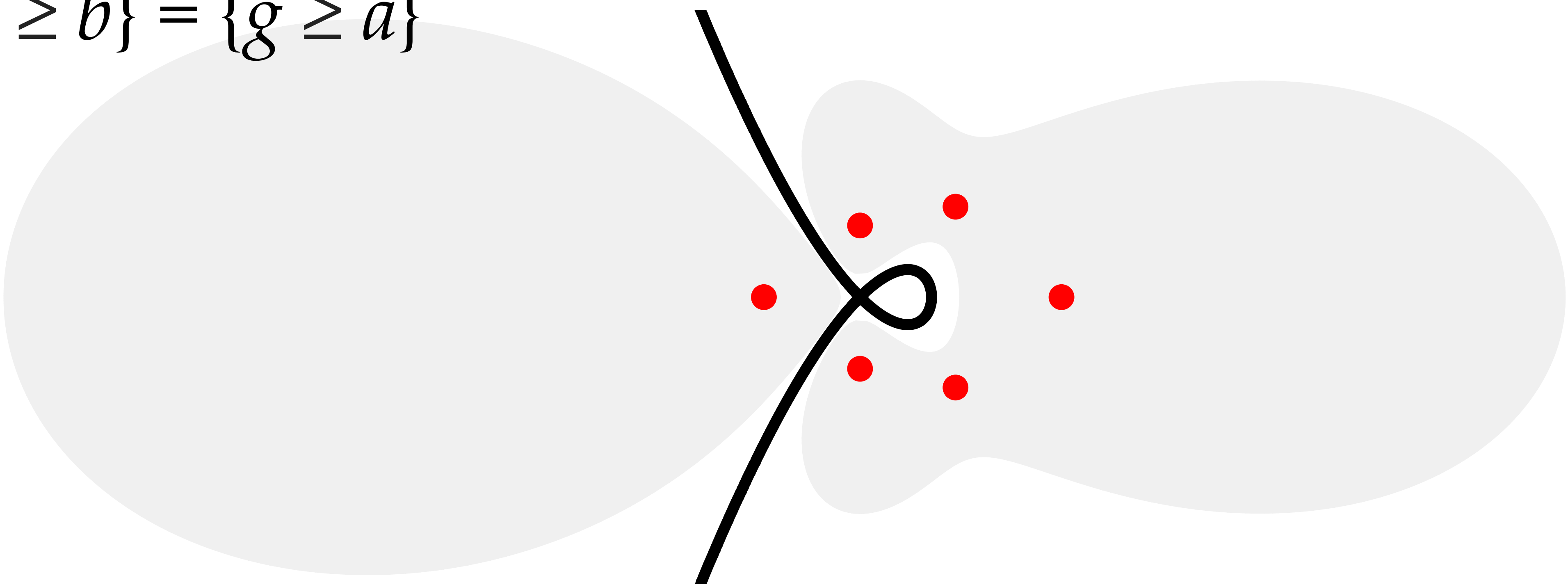

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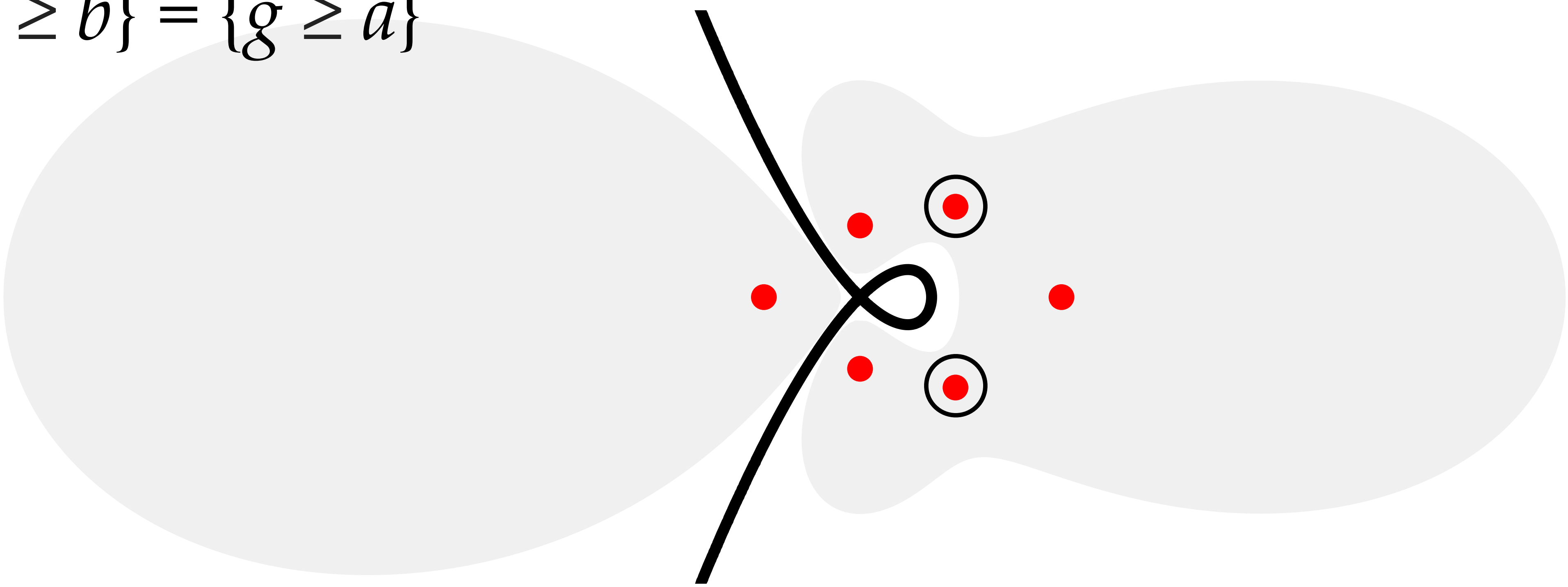

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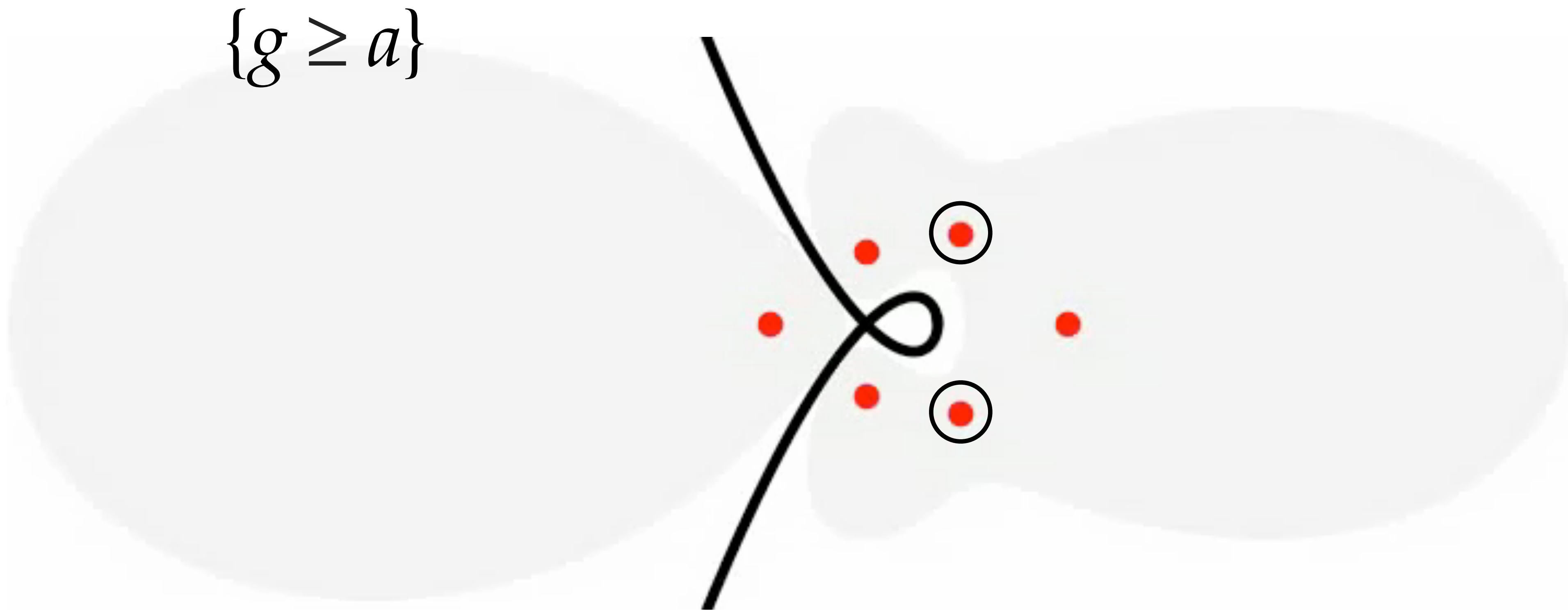

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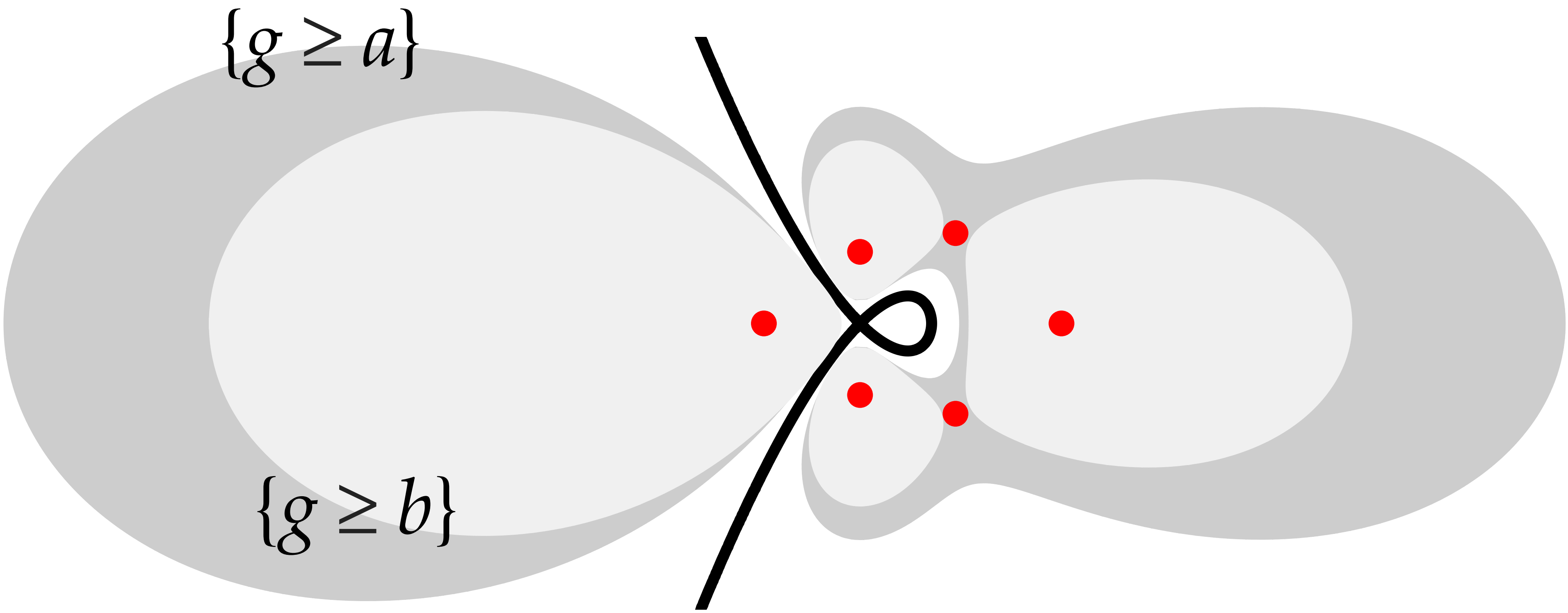

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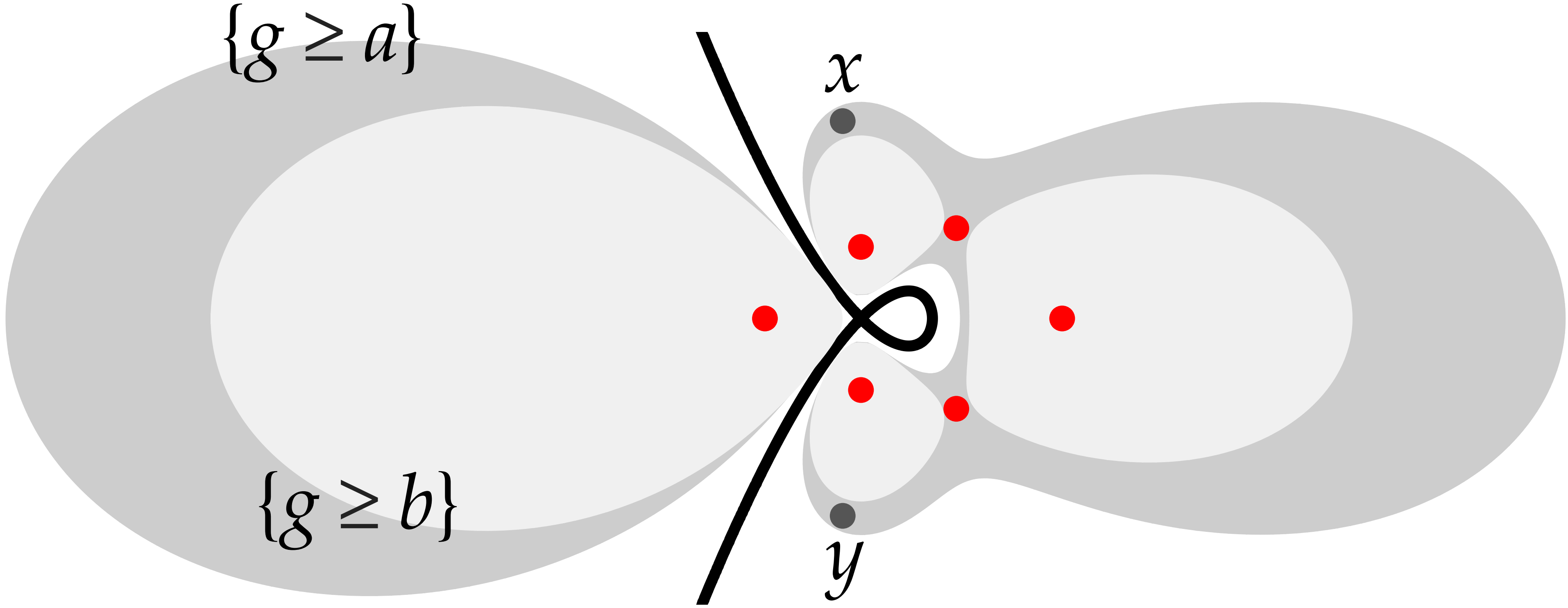

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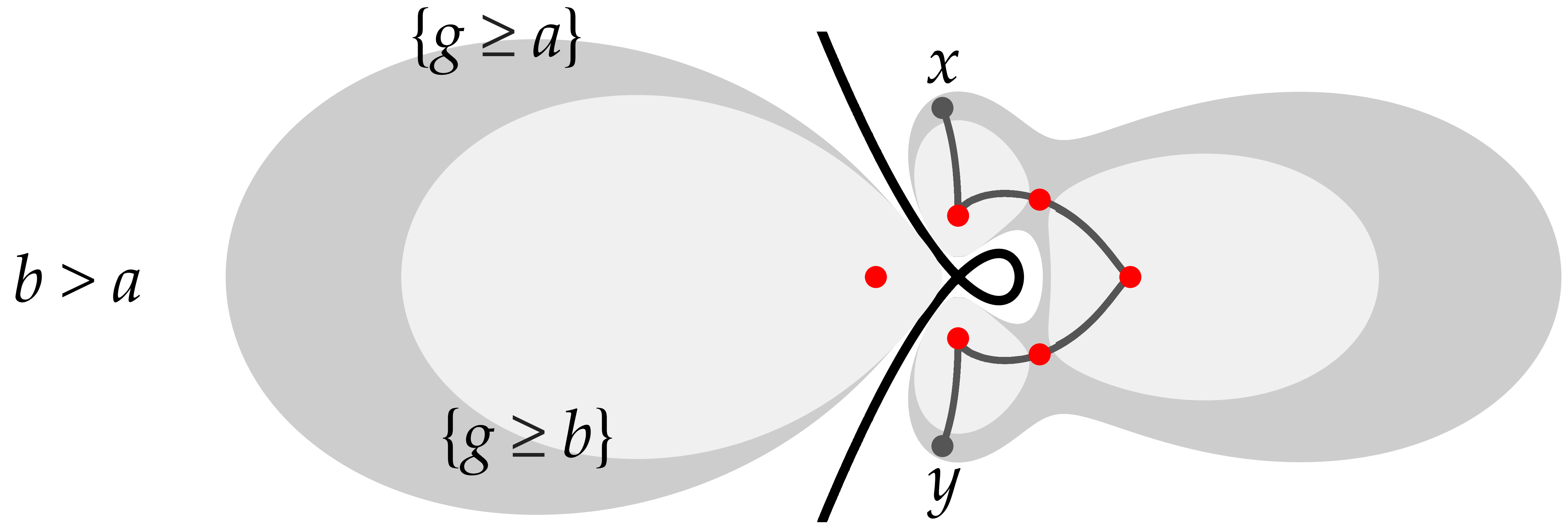

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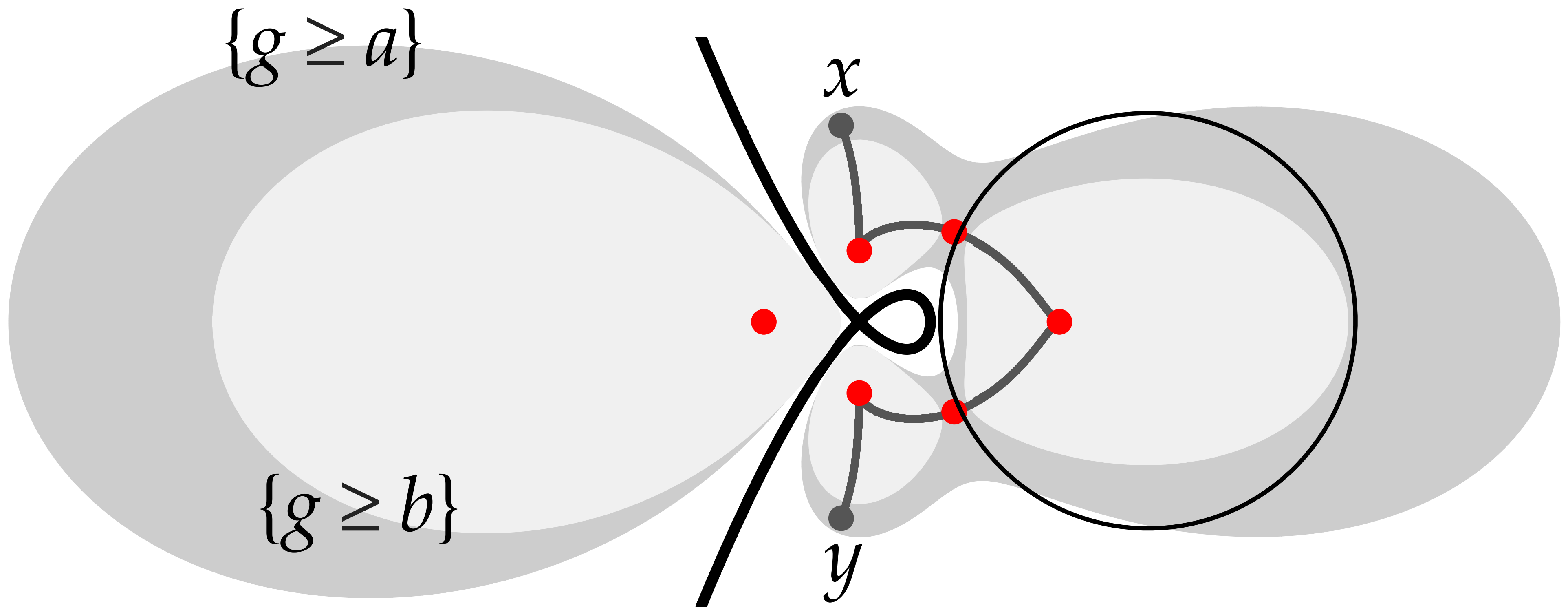

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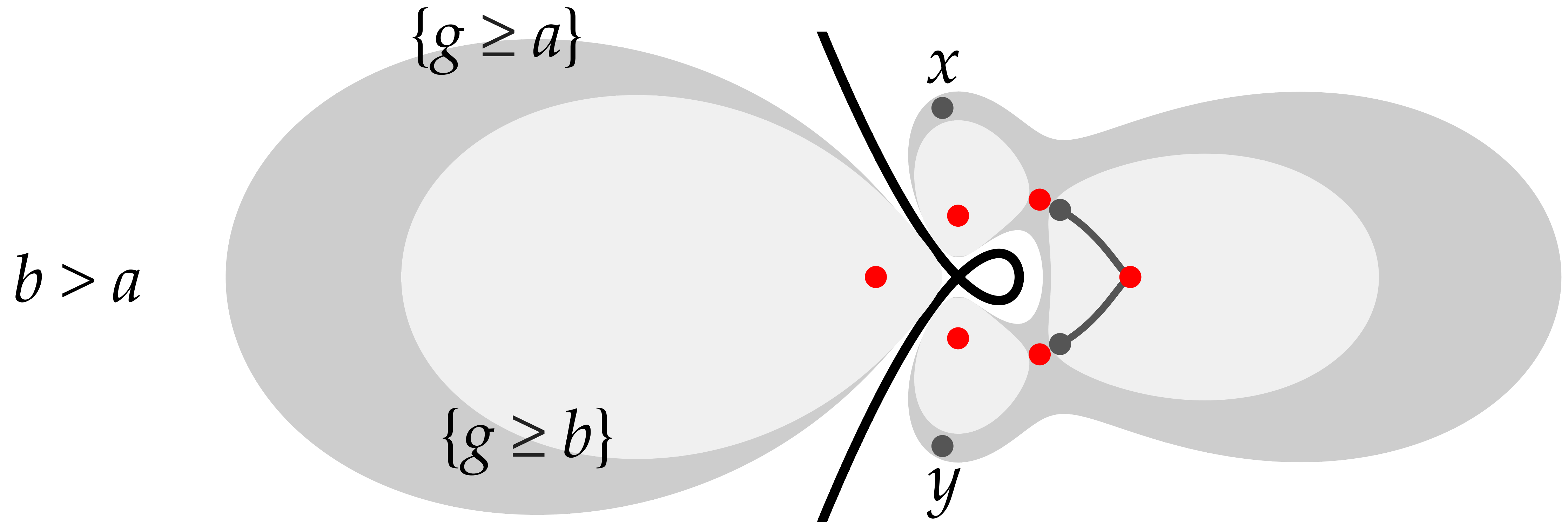

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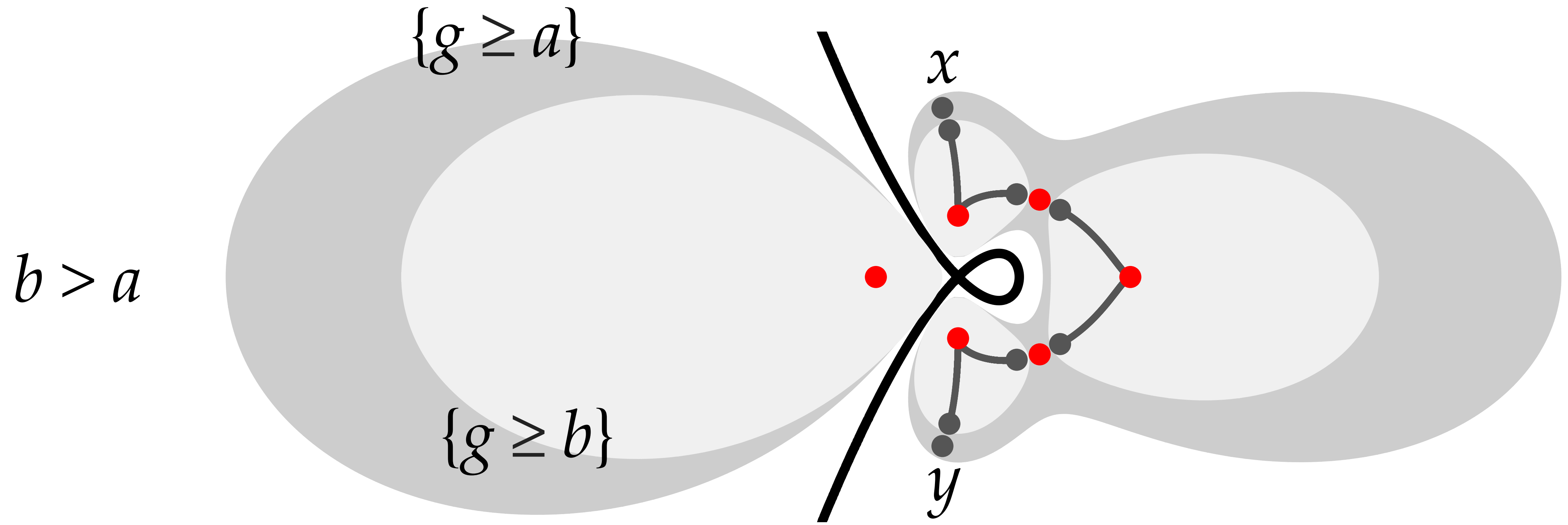

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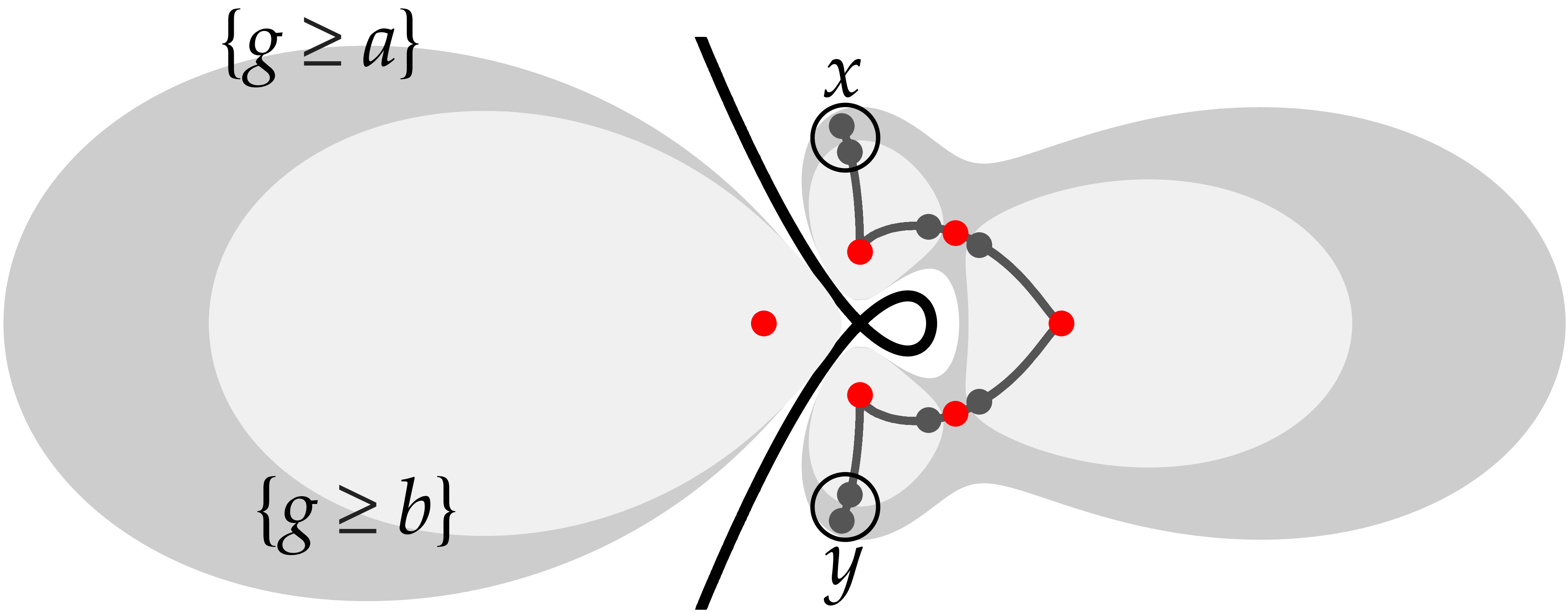

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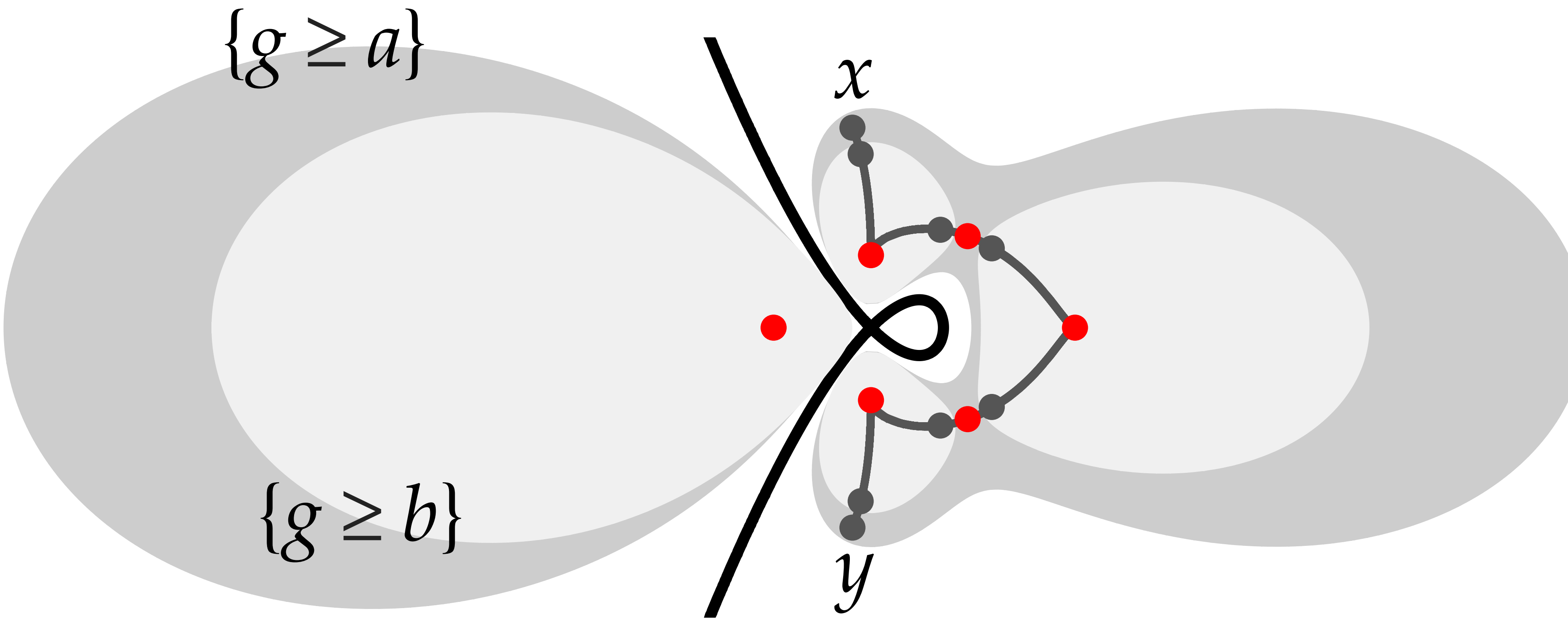
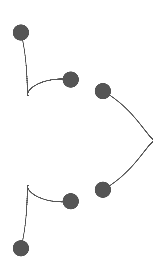


Total Length $\leq$ Length  + Length 

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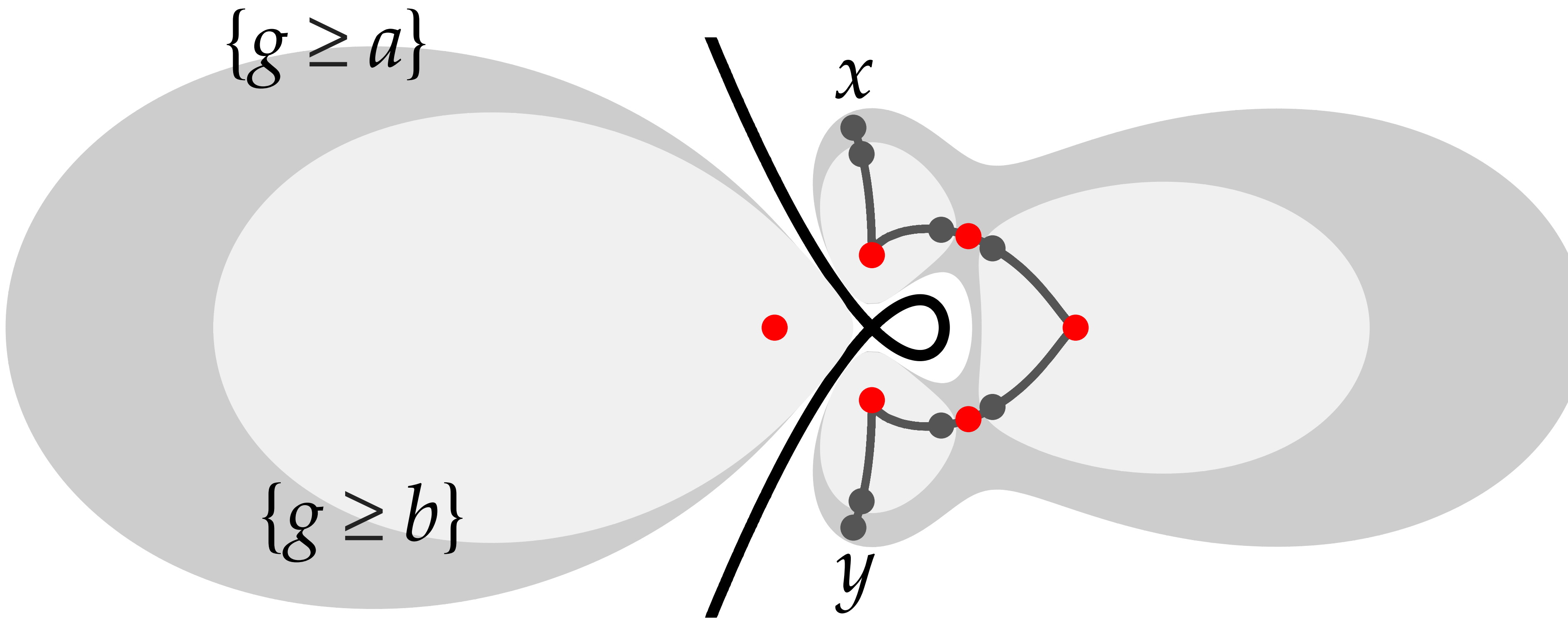
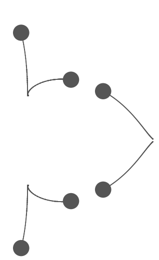


Total Length $\leq$ Length  + Length 

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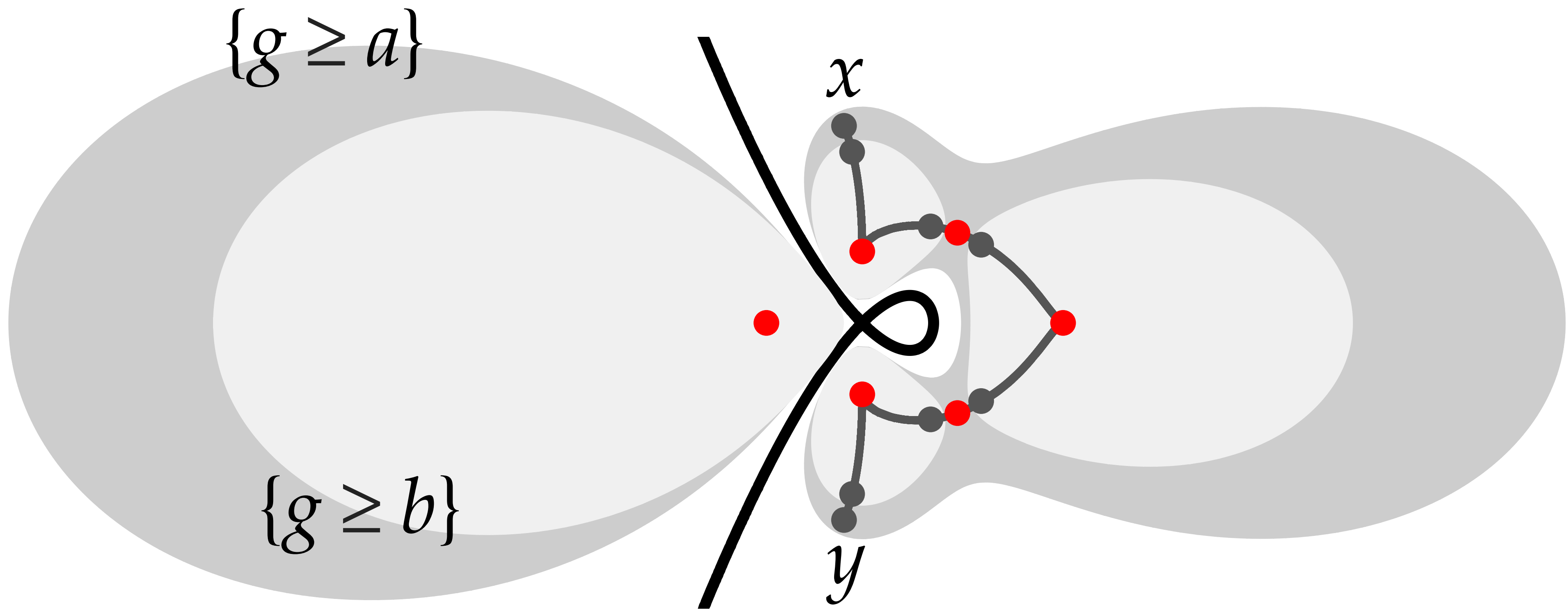
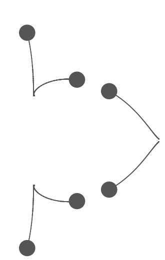


Total Length $\leq$ Length  + Length 

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Future Work

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- Rigorously tracing steepest ascent paths

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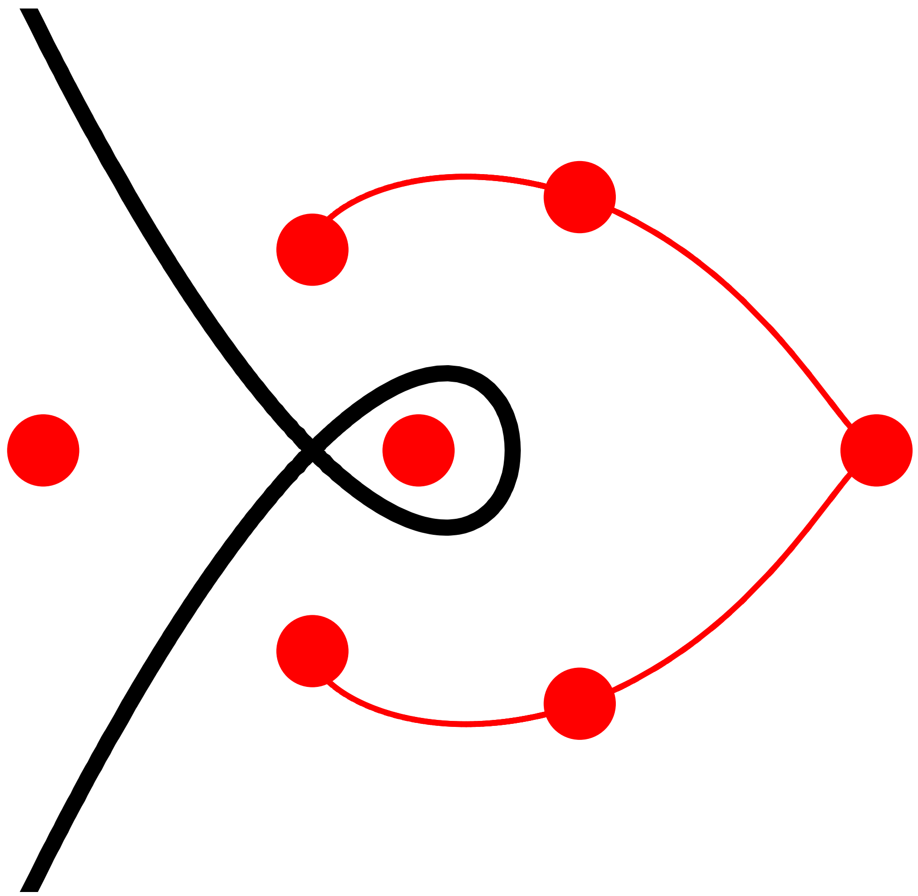

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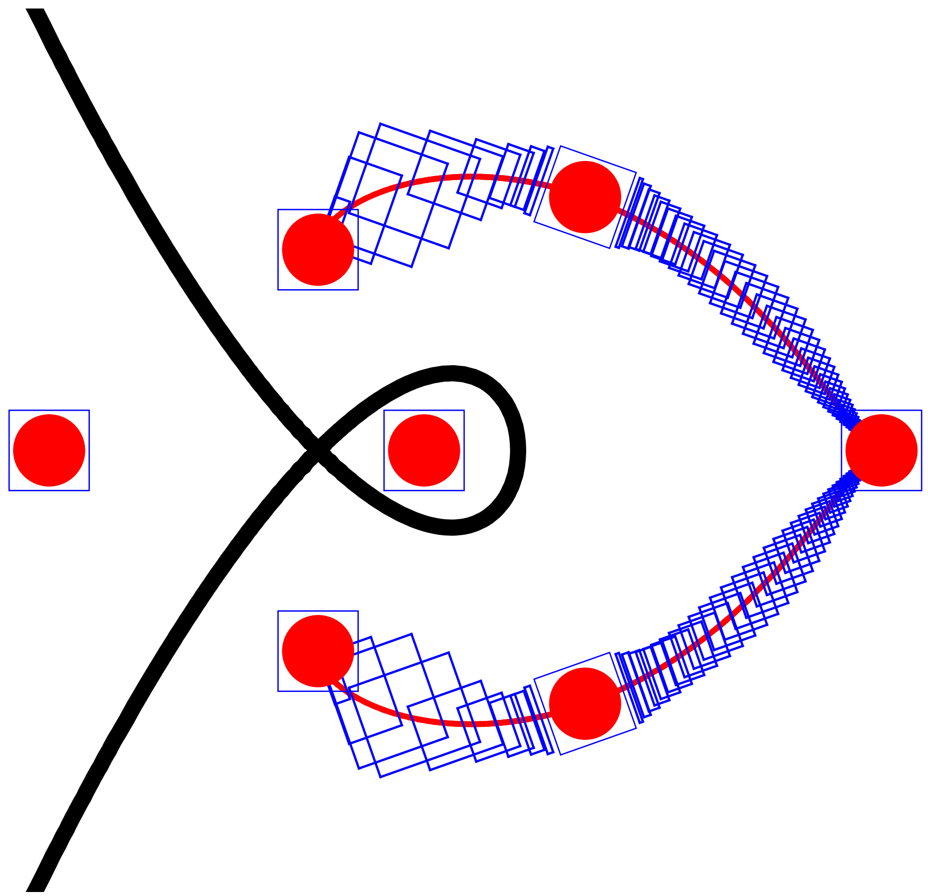

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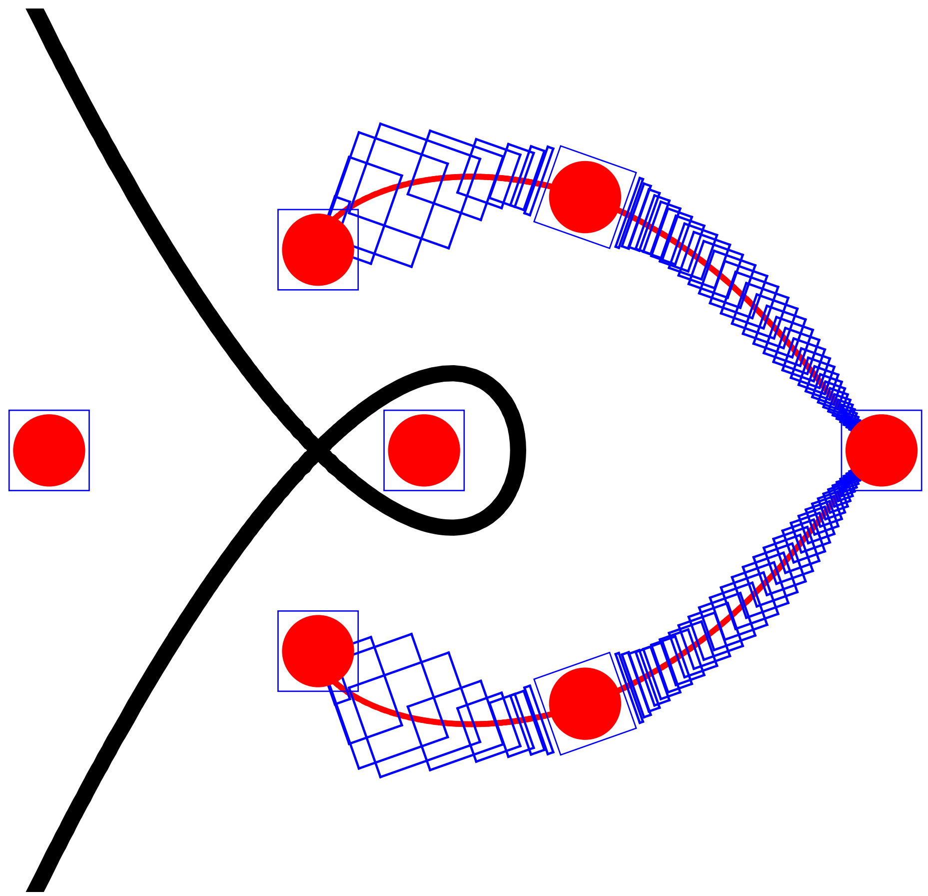

- Improve bounds

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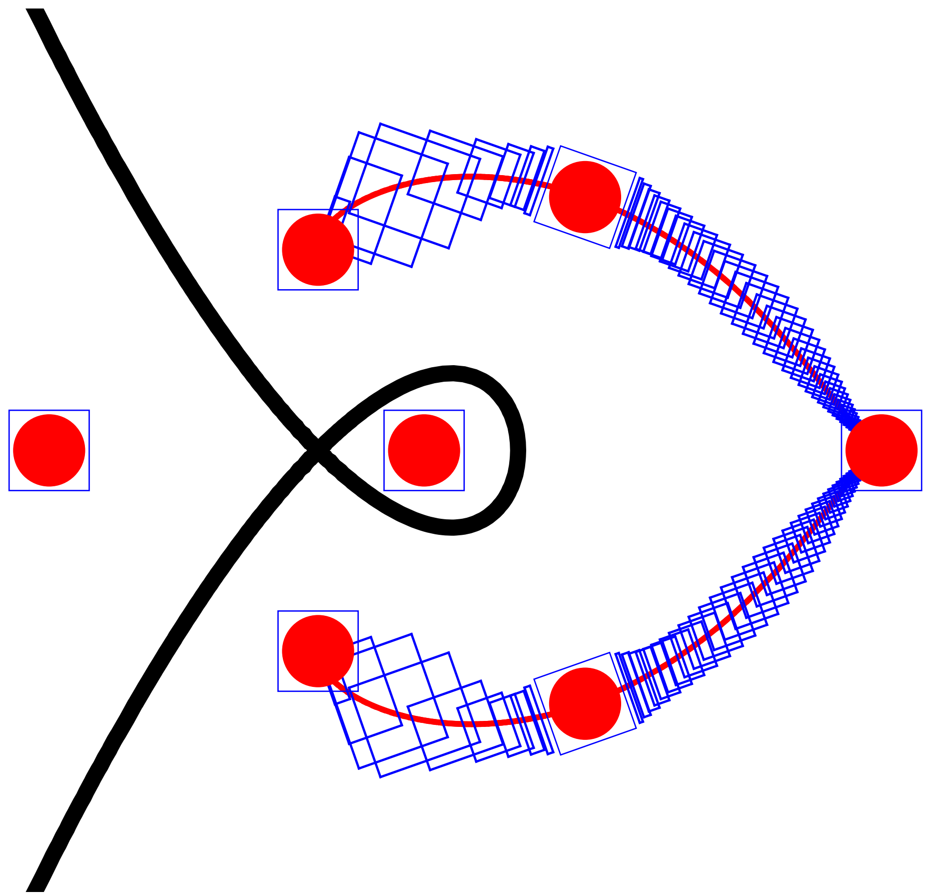

- Improve bounds
- Complexity analysis