

# Connectivity in Semialgebraic Sets

by

James Rohal ([jjrohal@ncsu.edu](mailto:jjrohal@ncsu.edu))

Preliminary exam for partial fulfillment of the  
requirements for the Degree of Doctor of  
Philosophy at North Carolina State University

Applied Mathematics

August 8, 2013

# Committee

|                     |                                  |                        |
|---------------------|----------------------------------|------------------------|
| Chair               | Hoon Hong <sup>1</sup>           | hong@ncsu.edu          |
| Member              | Jonathan Hauenstein <sup>1</sup> | hauenstein@ncsu.edu    |
| Member              | Erich Kaltofen <sup>1</sup>      | kaltofen@ncsu.edu      |
| Member              | Agnes Szanto <sup>1</sup>        | szanto@ncsu.edu        |
| External Member     | Mohab Safey El Din <sup>2</sup>  | Mohab.Safey@lip6.fr    |
| Proxy               | Irina Kogan <sup>1</sup>         | iakogan@ncsu.edu       |
| Graduate School Rep | Edgar Lobaton <sup>1</sup>       | edgar.lobaton@ncsu.edu |

<sup>1</sup> North Carolina State University, Raleigh, NC 27695, USA

<sup>2</sup> Université Pierre et Marie Curie, 75005 Paris, France

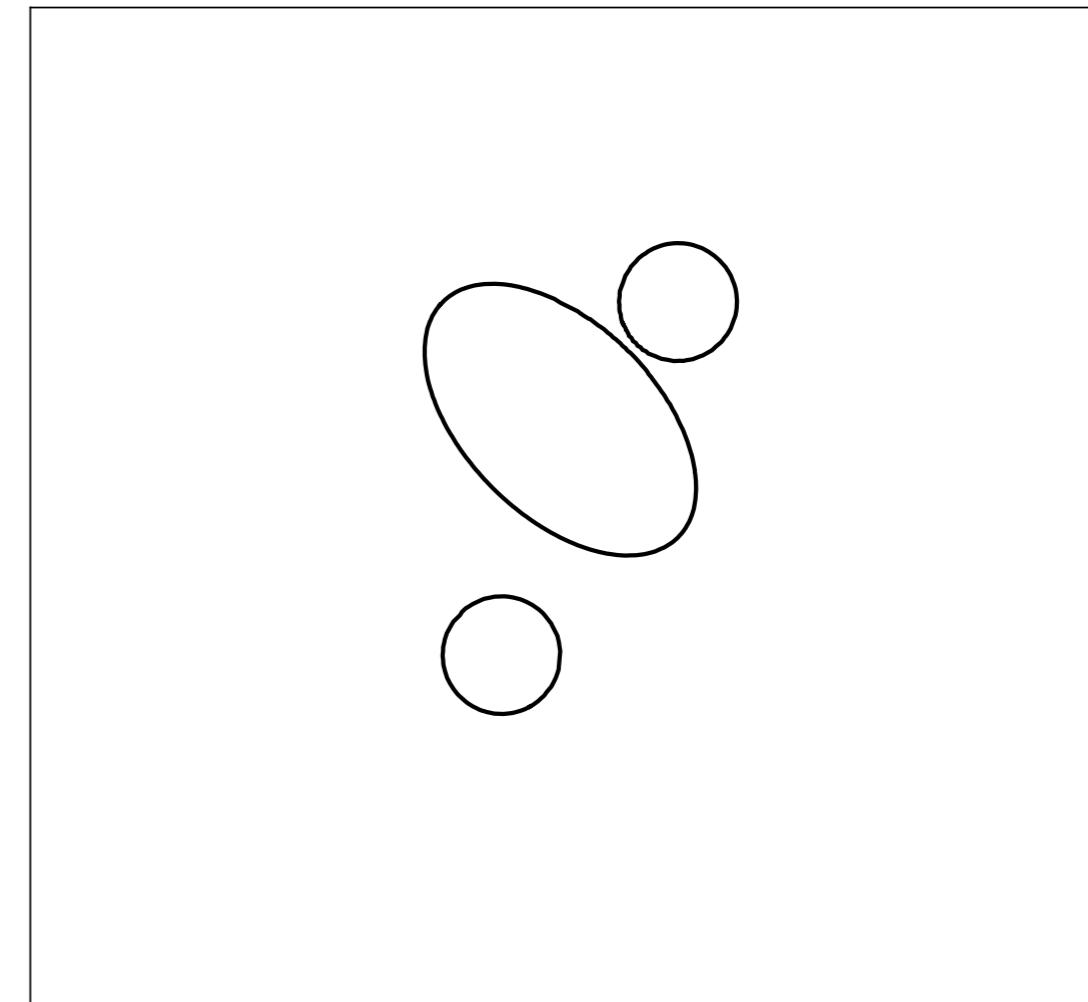
# Problem: Connectivity

# Problem: Connectivity

$$f = \left( (x_1 + 1)^2 + (x_2 + 4)^2 - 1 \right) \left( x_1^2 + x_1 x_2 + x_2^2 - 4 \right) \left( (x_1 - 2)^2 + (x_2 - 2)^2 - 1 \right)$$

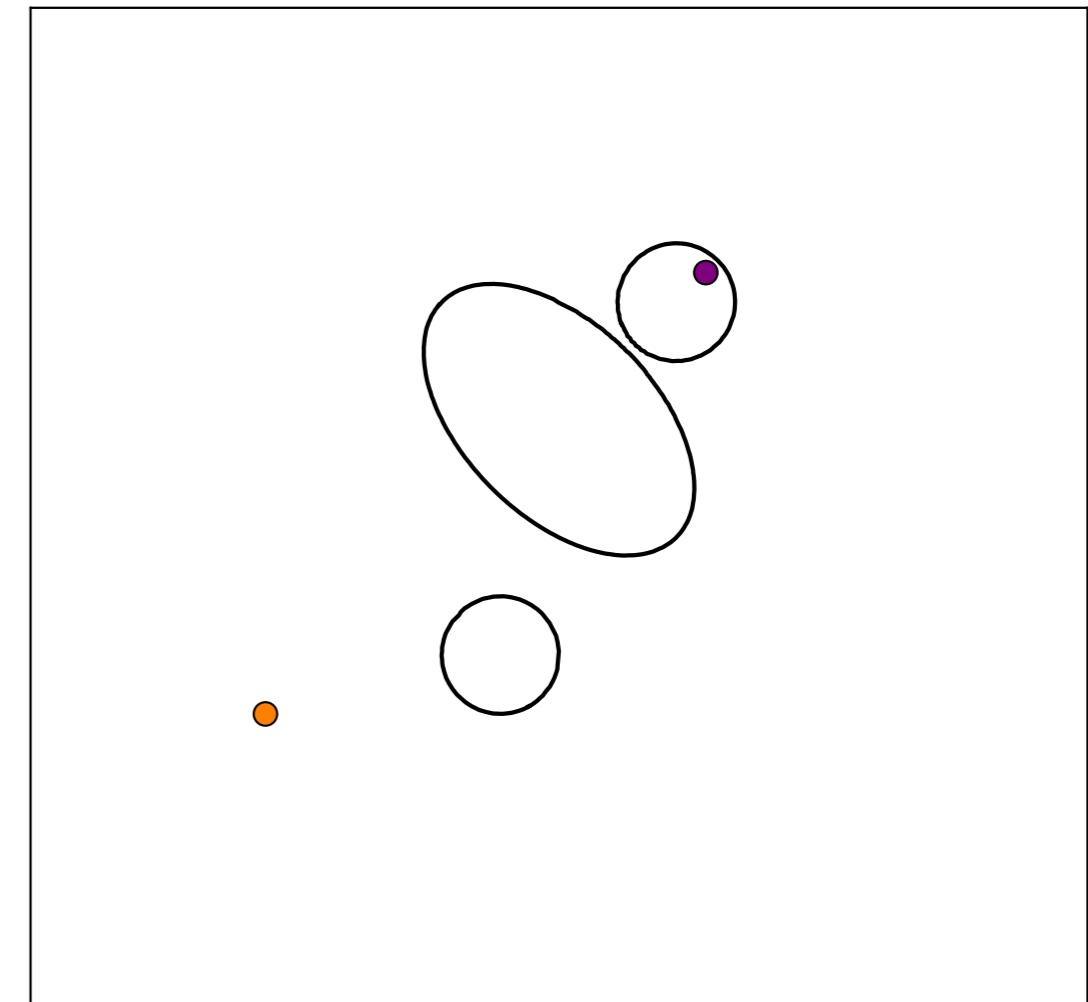
# Problem: Connectivity

$$f = \left( (x_1 + 1)^2 + (x_2 + 4)^2 - 1 \right) \left( x_1^2 + x_1 x_2 + x_2^2 - 4 \right) \left( (x_1 - 2)^2 + (x_2 - 2)^2 - 1 \right)$$



# Problem: Connectivity

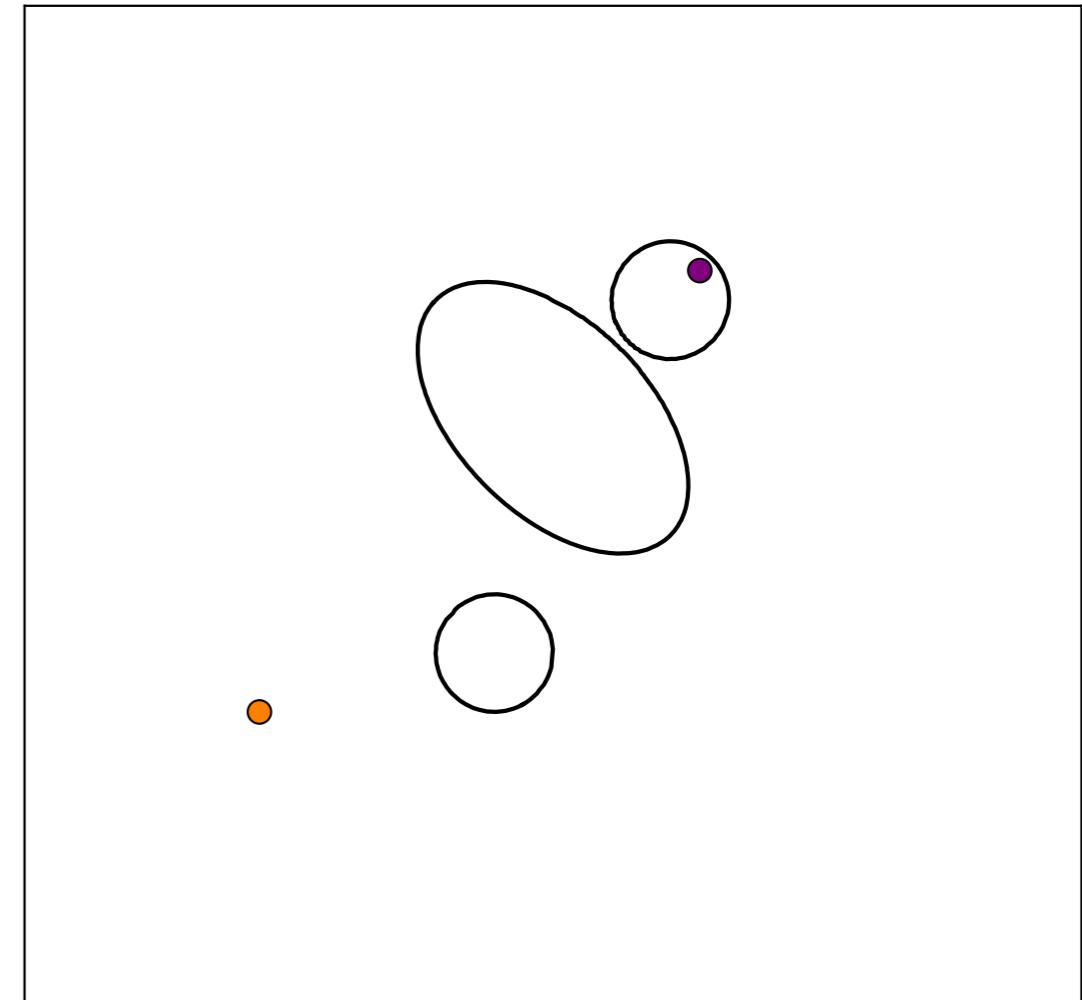
$$f = \left( (x_1 + 1)^2 + (x_2 + 4)^2 - 1 \right) \left( x_1^2 + x_1 x_2 + x_2^2 - 4 \right) \left( (x_1 - 2)^2 + (x_2 - 2)^2 - 1 \right)$$



# Problem: Connectivity

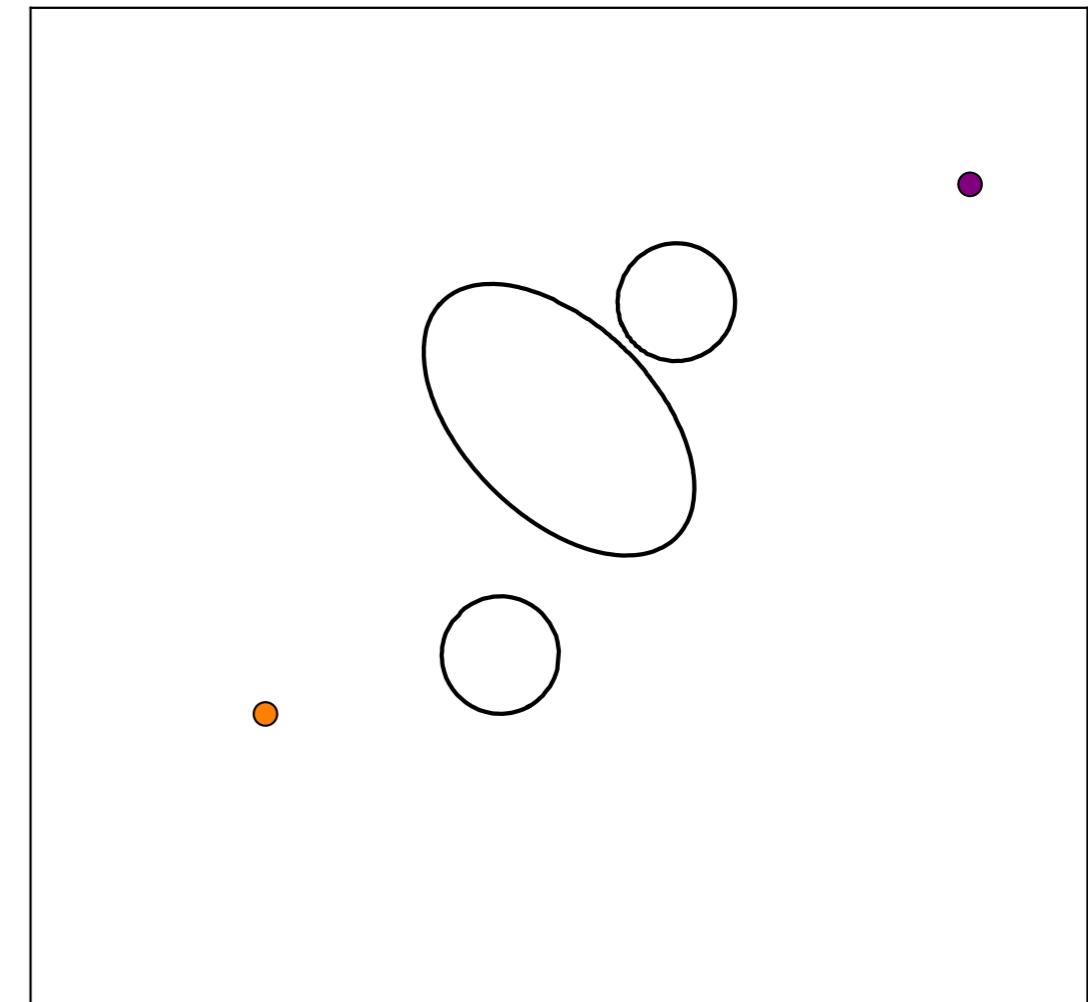
$$f = \left( (x_1 + 1)^2 + (x_2 + 4)^2 - 1 \right) \left( x_1^2 + x_1 x_2 + x_2^2 - 4 \right) \left( (x_1 - 2)^2 + (x_2 - 2)^2 - 1 \right)$$

**False**



# Problem: Connectivity

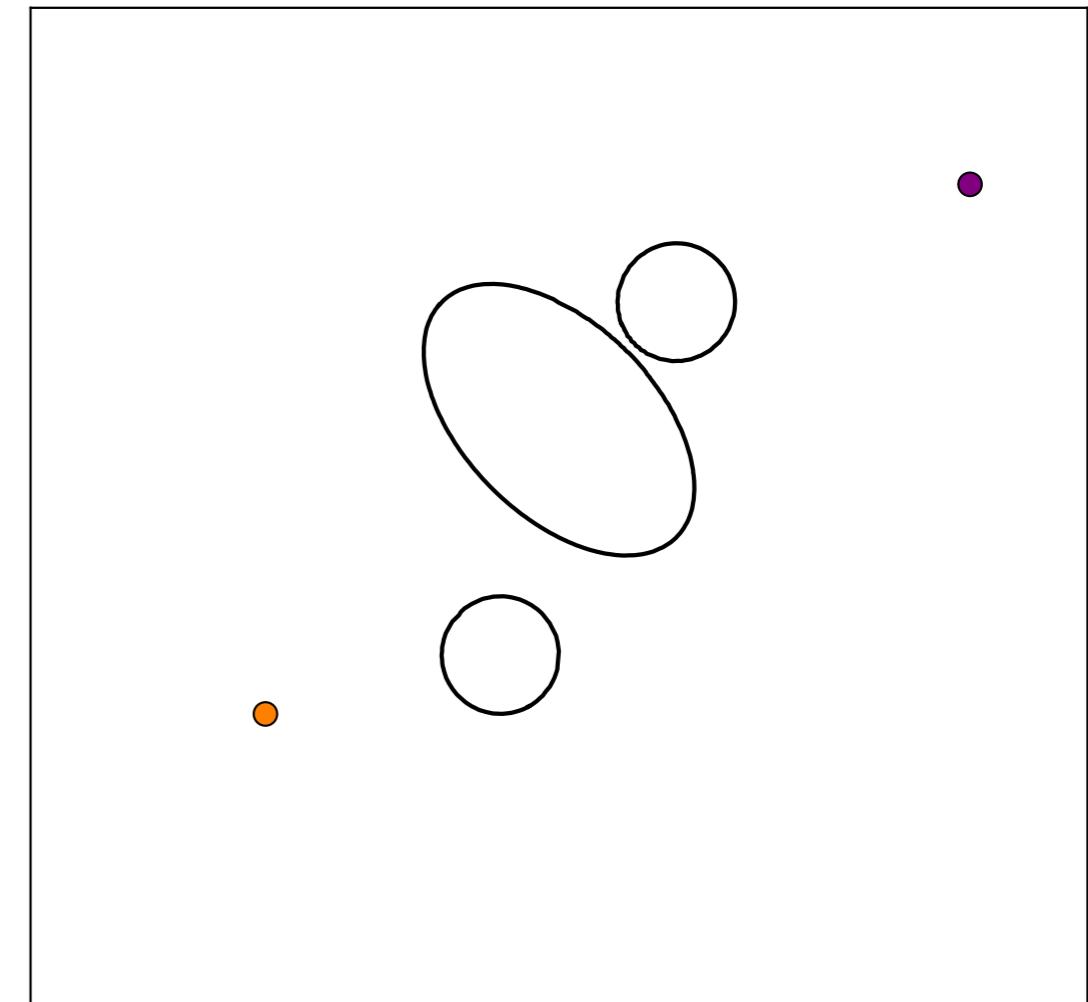
$$f = \left( (x_1 + 1)^2 + (x_2 + 4)^2 - 1 \right) \left( x_1^2 + x_1 x_2 + x_2^2 - 4 \right) \left( (x_1 - 2)^2 + (x_2 - 2)^2 - 1 \right)$$



# Problem: Connectivity

$$f = \left( (x_1 + 1)^2 + (x_2 + 4)^2 - 1 \right) \left( x_1^2 + x_1 x_2 + x_2^2 - 4 \right) \left( (x_1 - 2)^2 + (x_2 - 2)^2 - 1 \right)$$

True



# Problem: Connectivity

**Input**

$$f \in \mathbb{Z}[x_1, \dots, x_n]$$

$$\bullet, \bullet \in \mathbb{Q}^n \cap \{f \neq 0\}$$

**True**

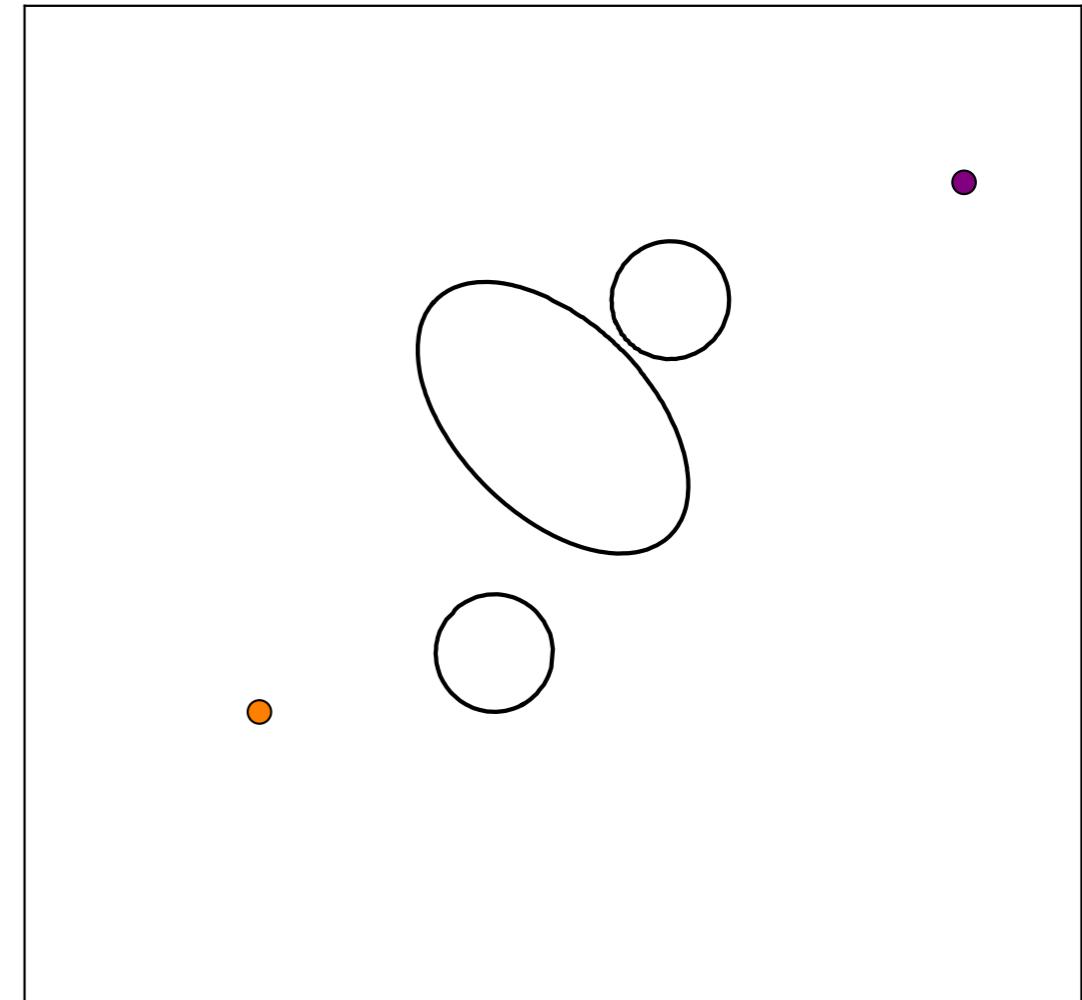
**Output**

**True**

if  $\bullet, \bullet$  are in a same  
semialgebraically  
connected component  
of  $\{f \neq 0\}$

**False**

otherwise



# Motivation

- Fundamental in computational real algebraic geometry.
- Many important applications in science and engineering.

- Previous work:

1975 Collins

1983 Schwartz, Sharir

1984 Arnon, Collins, McCallum

1987 Canny

1987 Roy

1988 Arnon, McCallum

1989 Alonso, Raimondo

1992 Feng

1992 Grigor'ev, Vorobjov

1993 Hong

1994 Heintz, Roy, Solerno

1996 Basu, Pollack, Roy

2008 Hong, Quinn

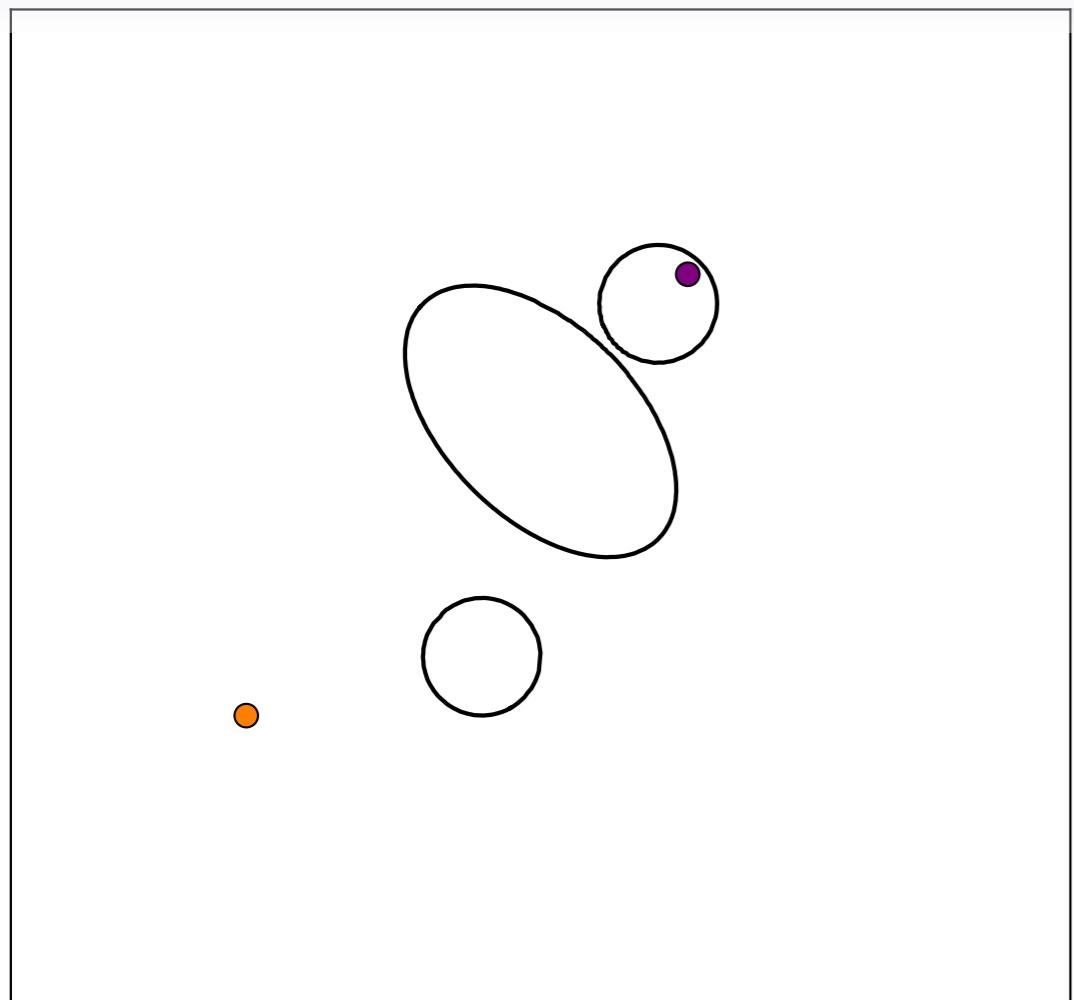
2009 Safey El Din, Schost

# Method: Overview

# Method: Overview

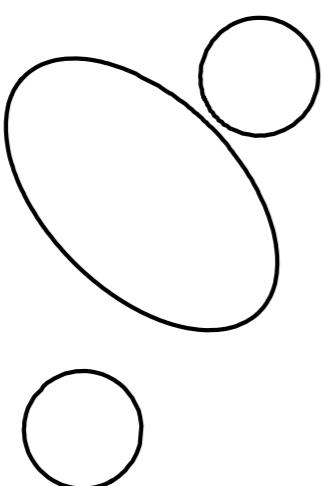
**Input:**  $f(x_1, x_2)$ ,  , 

# Method: Overview



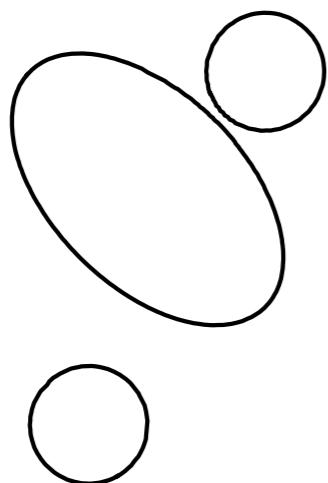
**Input:**  $f(x_1, x_2)$ ,  , 

# Method: Overview



**Input:**  $f(x_1, x_2)$ , ,

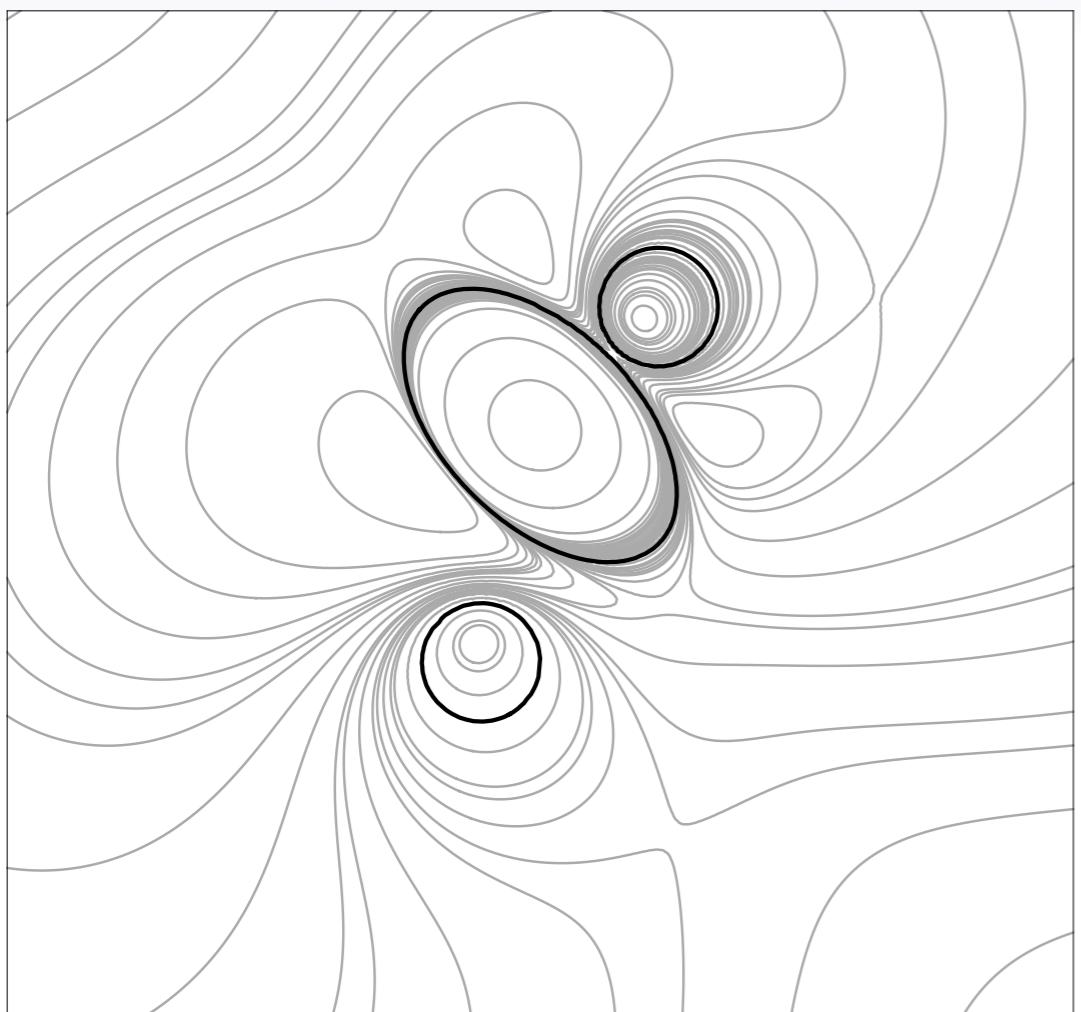
# Method: Overview



**Input:**  $f(x_1, x_2)$ , ,

$$1: g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$$

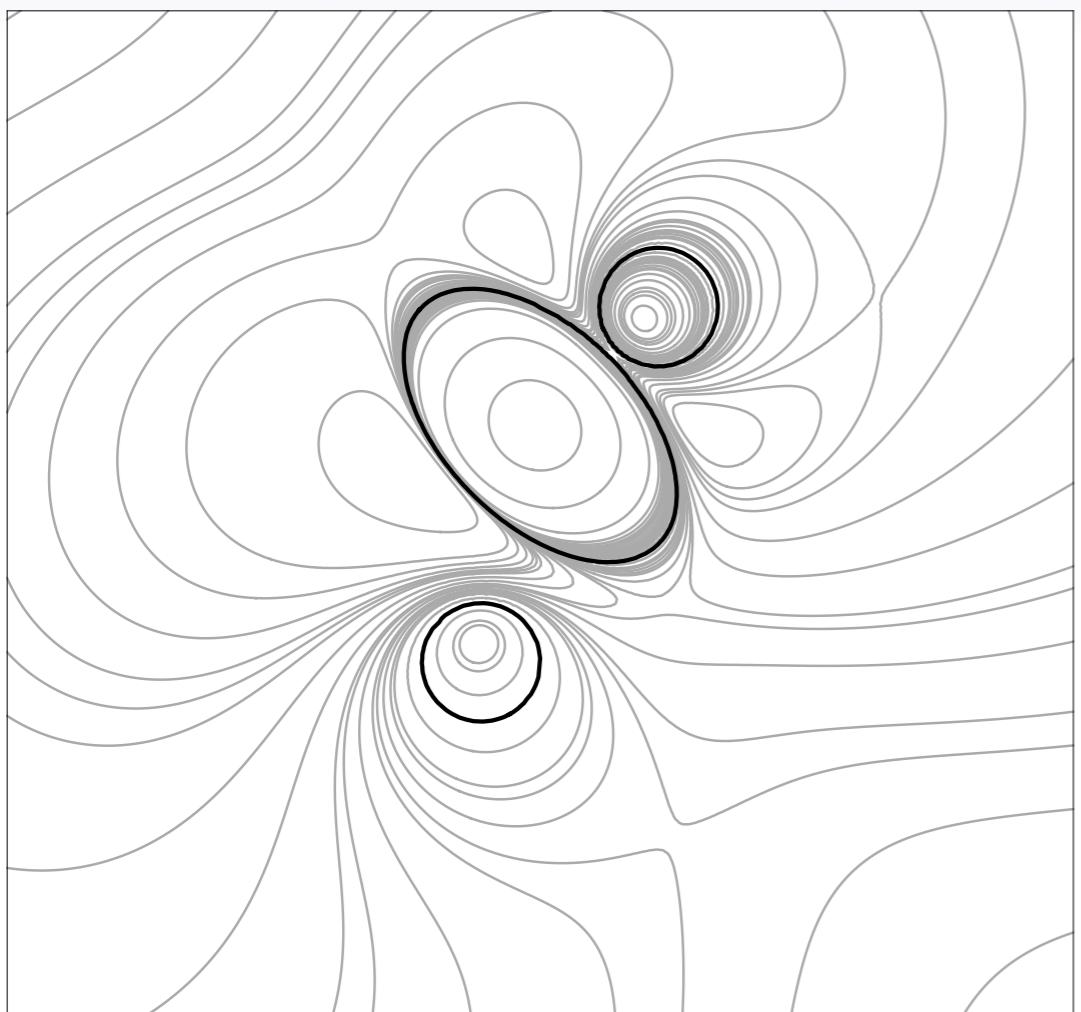
# Method: Overview



**Input:**  $f(x_1, x_2)$ , ,

**1:** 
$$g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$$

# Method: Overview

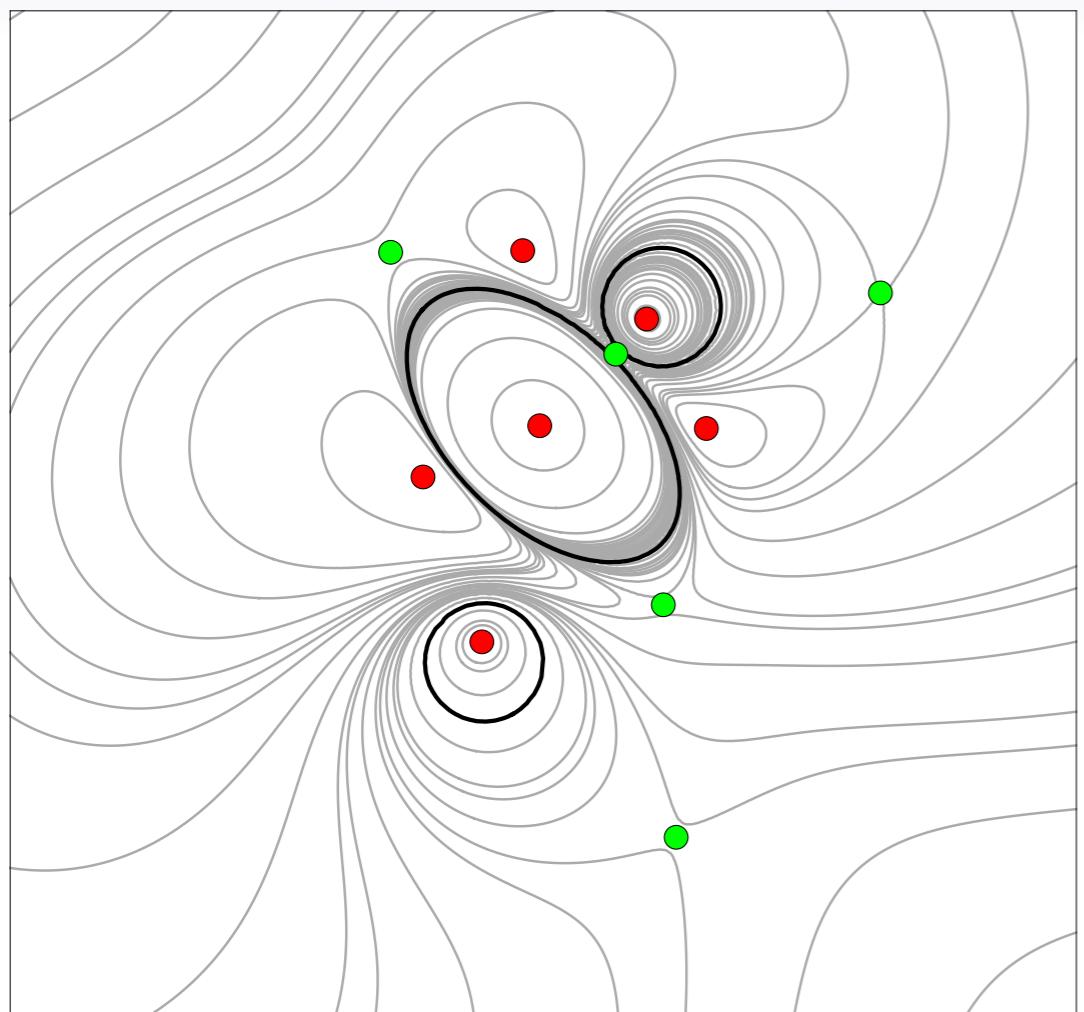


**Input:**  $f(x_1, x_2)$ ,  , 

**1:** 
$$g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$$

**2:** Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$

# Method: Overview

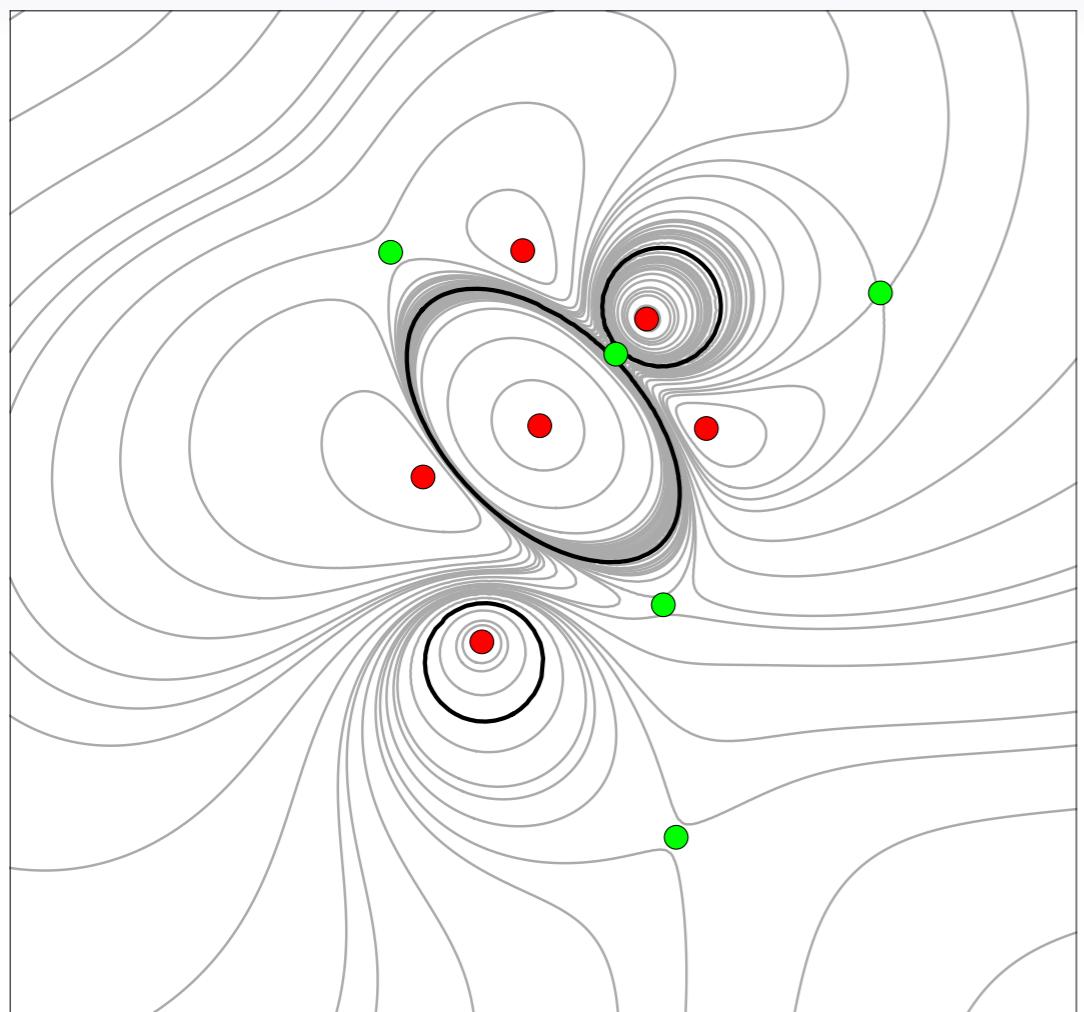


**Input:**  $f(x_1, x_2)$ , ,

**1:** 
$$g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$$

**2:** Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$

# Method: Overview



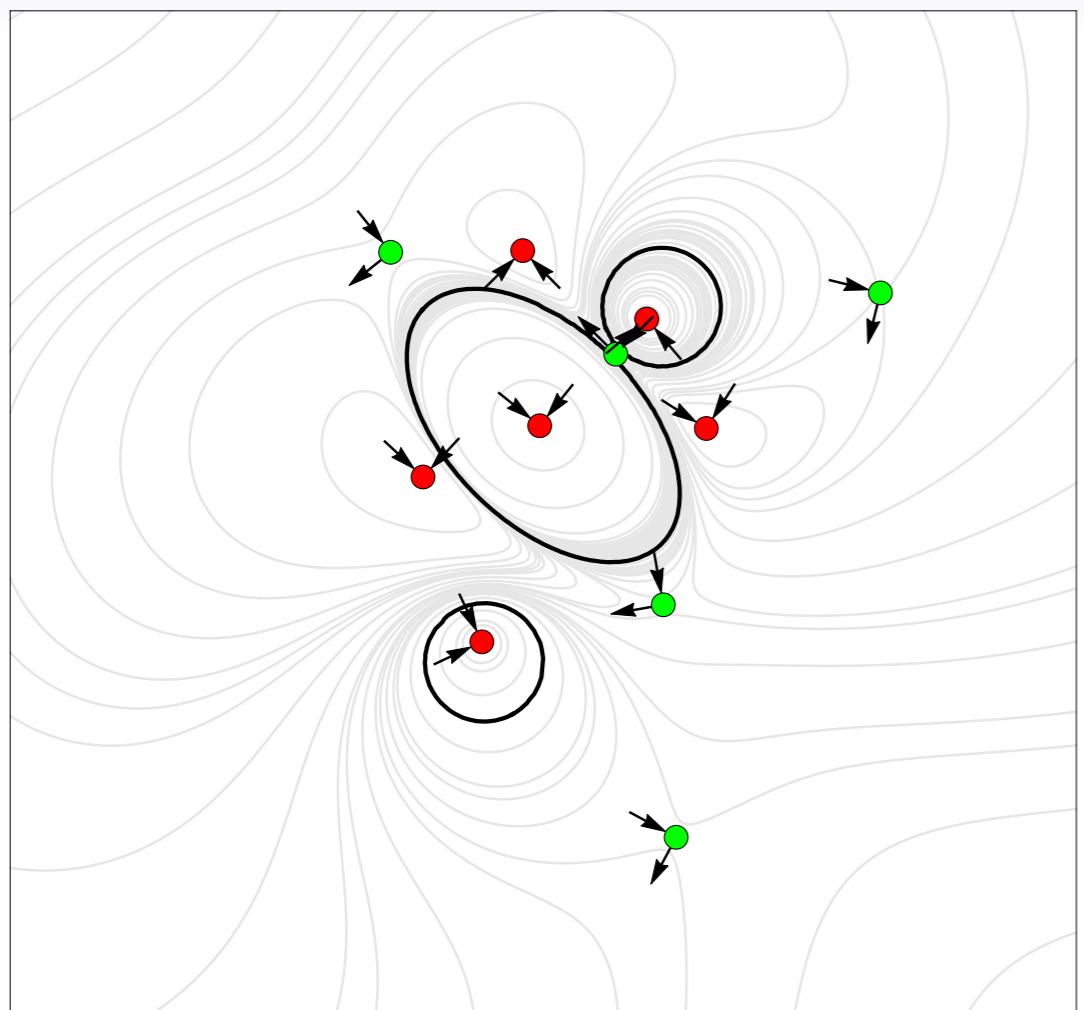
**Input:**  $f(x_1, x_2)$ , ,

**1:** 
$$g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$$

**2:** Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$

**3:** Find eigenvectors of  $(\text{Hess } g)(x)$

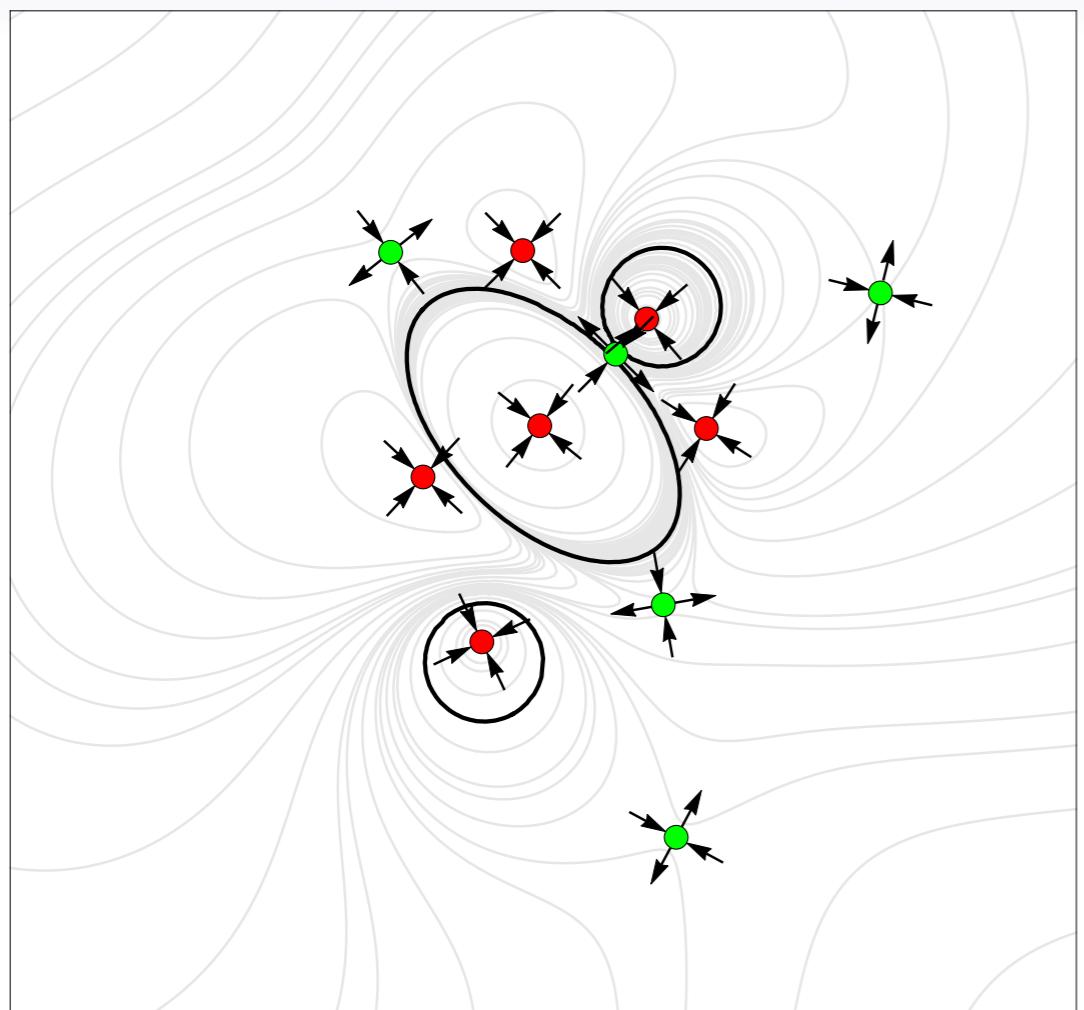
# Method: Overview



**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$

# Method: Overview



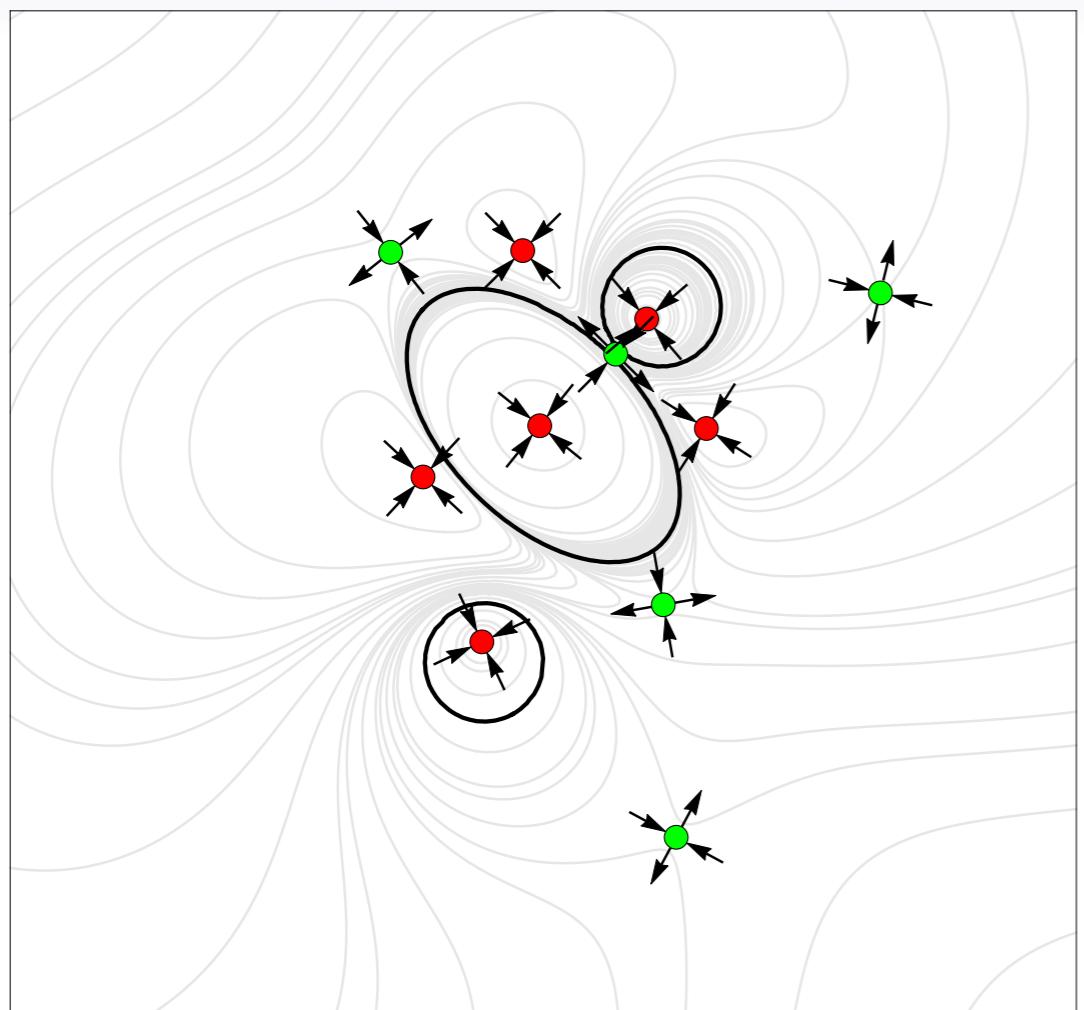
**Input:**  $f(x_1, x_2)$ , ,

**1:** 
$$g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$$

**2:** Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$

**3:** Find eigenvectors of  $(\text{Hess } g)(x)$

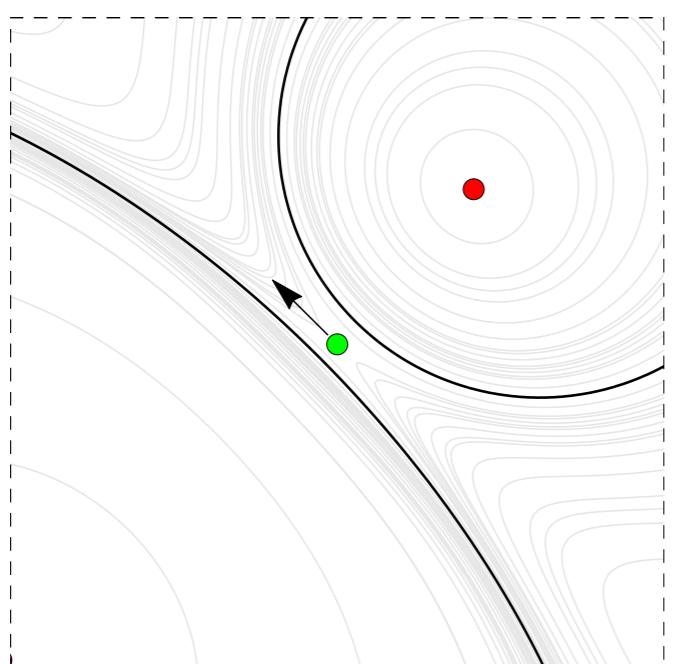
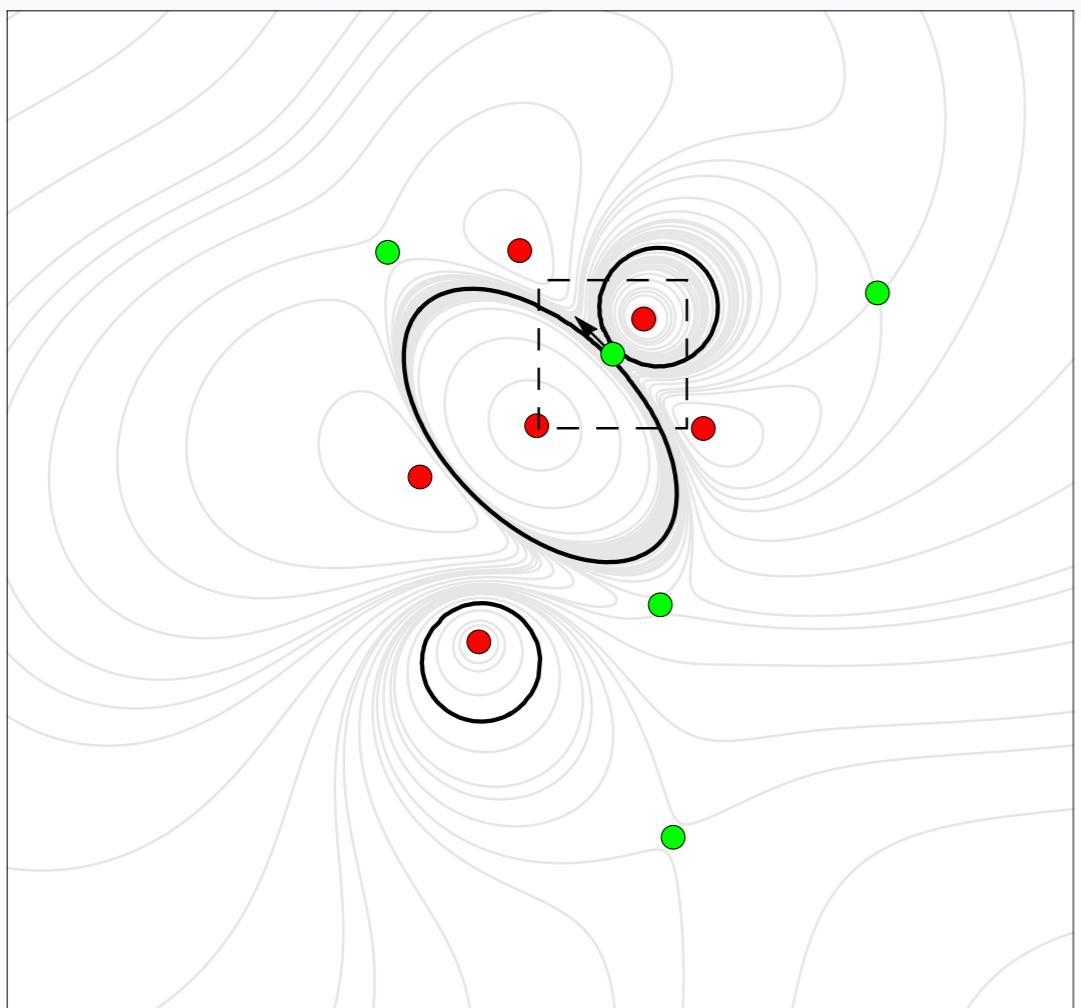
# Method: Overview



**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors

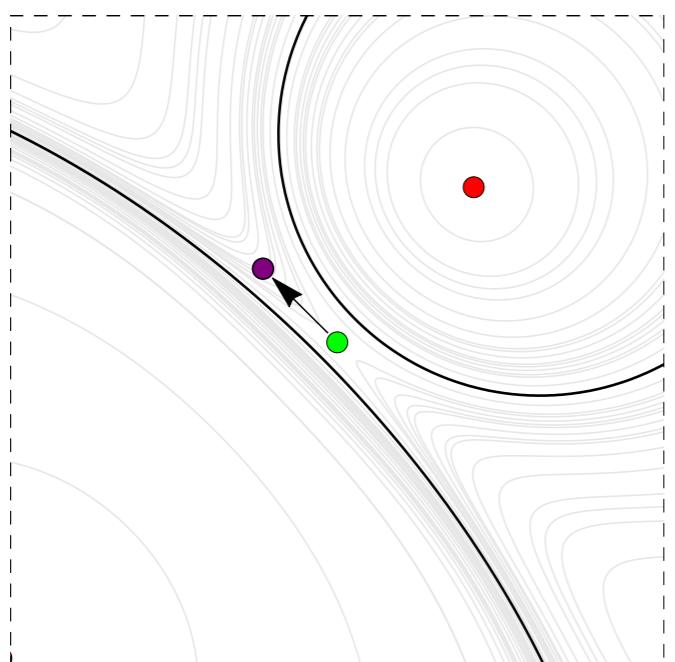
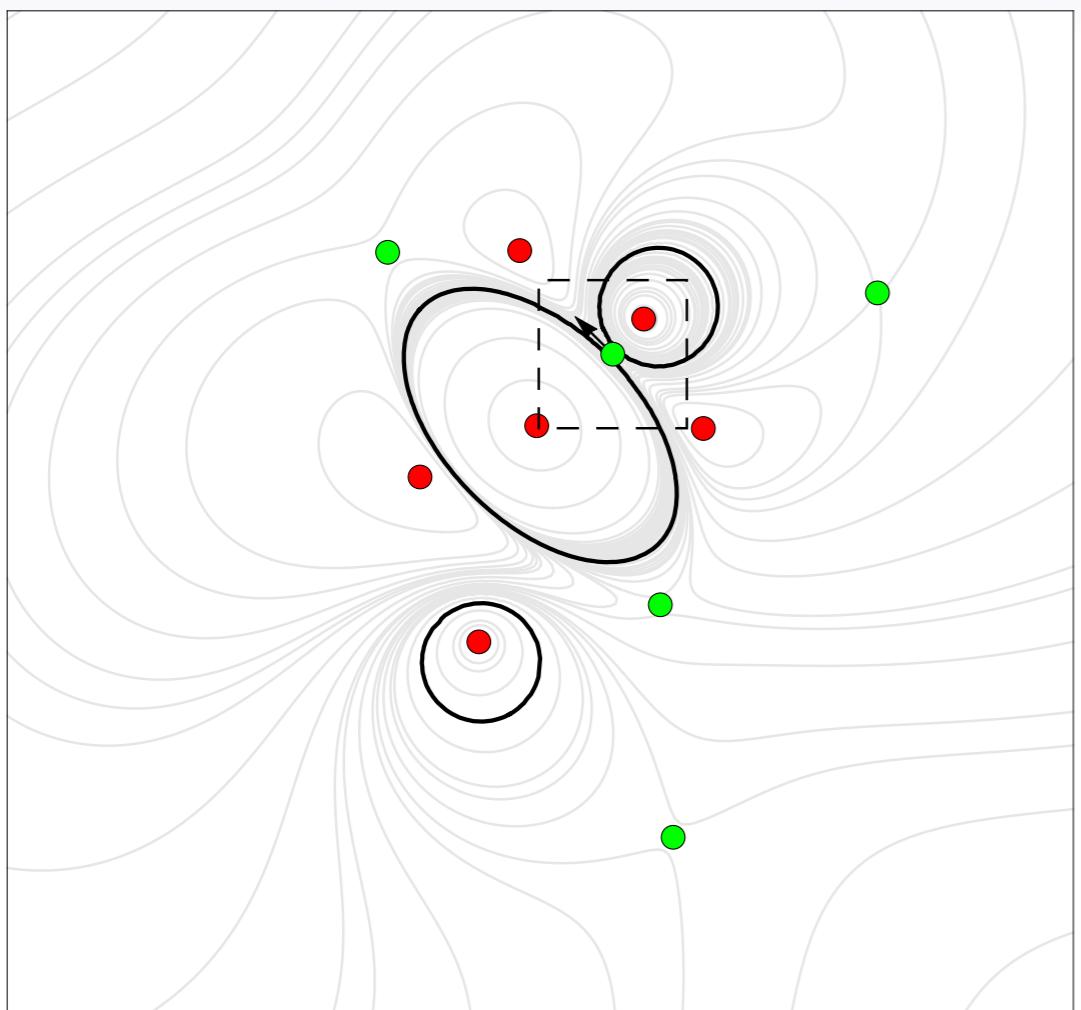
# Method: Overview



**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors

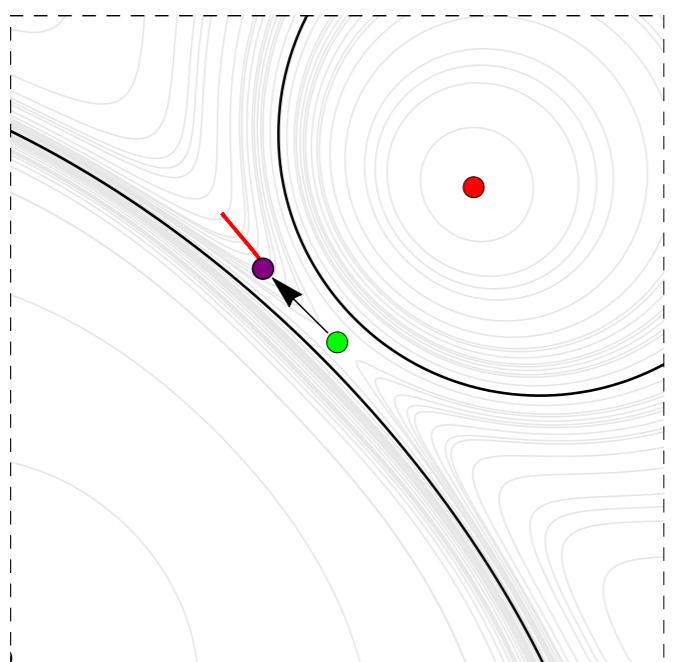
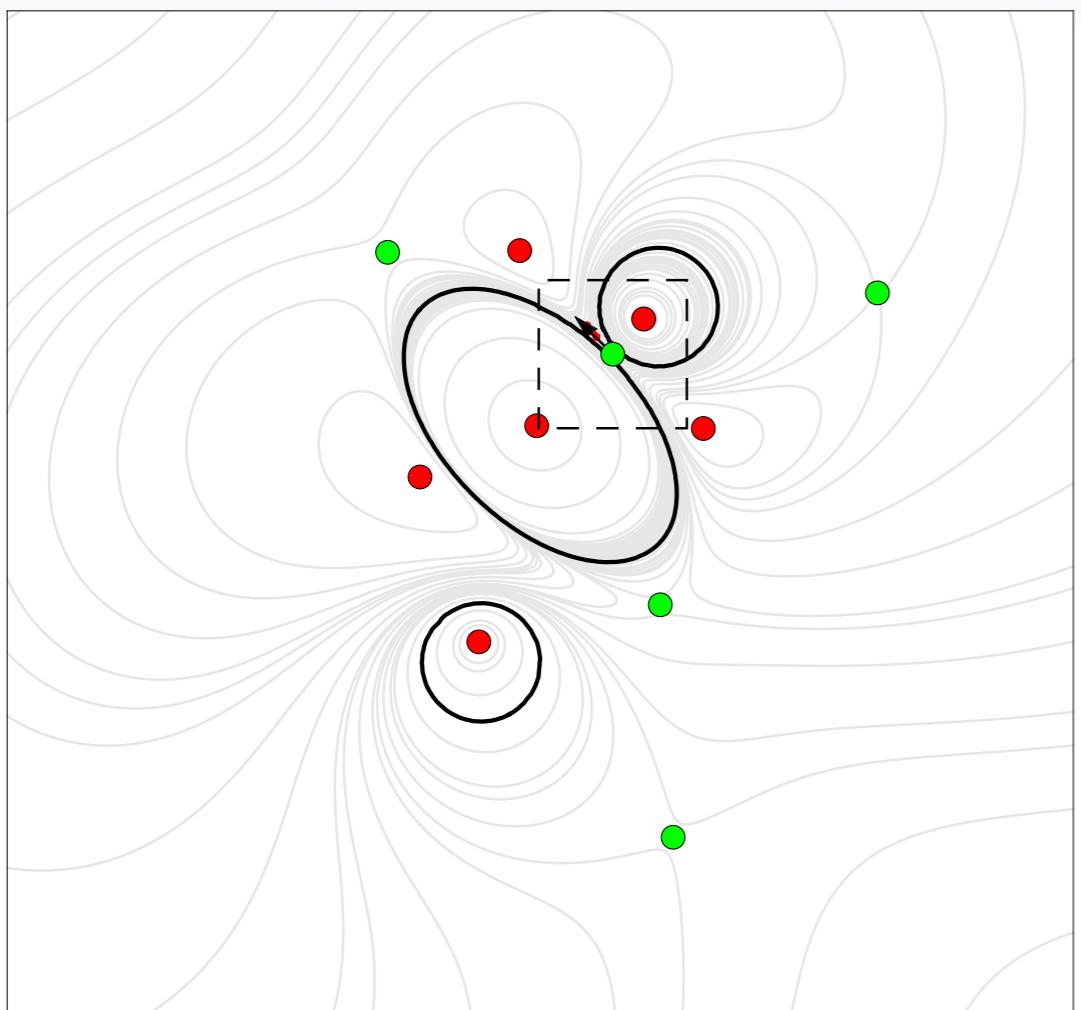
# Method: Overview



**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors

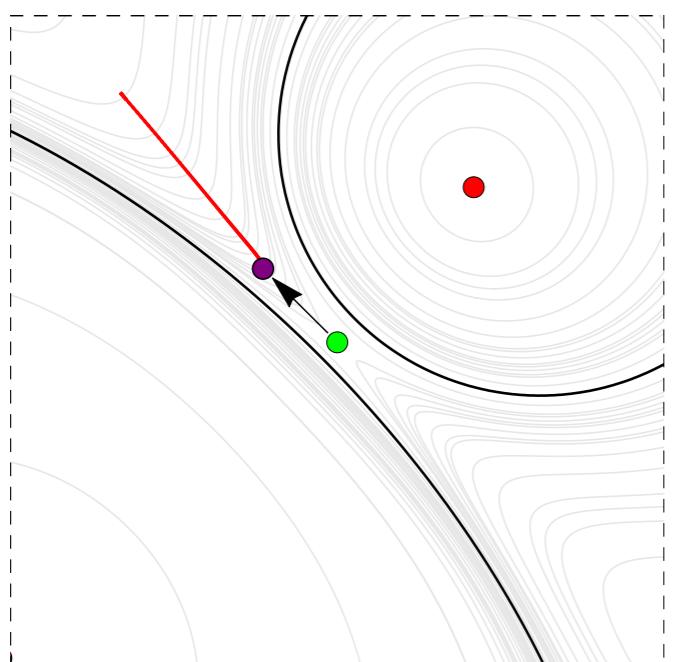
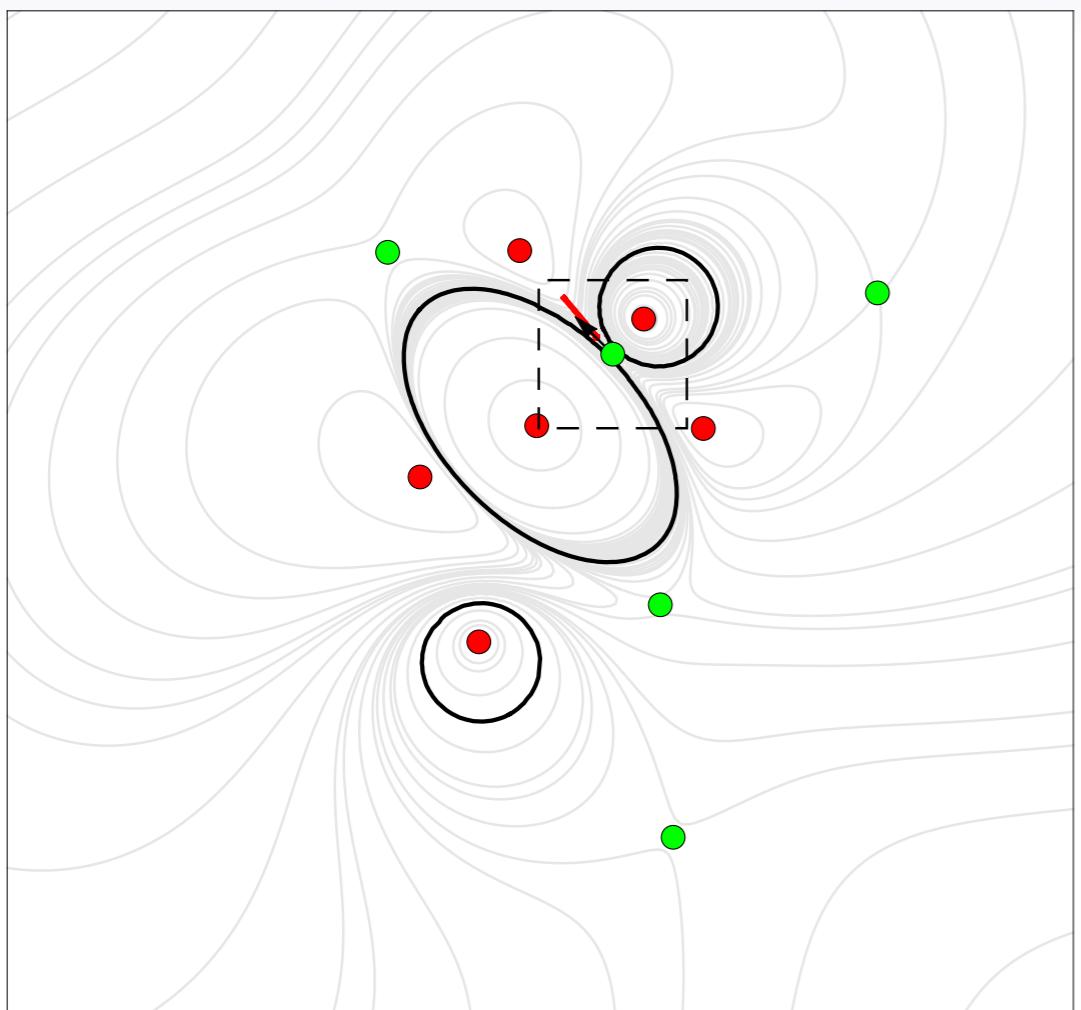
# Method: Overview



**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors

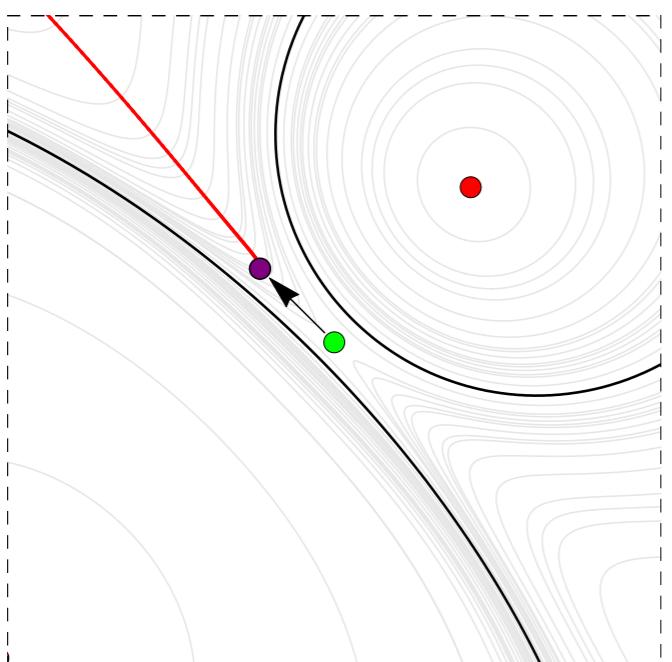
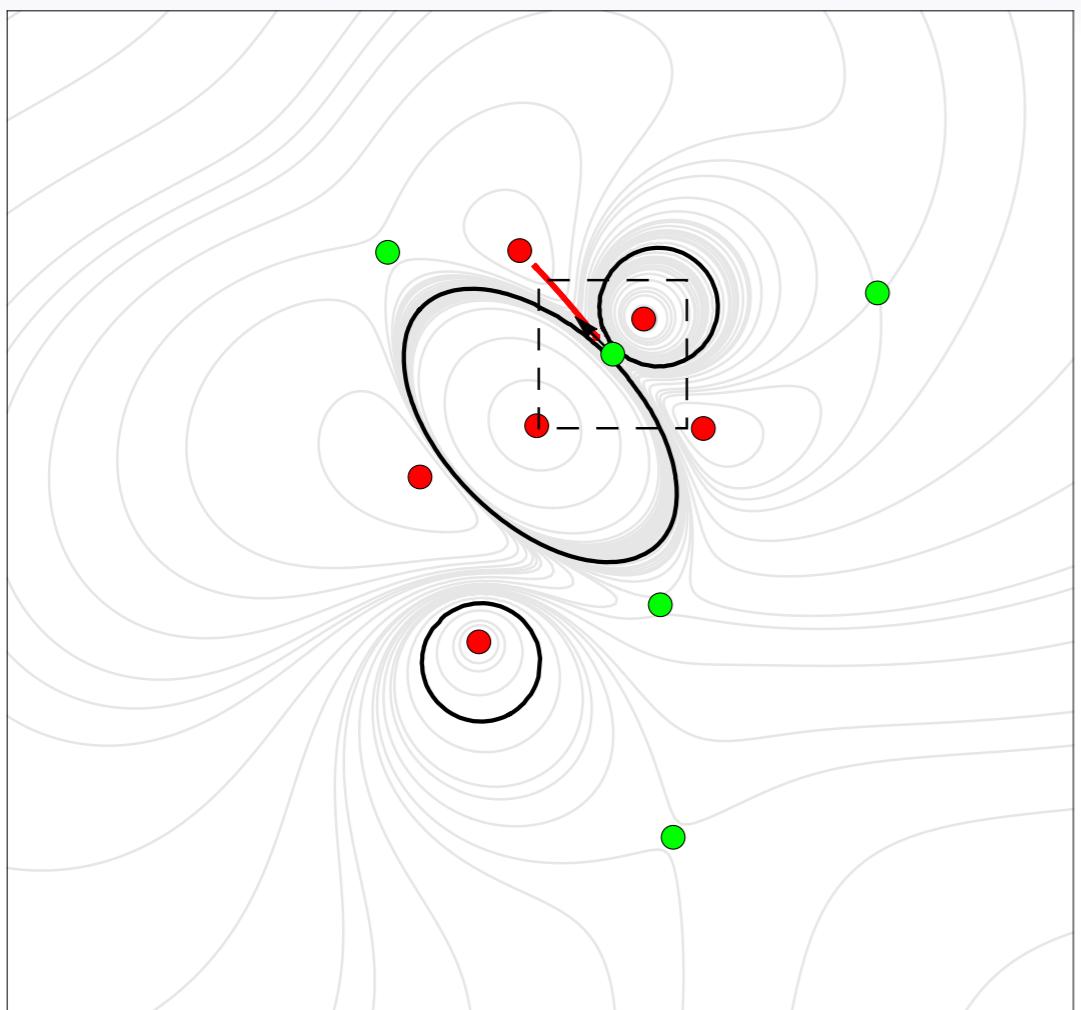
# Method: Overview



**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors

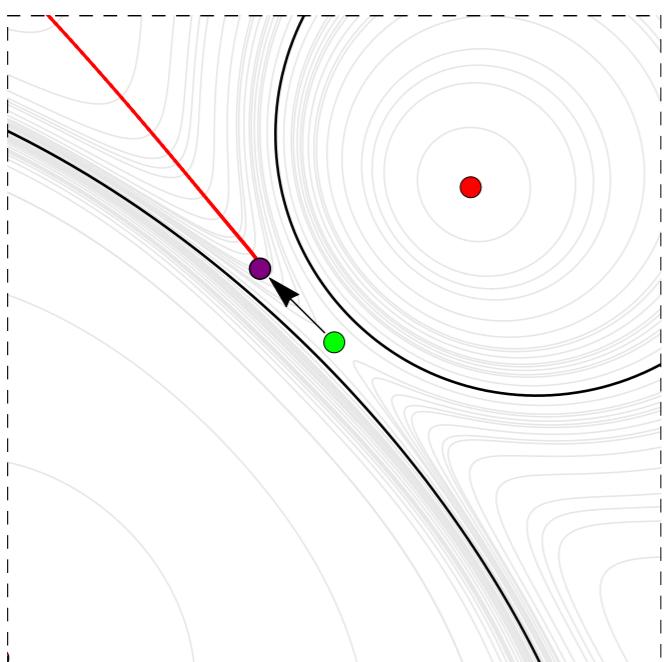
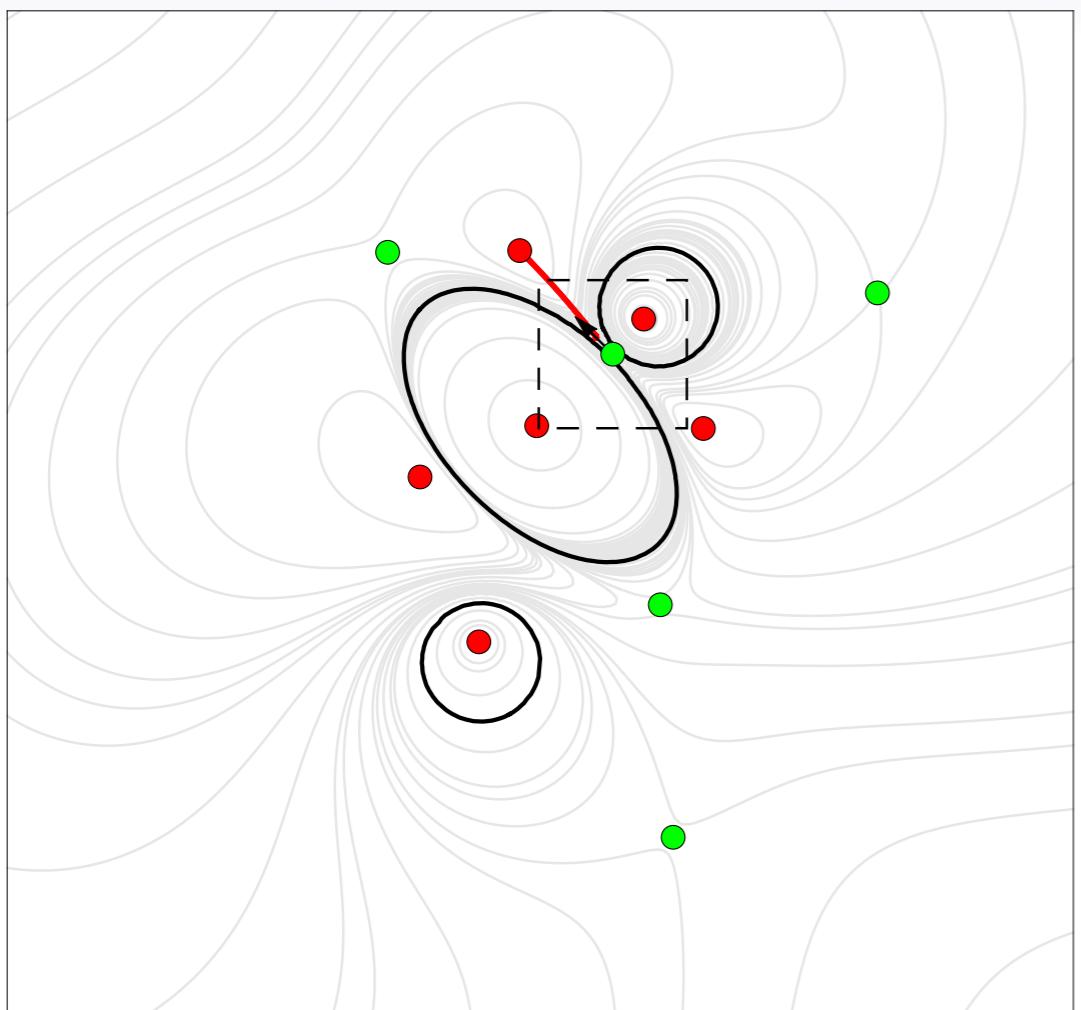
# Method: Overview



**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors

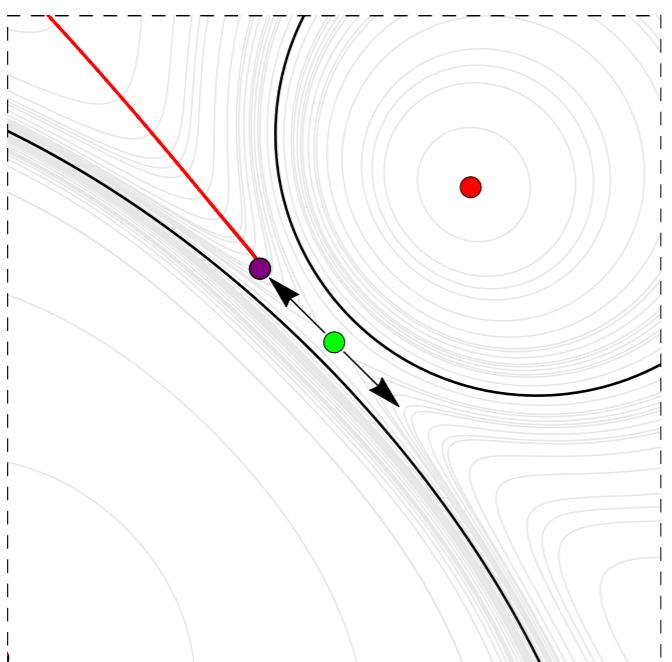
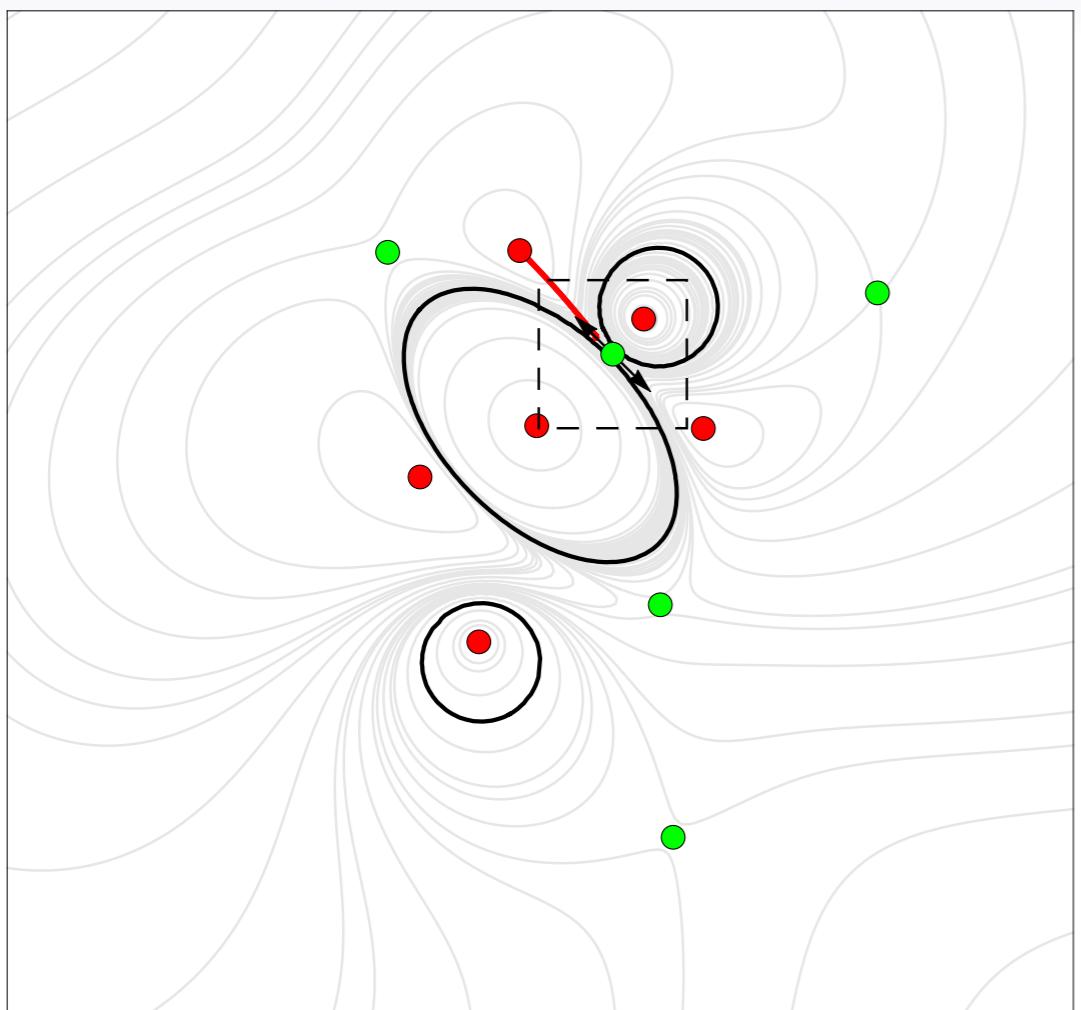
# Method: Overview



**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors

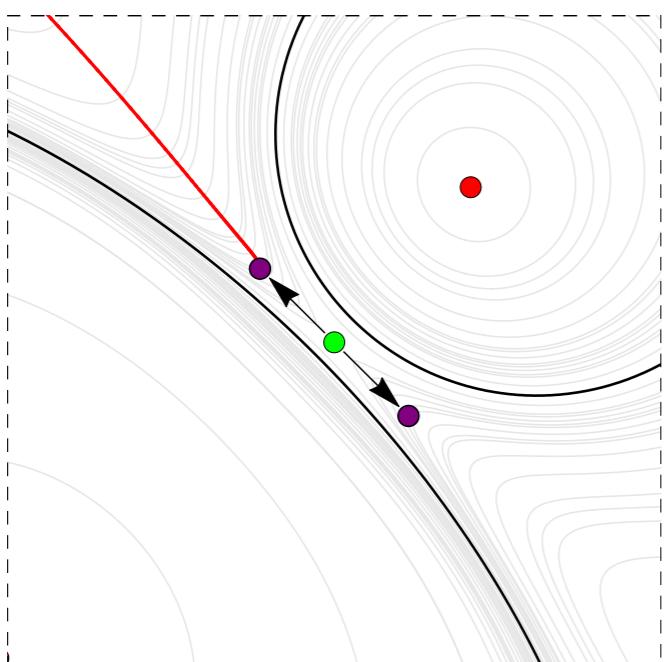
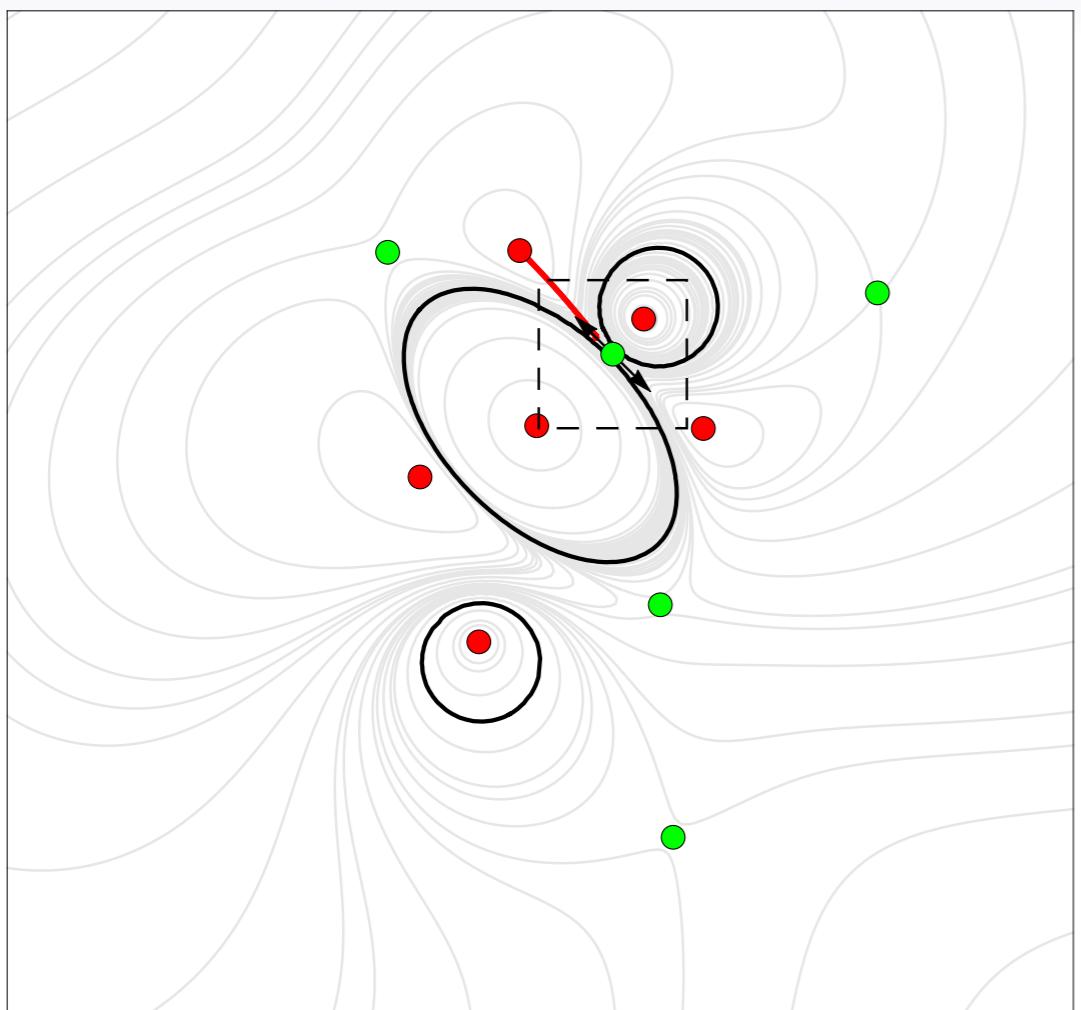
# Method: Overview



**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors

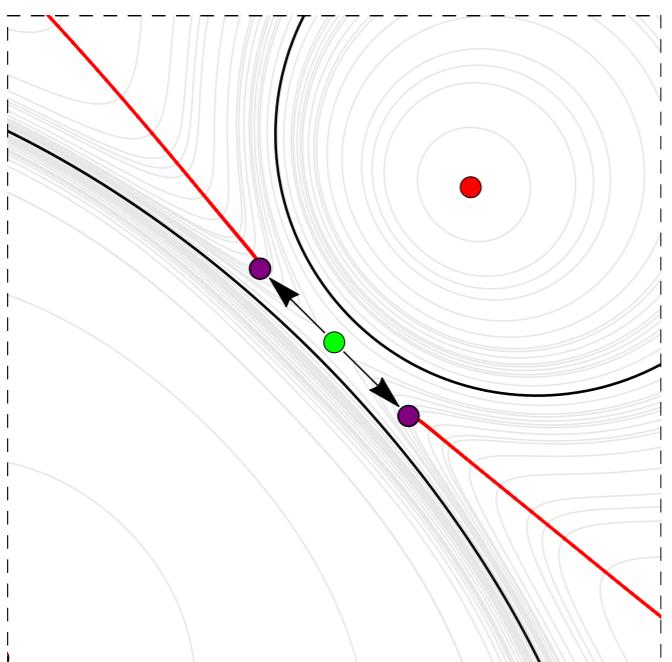
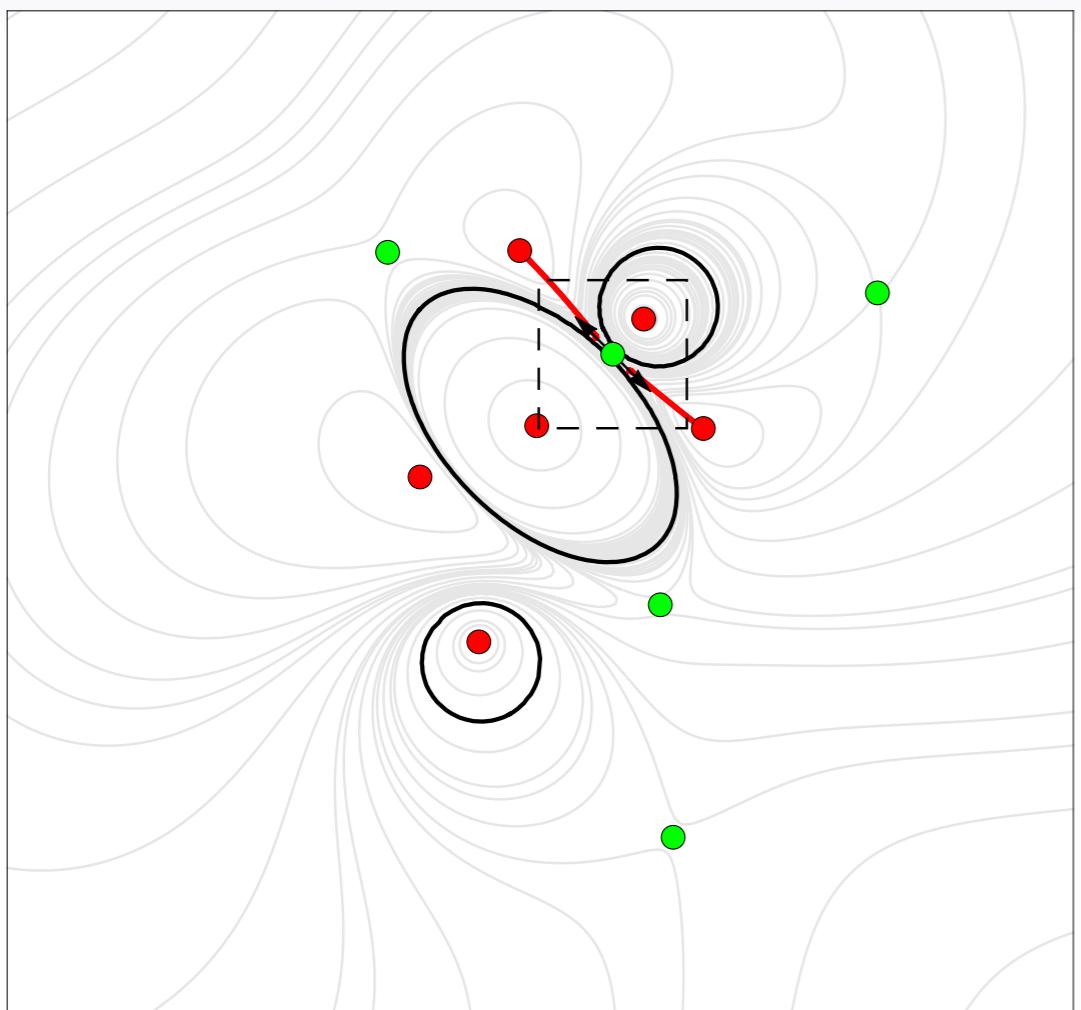
# Method: Overview



**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors

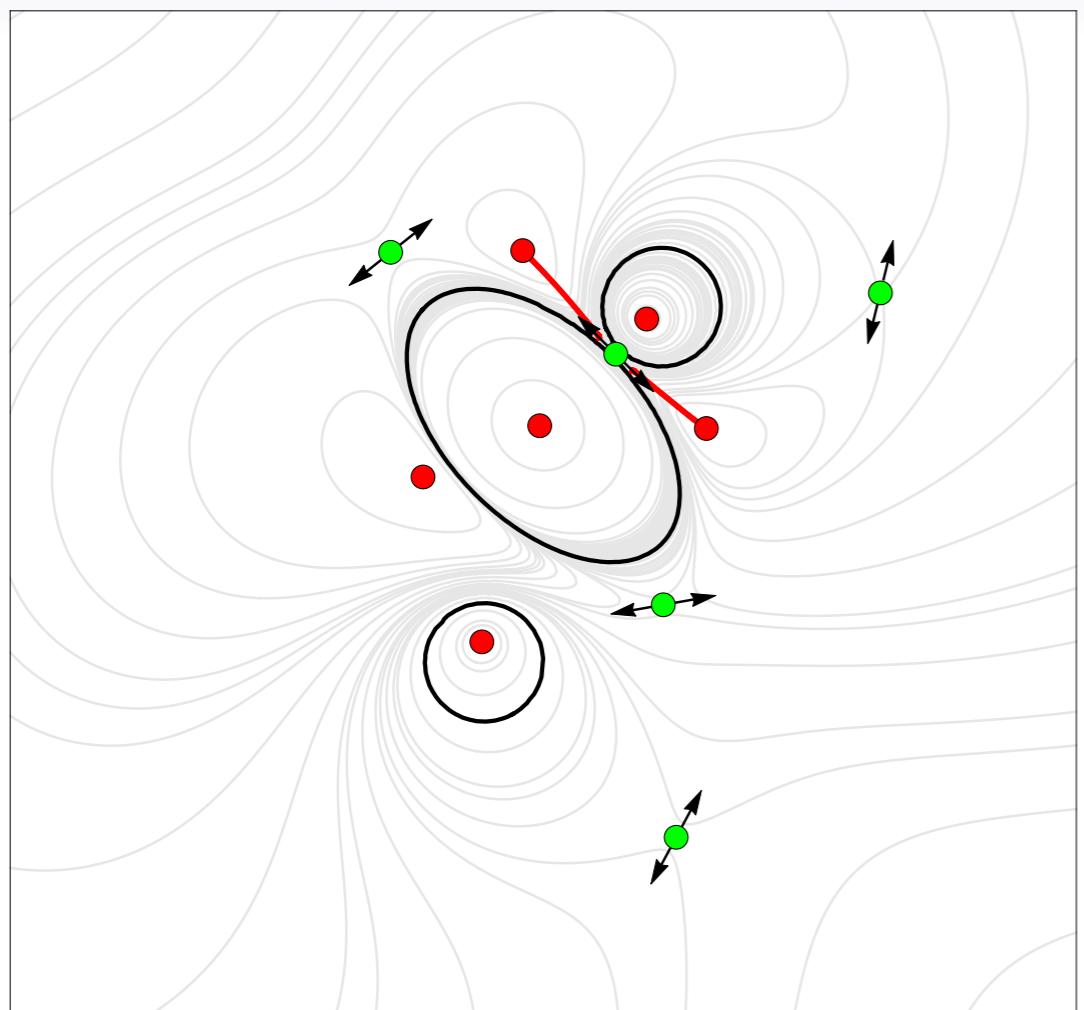
# Method: Overview



**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors

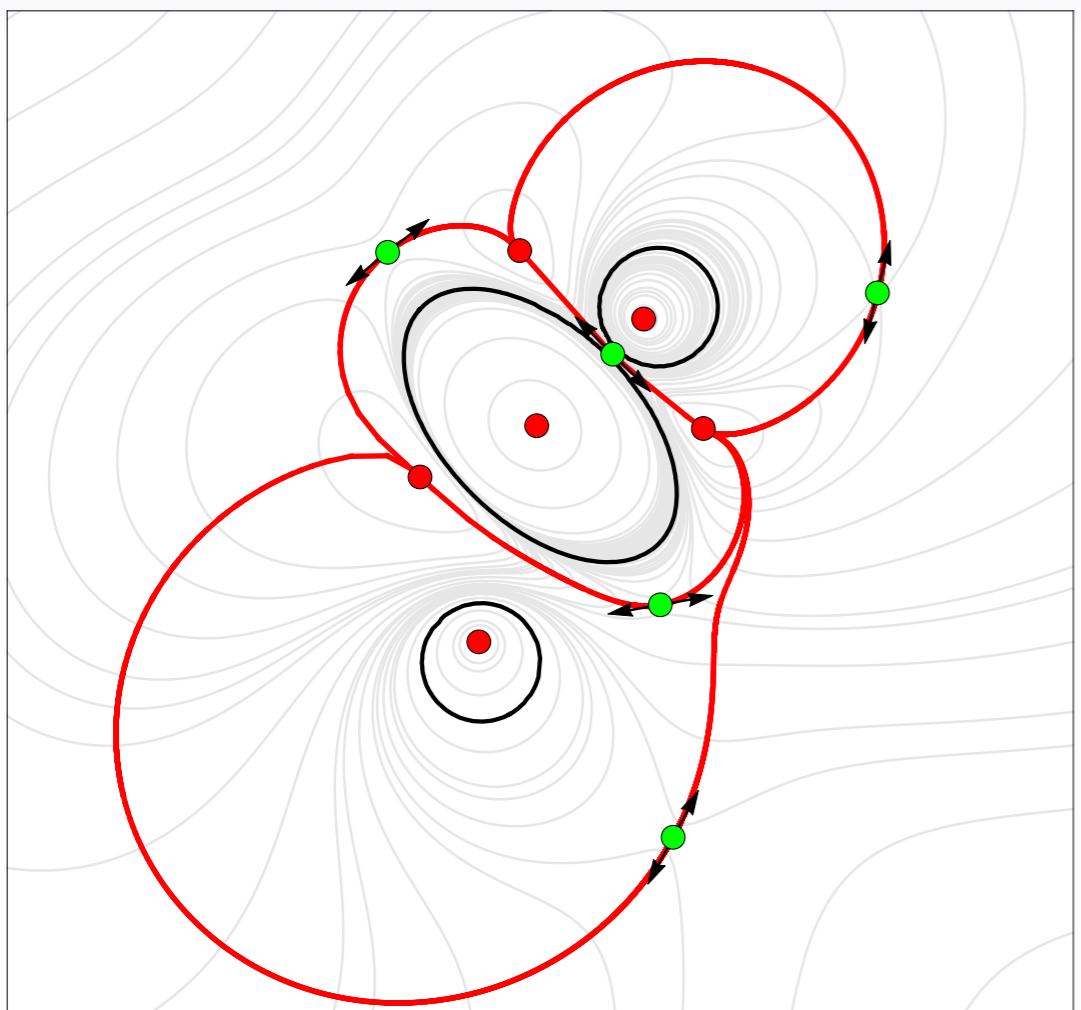
# Method: Overview



**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors

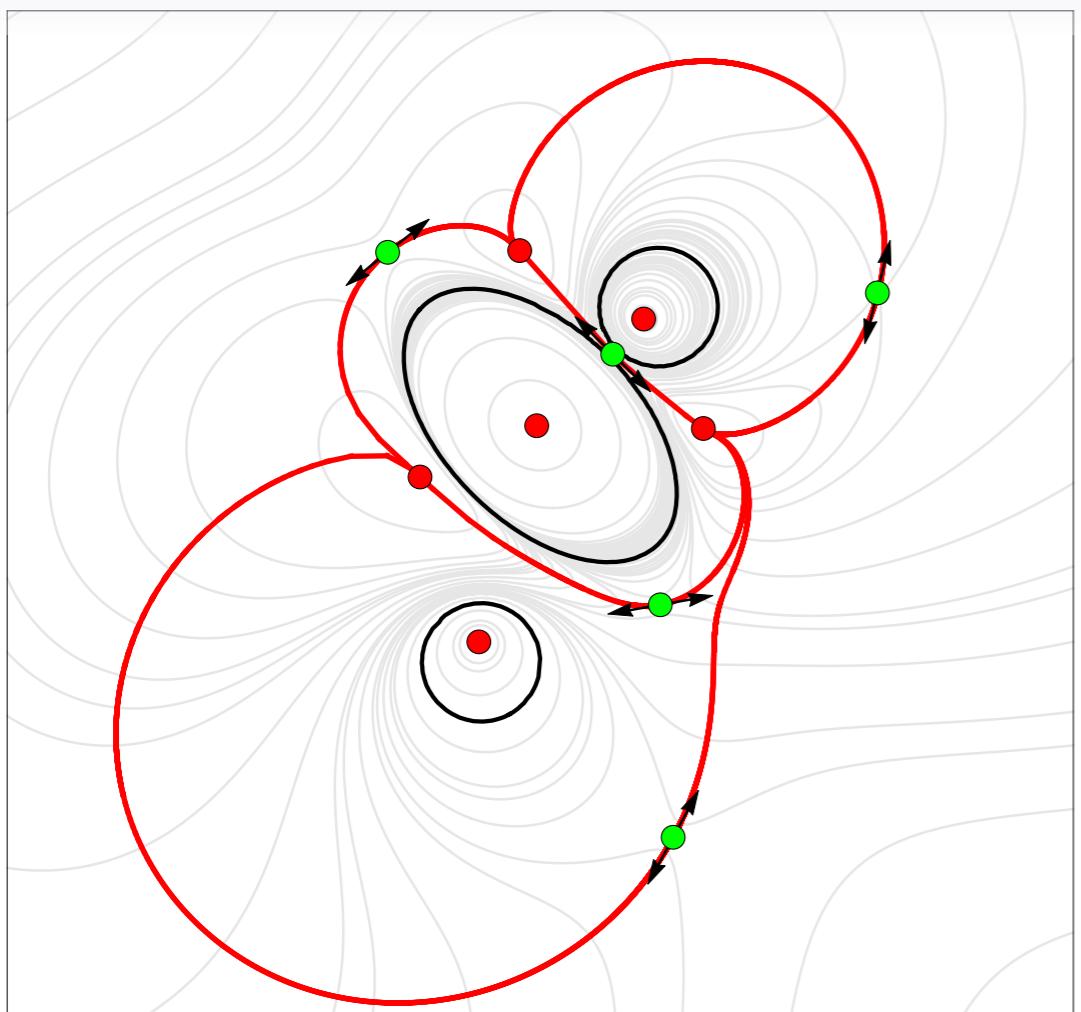
# Method: Overview



**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors

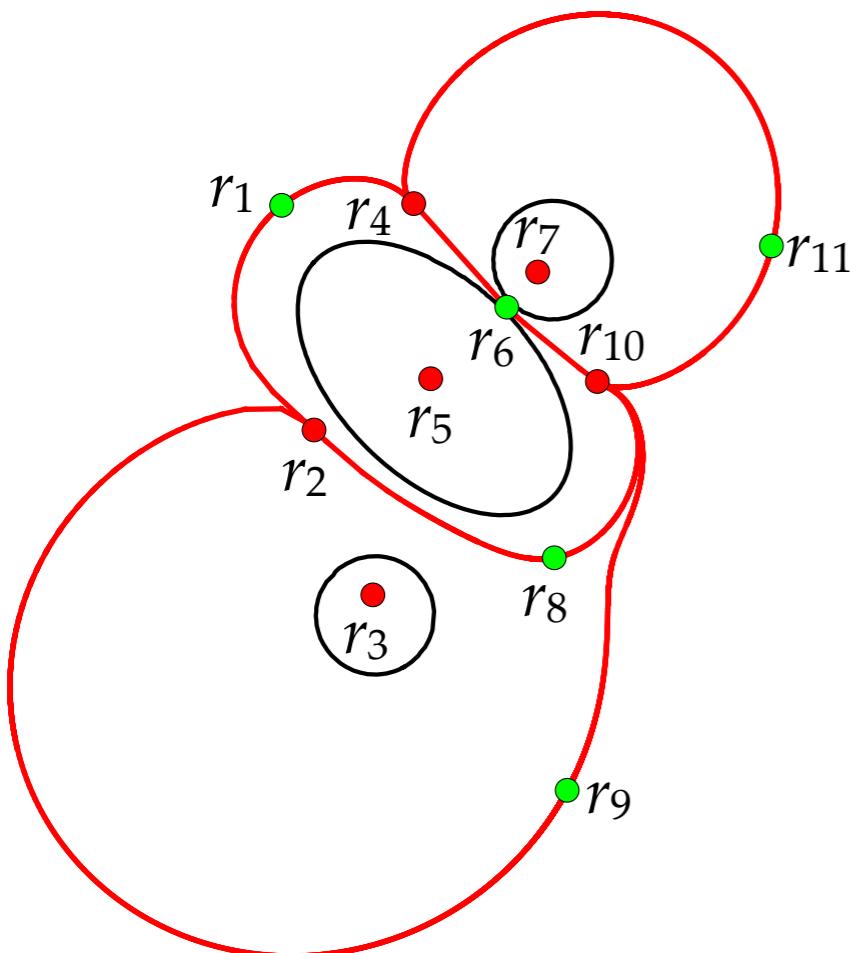
# Method: Overview



**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors
- 5: Form adjacency matrix

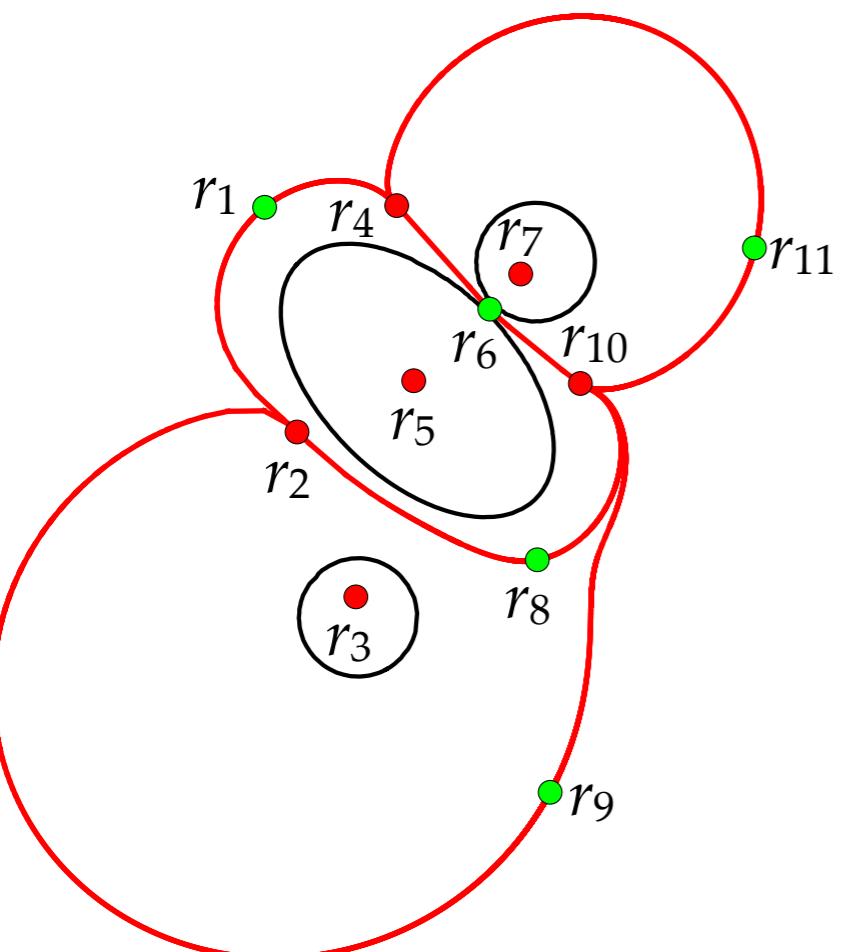
# Method: Overview



**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors
- 5: Form adjacency matrix

# Method: Overview

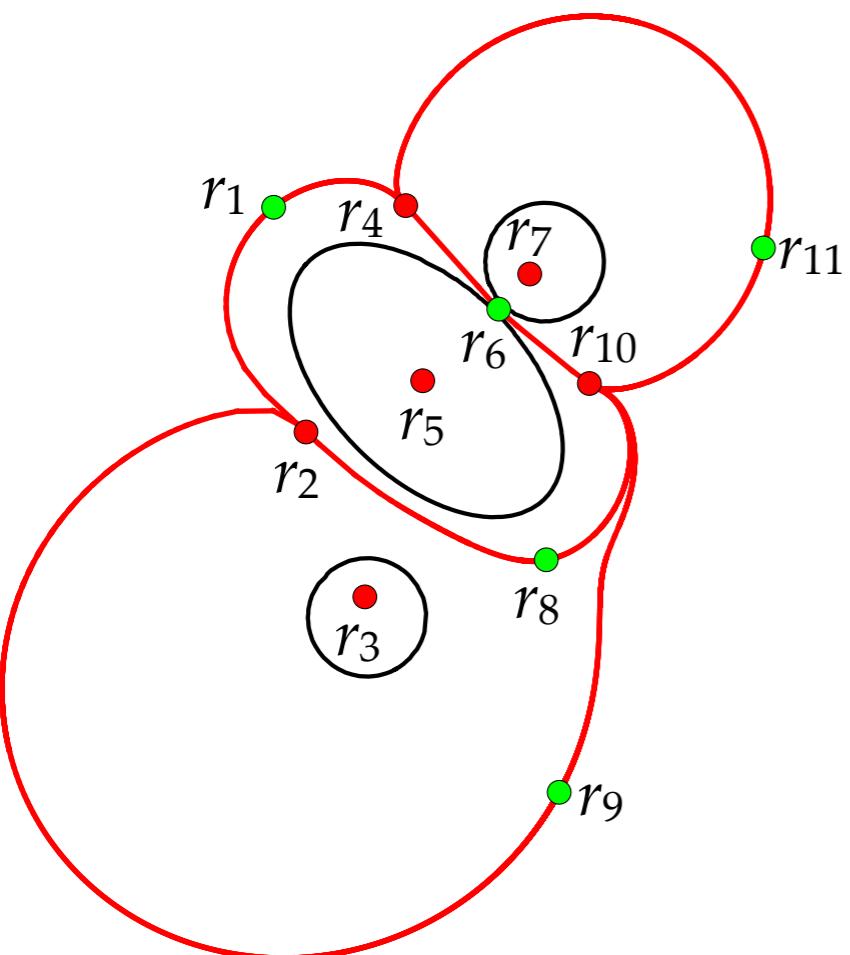


**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors
- 5: Form adjacency matrix

|          | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ | $r_6$ | $r_7$ | $r_8$ | $r_9$ | $r_{10}$ | $r_{11}$ |
|----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|----------|
| $r_1$    | 0     | 1     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_2$    | 1     | 0     | 0     | 0     | 0     | 0     | 1     | 1     | 0     | 0        | 0        |
| $r_3$    | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_4$    | 1     | 0     | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0        | 1        |
| $r_5$    | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_6$    | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 1        | 0        |
| $r_7$    | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_8$    | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0     | 1     | 0        | 0        |
| $r_9$    | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0     | 1     | 0        | 0        |
| $r_{10}$ | 0     | 0     | 0     | 0     | 1     | 0     | 1     | 1     | 0     | 1        | 0        |
| $r_{11}$ | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 1     | 0        | 0        |

# Method: Overview

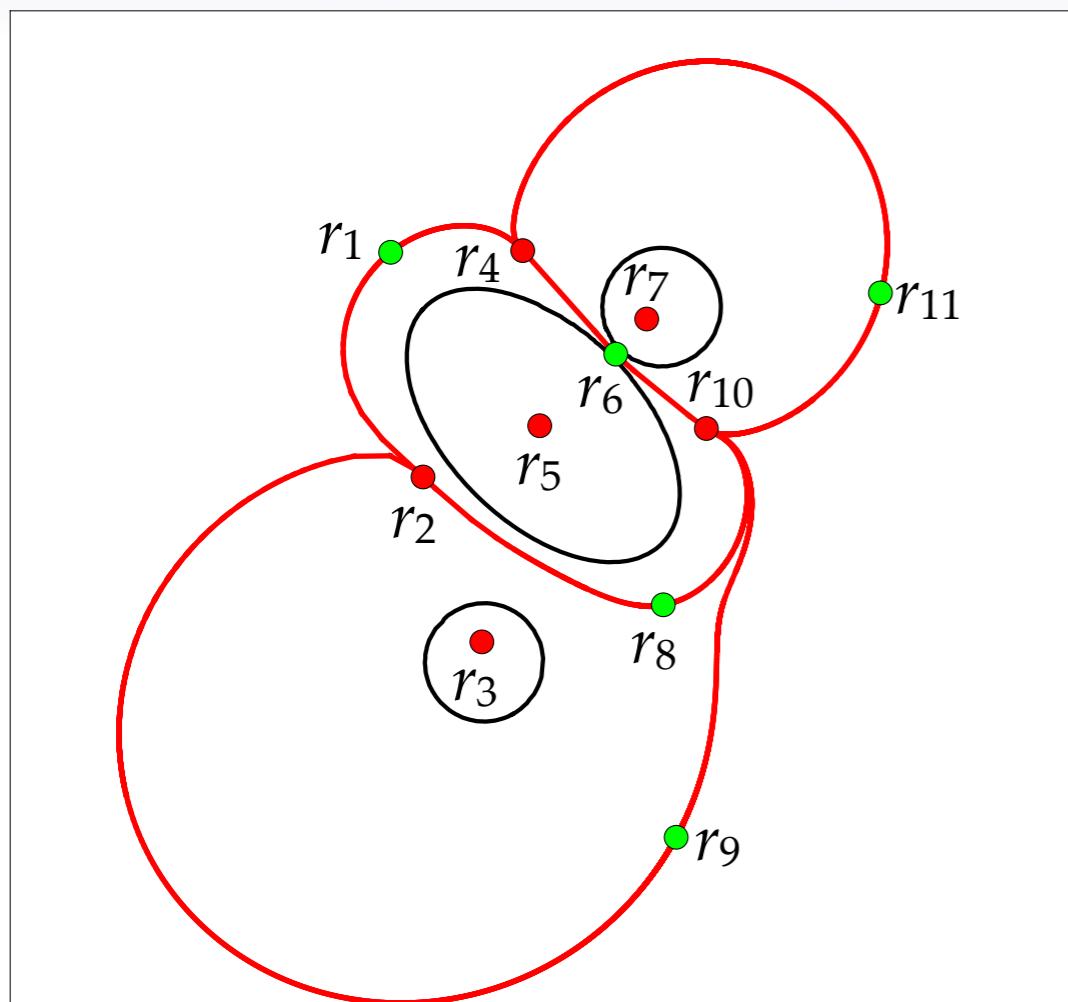


|          | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ | $r_6$ | $r_7$ | $r_8$ | $r_9$ | $r_{10}$ | $r_{11}$ |
|----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|----------|
| $r_1$    | 0     | 1     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_2$    | 1     | 0     | 0     | 0     | 0     | 0     | 1     | 1     | 0     | 0        | 0        |
| $r_3$    | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_4$    | 1     | 0     | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0        | 1        |
| $r_5$    | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_6$    | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 1        | 0        |
| $r_7$    | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_8$    | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0     | 1     | 0        | 0        |
| $r_9$    | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0     | 1     | 0        | 0        |
| $r_{10}$ | 0     | 0     | 0     | 0     | 1     | 0     | 1     | 1     | 0     | 1        | 0        |
| $r_{11}$ | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 1     | 0        | 0        |

**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors
- 5: Form adjacency matrix
- 6: Closure of adjacency matrix

# Method: Overview

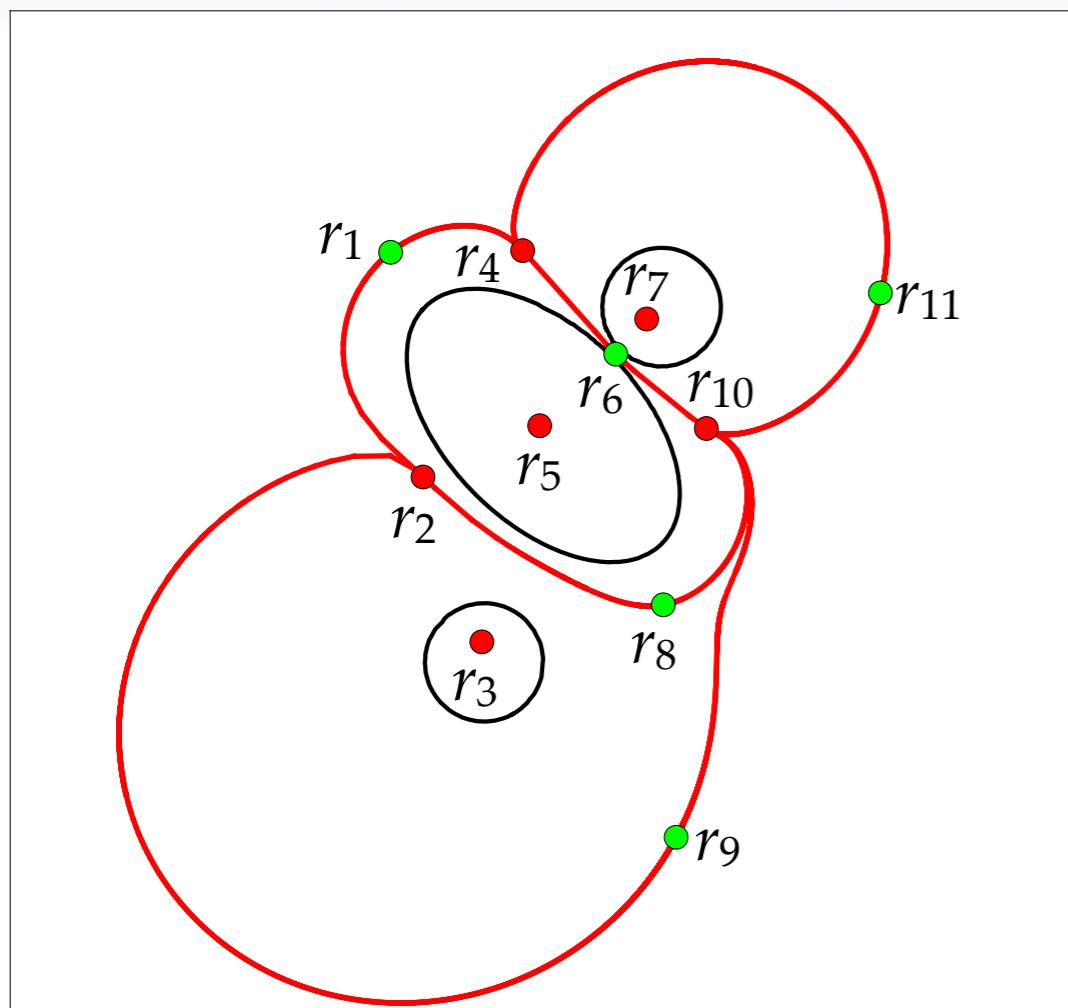


|          | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ | $r_6$ | $r_7$ | $r_8$ | $r_9$ | $r_{10}$ | $r_{11}$ |
|----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|----------|
| $r_1$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_2$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_3$    | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_4$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_5$    | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_6$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_7$    | 0     | 0     | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0        | 0        |
| $r_8$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_9$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{10}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{11}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |

**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors
- 5: Form adjacency matrix
- 6: Closure of adjacency matrix

# Method: Overview

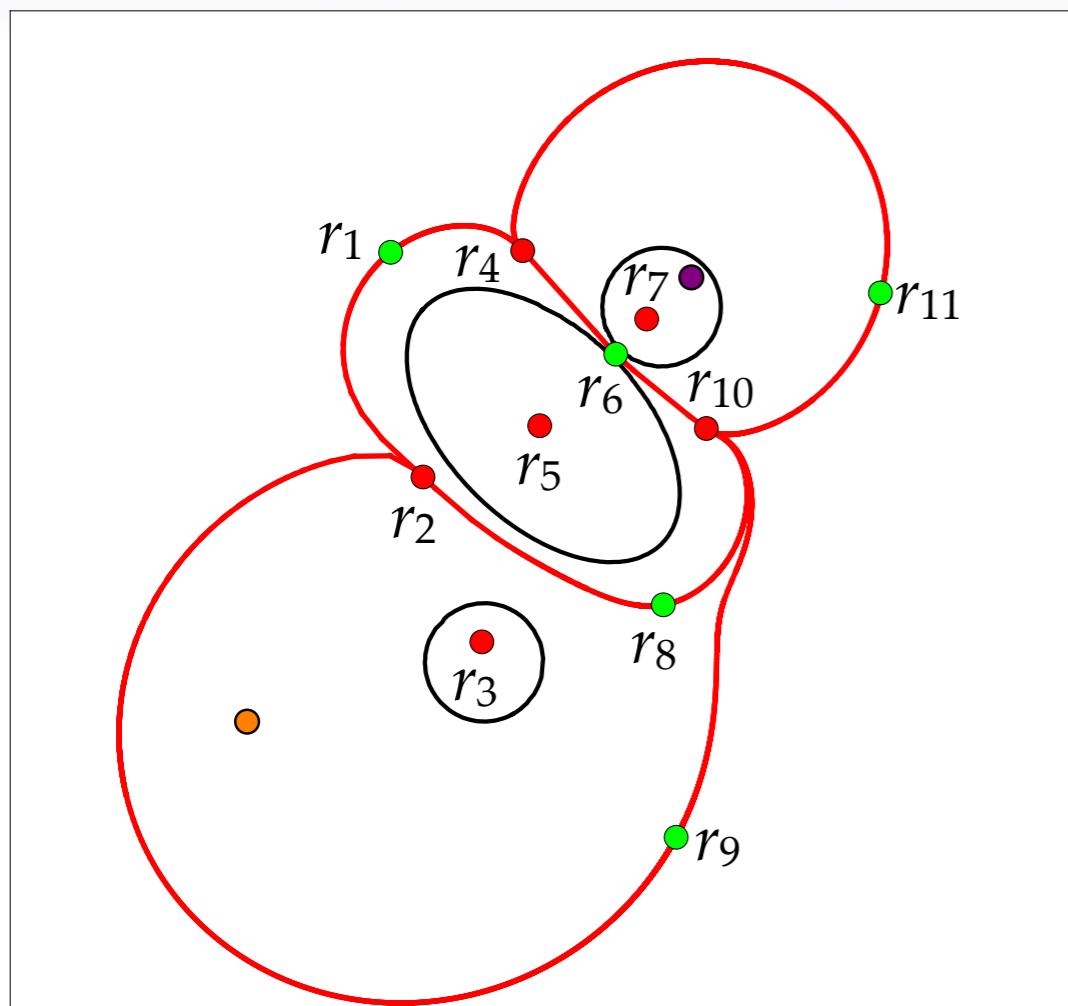


|          | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ | $r_6$ | $r_7$ | $r_8$ | $r_9$ | $r_{10}$ | $r_{11}$ |
|----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|----------|
| $r_1$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_2$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_3$    | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_4$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_5$    | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_6$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_7$    | 0     | 0     | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0        | 0        |
| $r_8$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_9$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{10}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{11}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |

**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors
- 5: Form adjacency matrix
- 6: Closure of adjacency matrix
- 7: Steepest ascent from

# Method: Overview

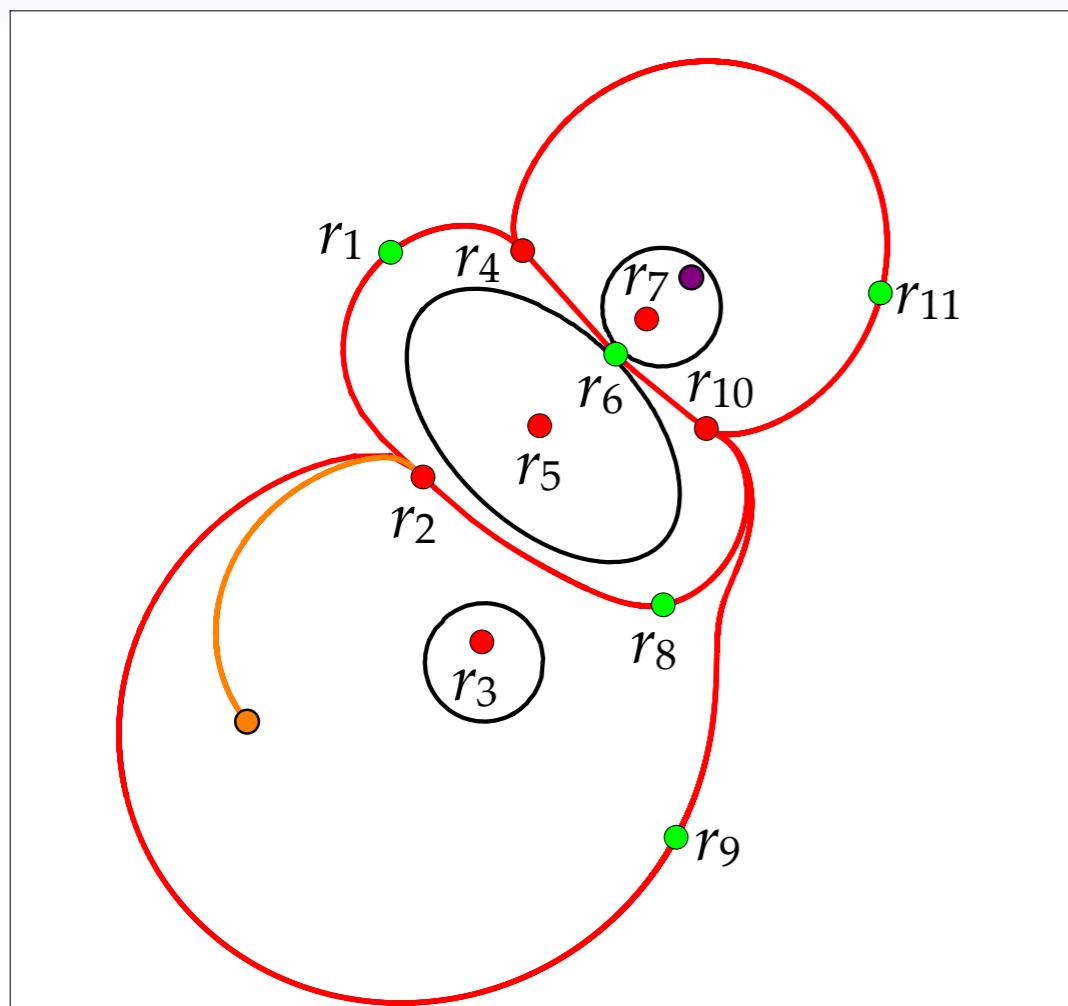


|          | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ | $r_6$ | $r_7$ | $r_8$ | $r_9$ | $r_{10}$ | $r_{11}$ |
|----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|----------|
| $r_1$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_2$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_3$    | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_4$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_5$    | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_6$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_7$    | 0     | 0     | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0        | 0        |
| $r_8$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_9$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{10}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{11}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |

**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors
- 5: Form adjacency matrix
- 6: Closure of adjacency matrix
- 7: Steepest ascent from

# Method: Overview

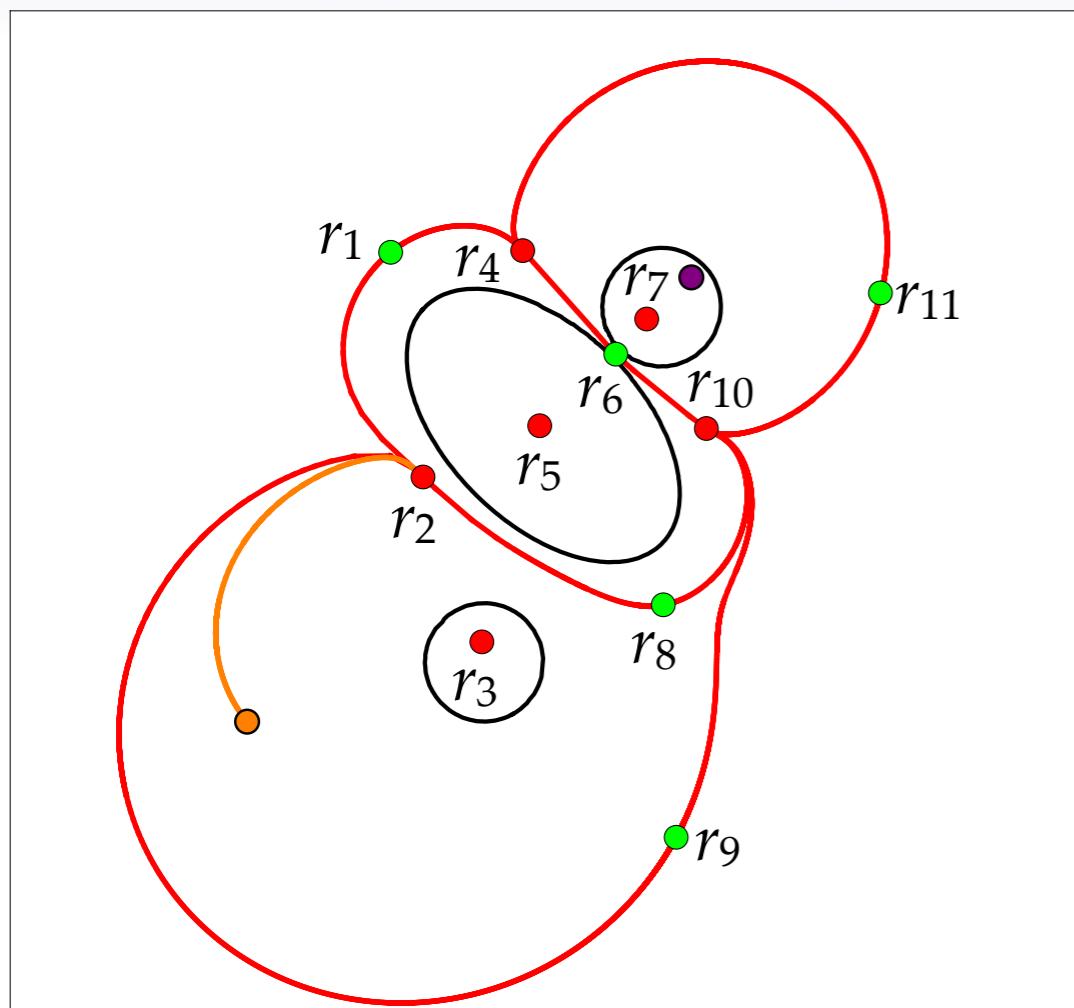


|          | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ | $r_6$ | $r_7$ | $r_8$ | $r_9$ | $r_{10}$ | $r_{11}$ |
|----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|----------|
| $r_1$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_2$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_3$    | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_4$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_5$    | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_6$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_7$    | 0     | 0     | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0        | 0        |
| $r_8$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_9$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{10}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{11}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |

**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors
- 5: Form adjacency matrix
- 6: Closure of adjacency matrix
- 7: Steepest ascent from

# Method: Overview

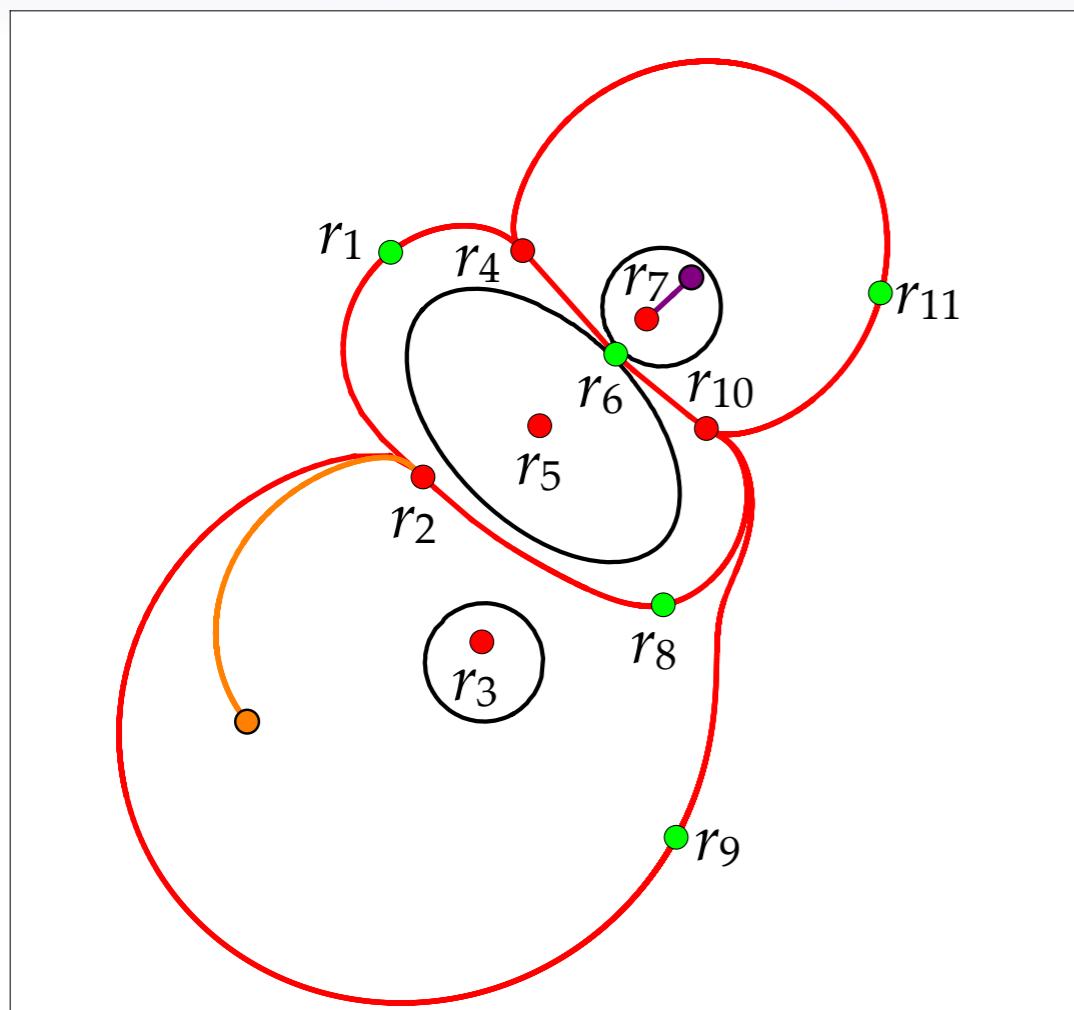


|          | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ | $r_6$ | $r_7$ | $r_8$ | $r_9$ | $r_{10}$ | $r_{11}$ |
|----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|----------|
| $r_1$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_2$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_3$    | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_4$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_5$    | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_6$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_7$    | 0     | 0     | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0        | 0        |
| $r_8$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_9$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{10}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{11}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |

**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors
- 5: Form adjacency matrix
- 6: Closure of adjacency matrix
- 7: Steepest ascent from
- 8: Steepest ascent from

# Method: Overview

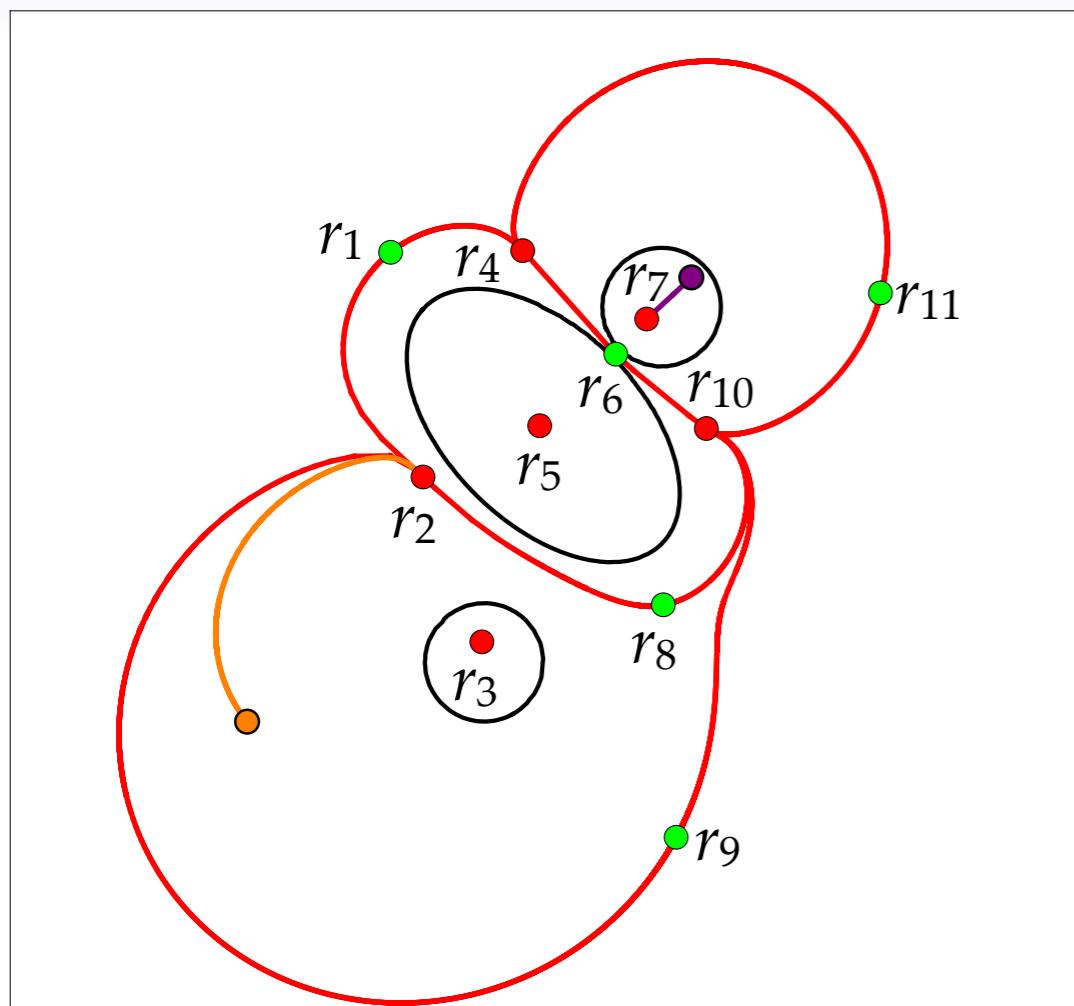


|          | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ | $r_6$ | $r_7$ | $r_8$ | $r_9$ | $r_{10}$ | $r_{11}$ |
|----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|----------|
| $r_1$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_2$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_3$    | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_4$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_5$    | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_6$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_7$    | 0     | 0     | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0        | 0        |
| $r_8$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_9$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{10}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{11}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |

**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors
- 5: Form adjacency matrix
- 6: Closure of adjacency matrix
- 7: Steepest ascent from
- 8: Steepest ascent from

# Method: Overview

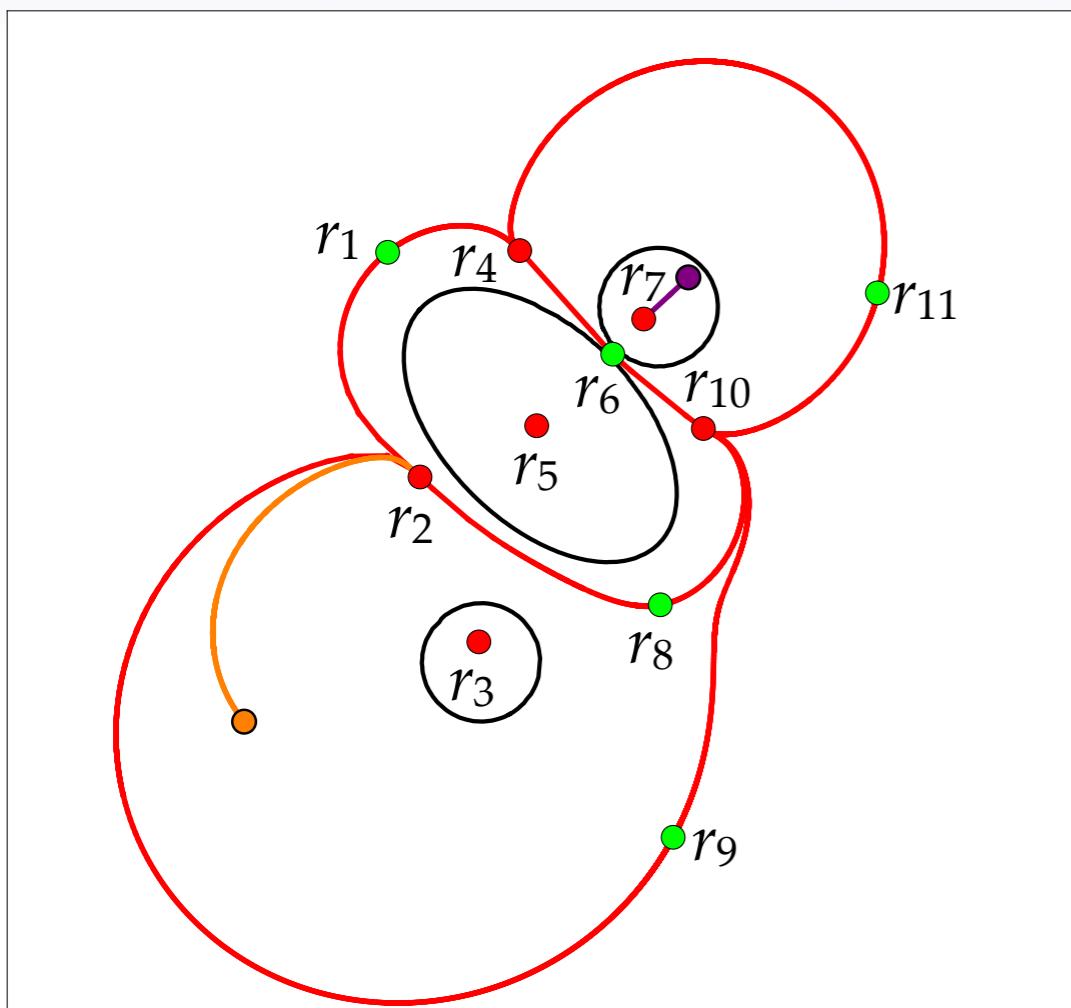


|          | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ | $r_6$ | $r_7$ | $r_8$ | $r_9$ | $r_{10}$ | $r_{11}$ |
|----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|----------|
| $r_1$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_2$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_3$    | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_4$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_5$    | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_6$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_7$    | 0     | 0     | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0        | 0        |
| $r_8$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_9$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{10}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{11}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |

**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors
- 5: Form adjacency matrix
- 6: Closure of adjacency matrix
- 7: Steepest ascent from
- 8: Steepest ascent from
- 9: Read matrix

# Method: Overview

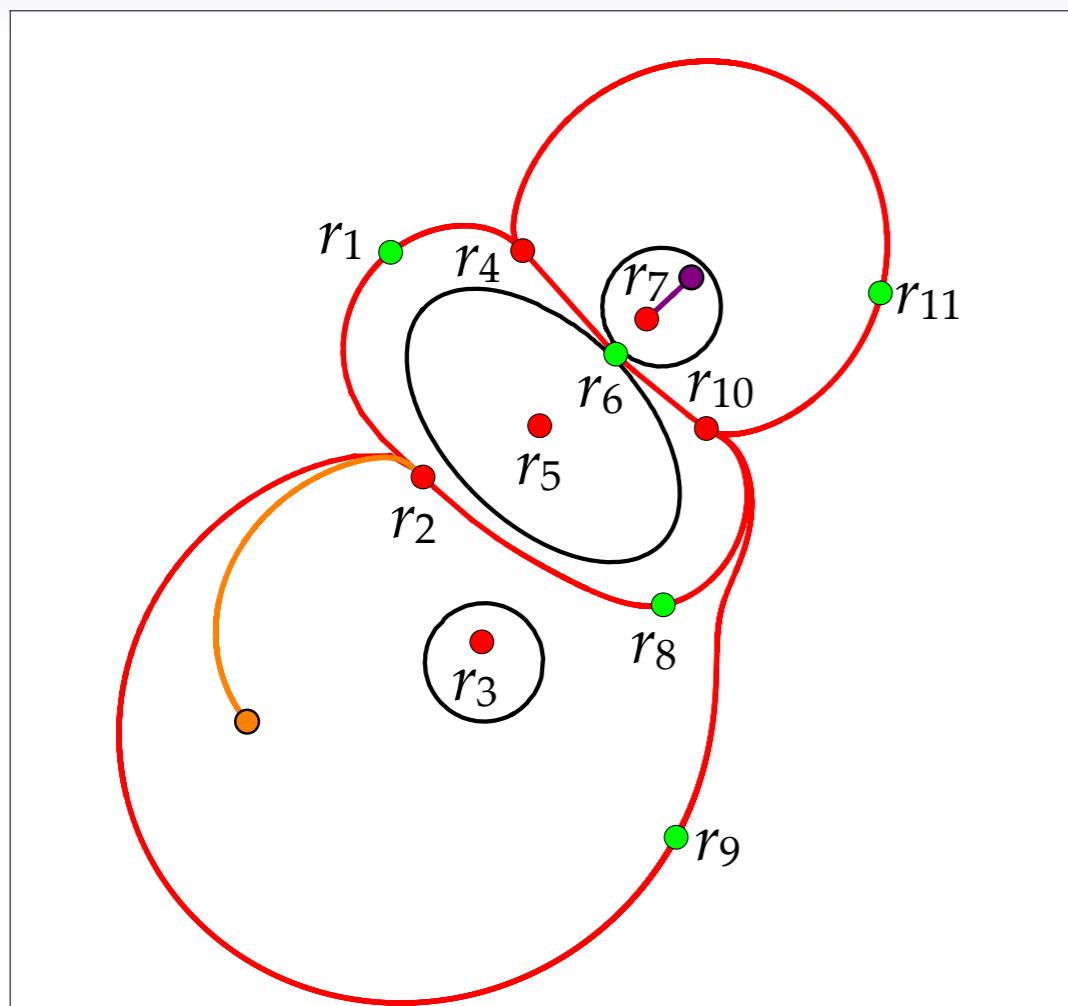


|          | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ | $r_6$ | $r_7$ | $r_8$ | $r_9$ | $r_{10}$ | $r_{11}$ |
|----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|----------|
| $r_1$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_2$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_3$    | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_4$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_5$    | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_6$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_7$    | 0     | 0     | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0        | 0        |
| $r_8$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_9$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{10}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{11}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |

**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors
- 5: Form adjacency matrix
- 6: Closure of adjacency matrix
- 7: Steepest ascent from
- 8: Steepest ascent from
- 9: Read matrix

# Method: Overview



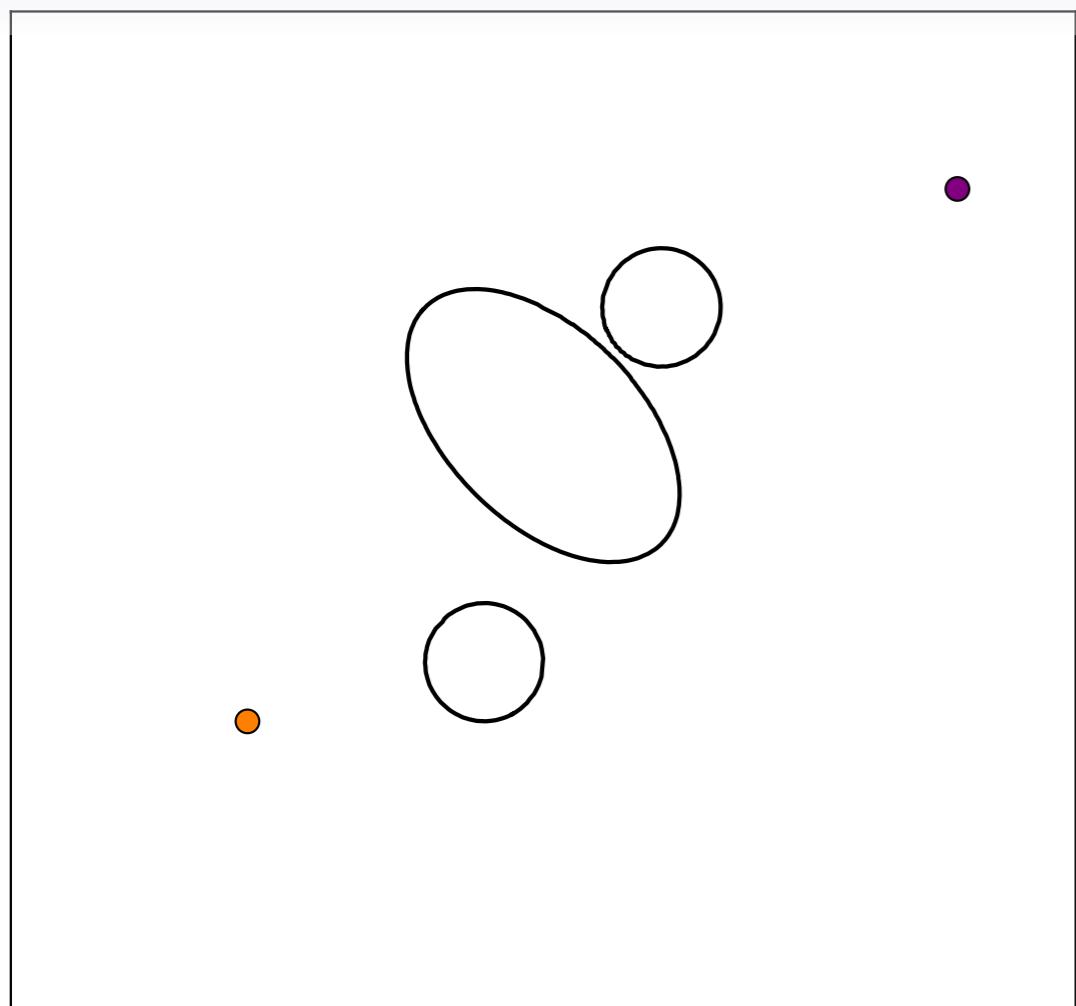
|          | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ | $r_6$ | $r_7$ | $r_8$ | $r_9$ | $r_{10}$ | $r_{11}$ |
|----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|----------|
| $r_1$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_2$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_3$    | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_4$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_5$    | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_6$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_7$    | 0     | 0     | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0        | 0        |
| $r_8$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_9$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{10}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{11}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |

**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors
- 5: Form adjacency matrix
- 6: Closure of adjacency matrix
- 7: Steepest ascent from
- 8: Steepest ascent from
- 9: Read matrix

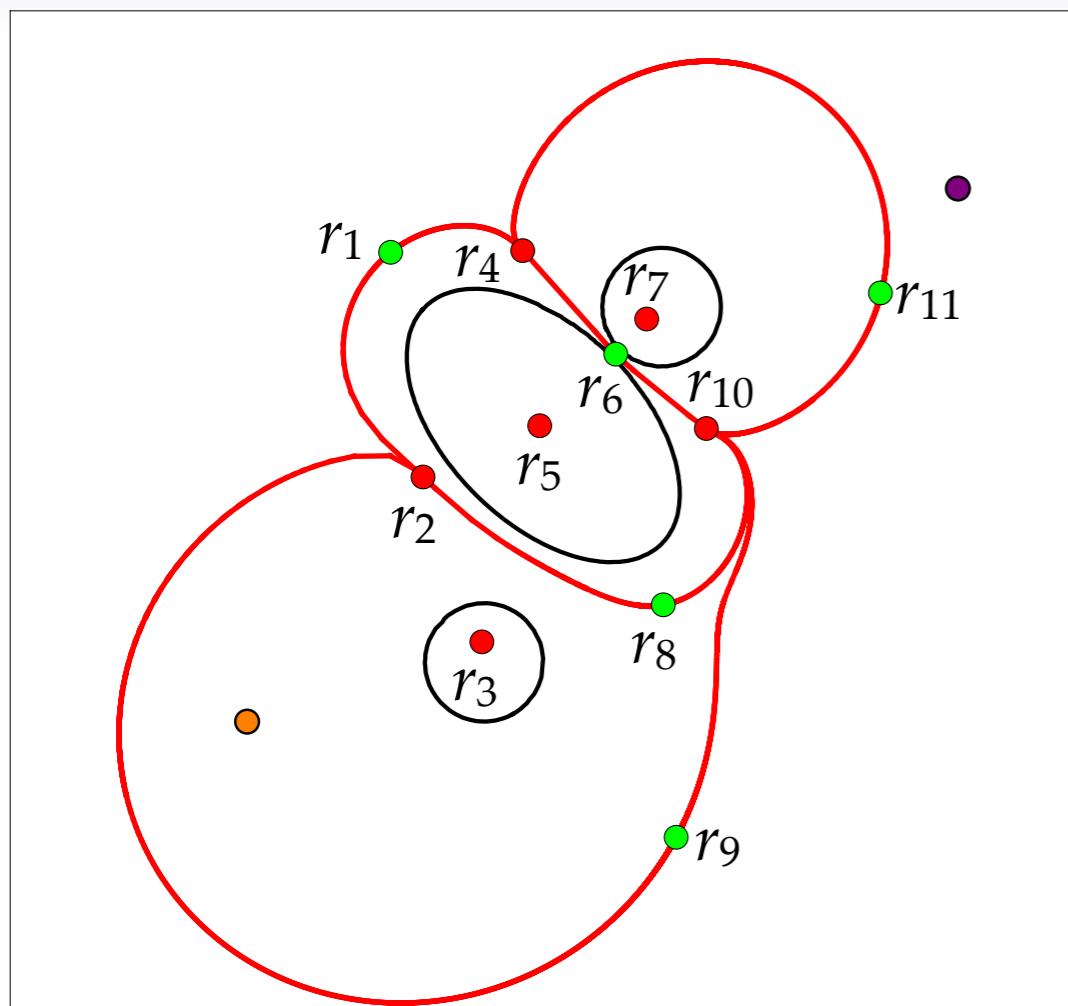
**Output:** False

# Method: Overview



**Input:**  $f(x_1, x_2)$ , ,

# Method: Overview

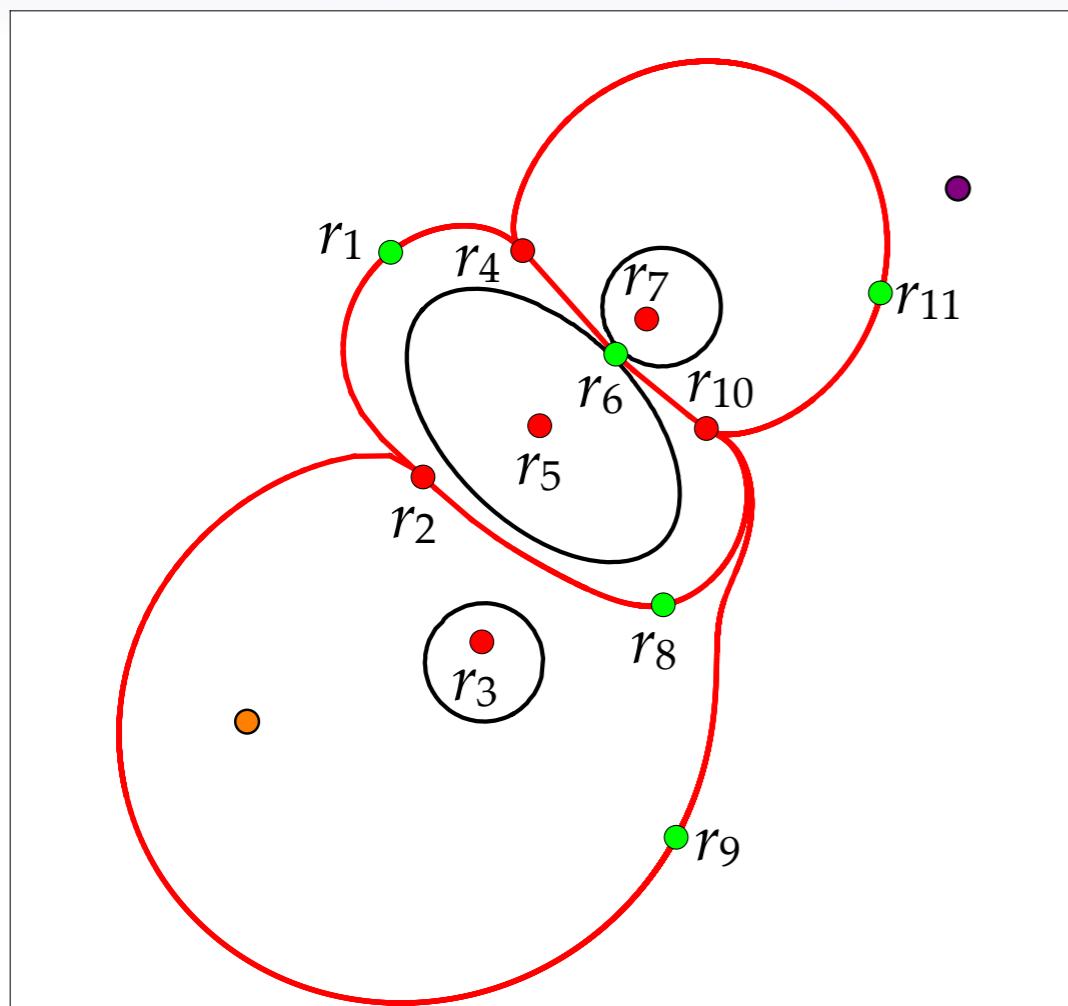


|          | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ | $r_6$ | $r_7$ | $r_8$ | $r_9$ | $r_{10}$ | $r_{11}$ |
|----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|----------|
| $r_1$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_2$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_3$    | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_4$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_5$    | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_6$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_7$    | 0     | 0     | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0        | 0        |
| $r_8$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_9$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{10}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{11}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |

**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors
- 5: Form adjacency matrix
- 6: Closure of adjacency matrix

# Method: Overview

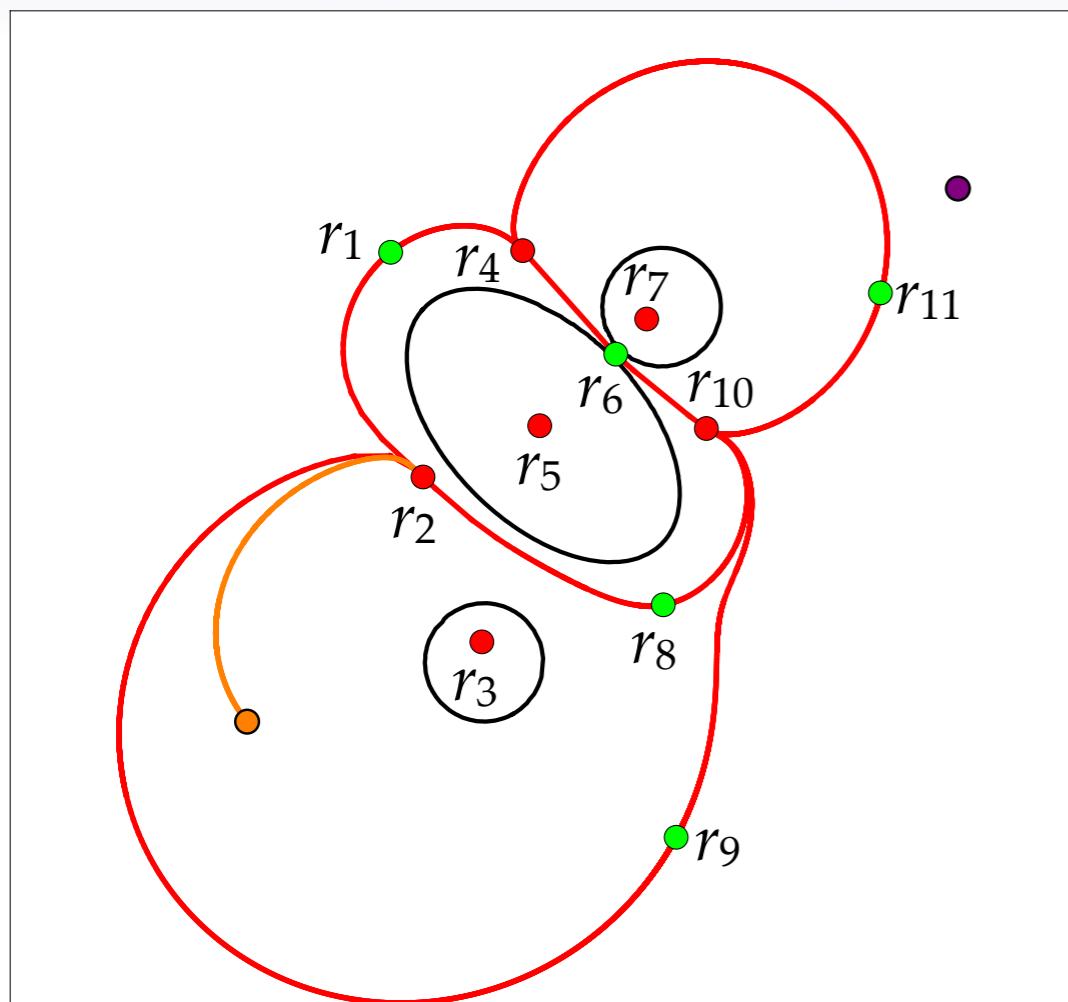


|          | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ | $r_6$ | $r_7$ | $r_8$ | $r_9$ | $r_{10}$ | $r_{11}$ |
|----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|----------|
| $r_1$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_2$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_3$    | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_4$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_5$    | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_6$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_7$    | 0     | 0     | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0        | 0        |
| $r_8$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_9$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{10}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{11}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |

**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors
- 5: Form adjacency matrix
- 6: Closure of adjacency matrix
- 7: Steepest ascent from

# Method: Overview

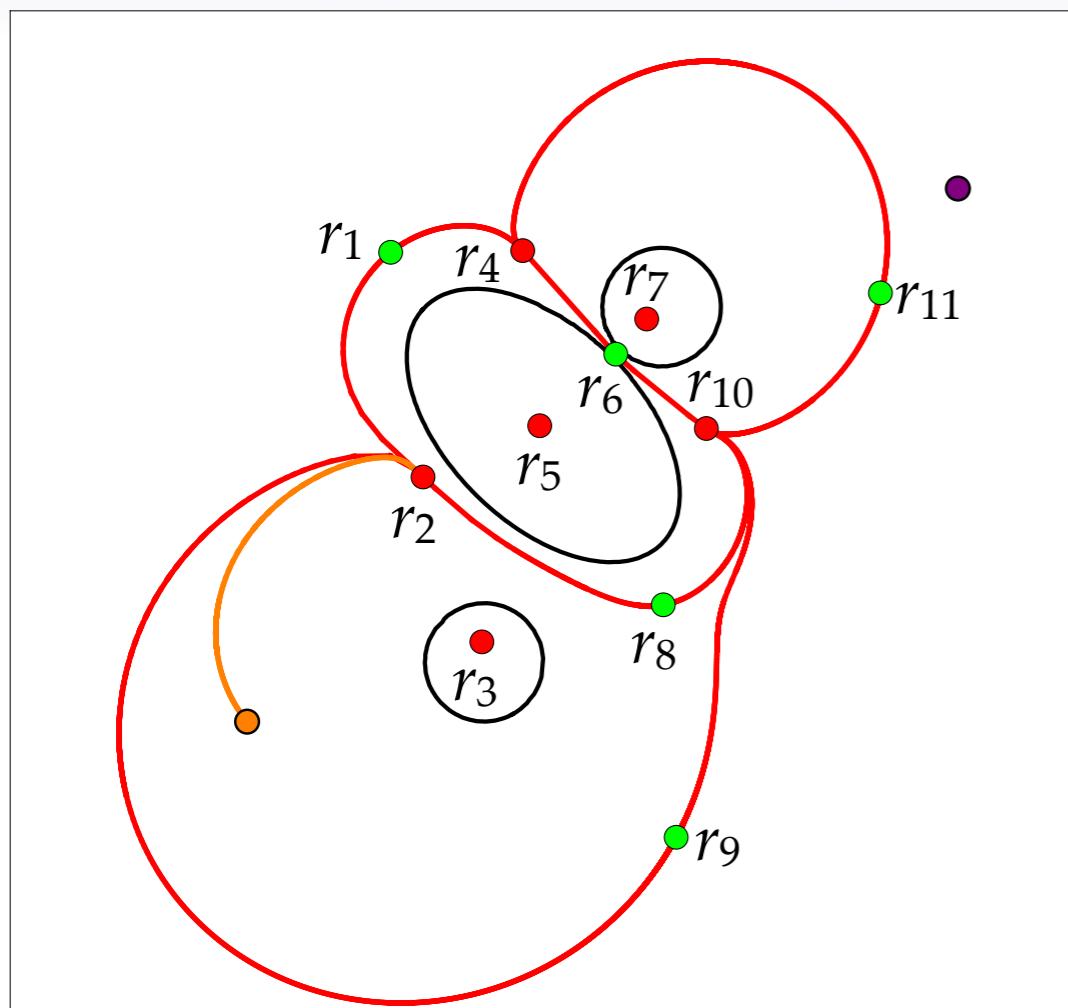


|          | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ | $r_6$ | $r_7$ | $r_8$ | $r_9$ | $r_{10}$ | $r_{11}$ |
|----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|----------|
| $r_1$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_2$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_3$    | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_4$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_5$    | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_6$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_7$    | 0     | 0     | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0        | 0        |
| $r_8$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_9$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{10}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{11}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |

**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors
- 5: Form adjacency matrix
- 6: Closure of adjacency matrix
- 7: Steepest ascent from

# Method: Overview

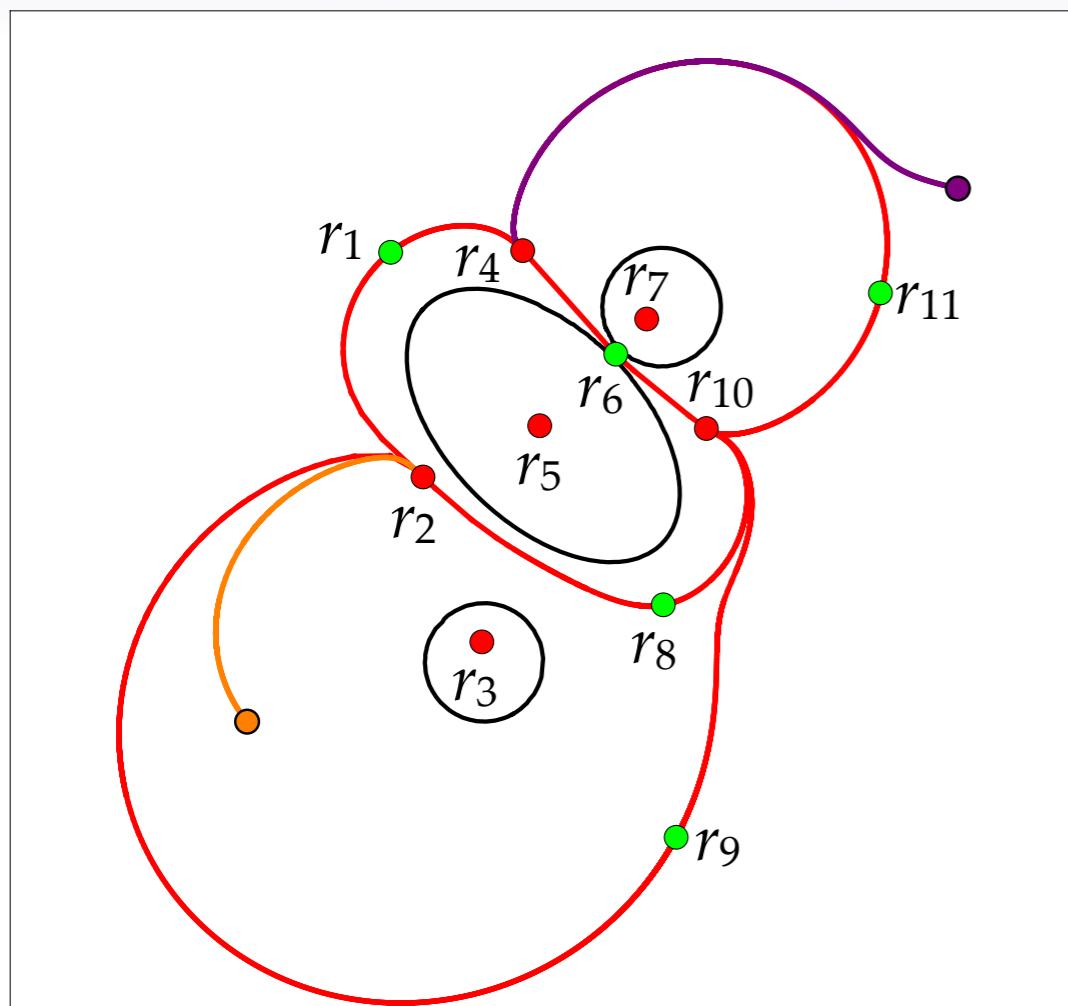


|          | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ | $r_6$ | $r_7$ | $r_8$ | $r_9$ | $r_{10}$ | $r_{11}$ |
|----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|----------|
| $r_1$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_2$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_3$    | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_4$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_5$    | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_6$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_7$    | 0     | 0     | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0        | 0        |
| $r_8$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_9$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{10}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{11}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |

**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors
- 5: Form adjacency matrix
- 6: Closure of adjacency matrix
- 7: Steepest ascent from
- 8: Steepest ascent from

# Method: Overview

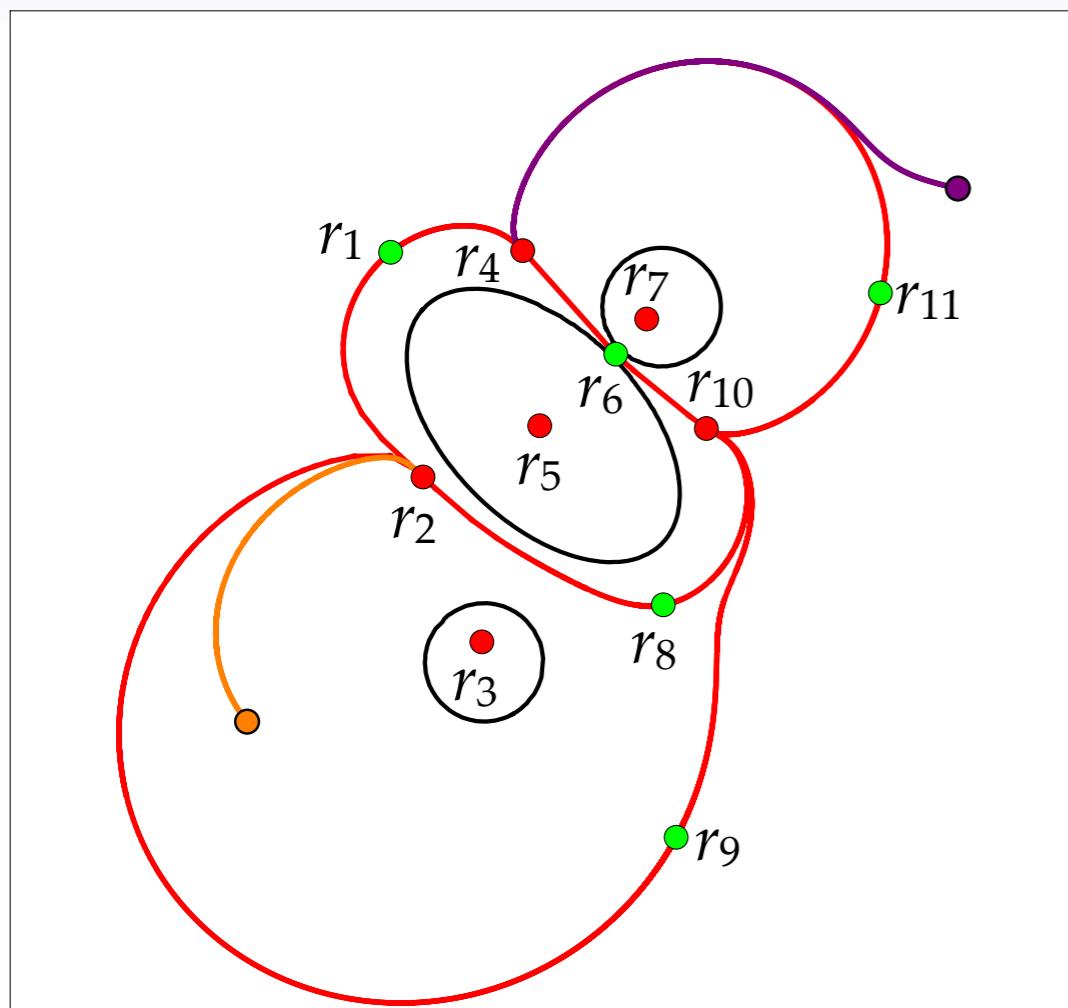


|          | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ | $r_6$ | $r_7$ | $r_8$ | $r_9$ | $r_{10}$ | $r_{11}$ |
|----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|----------|
| $r_1$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_2$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_3$    | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_4$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_5$    | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_6$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_7$    | 0     | 0     | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0        | 0        |
| $r_8$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_9$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{10}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{11}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |

**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors
- 5: Form adjacency matrix
- 6: Closure of adjacency matrix
- 7: Steepest ascent from
- 8: Steepest ascent from

# Method: Overview

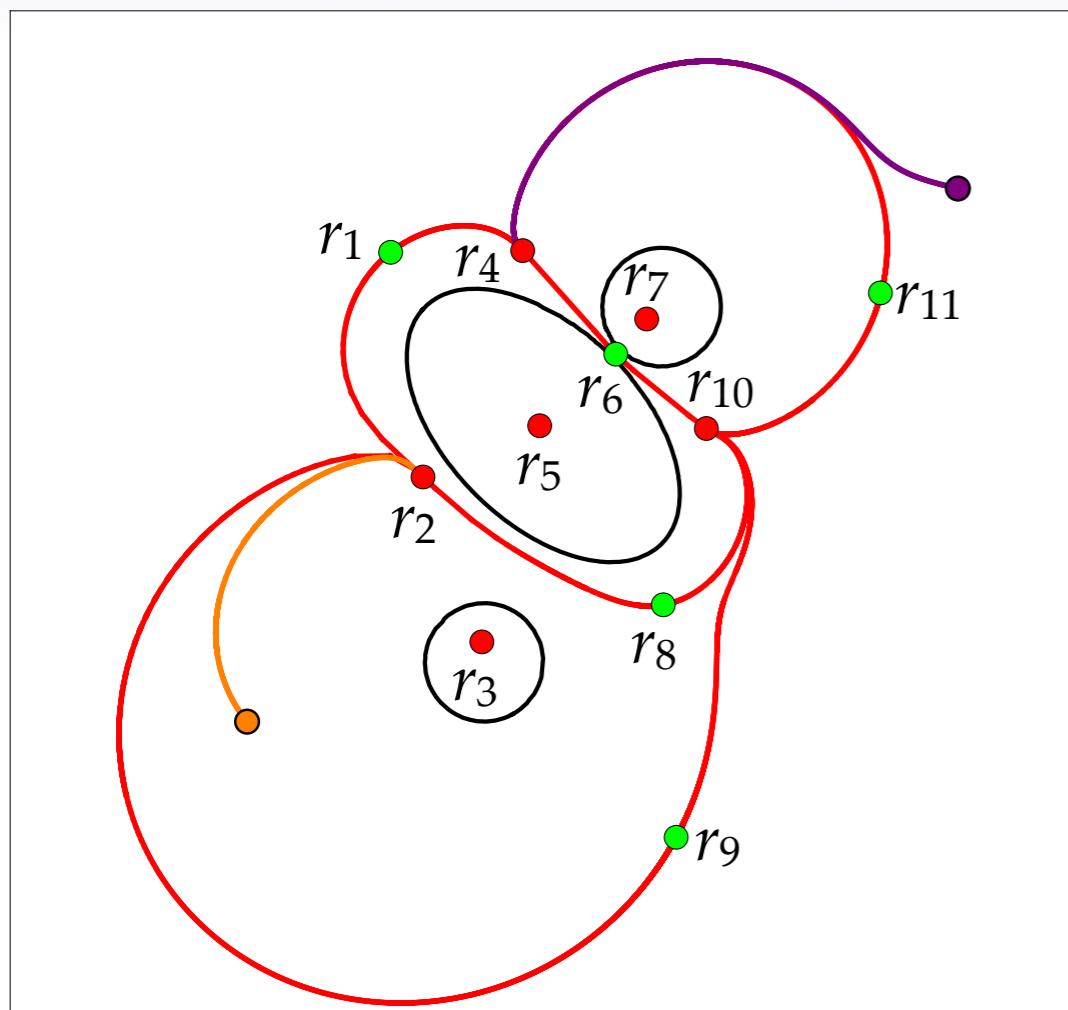


|          | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ | $r_6$ | $r_7$ | $r_8$ | $r_9$ | $r_{10}$ | $r_{11}$ |
|----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|----------|
| $r_1$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_2$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_3$    | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_4$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_5$    | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_6$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_7$    | 0     | 0     | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0        | 0        |
| $r_8$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_9$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{10}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{11}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |

**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors
- 5: Form adjacency matrix
- 6: Closure of adjacency matrix
- 7: Steepest ascent from
- 8: Steepest ascent from
- 9: Read matrix

# Method: Overview

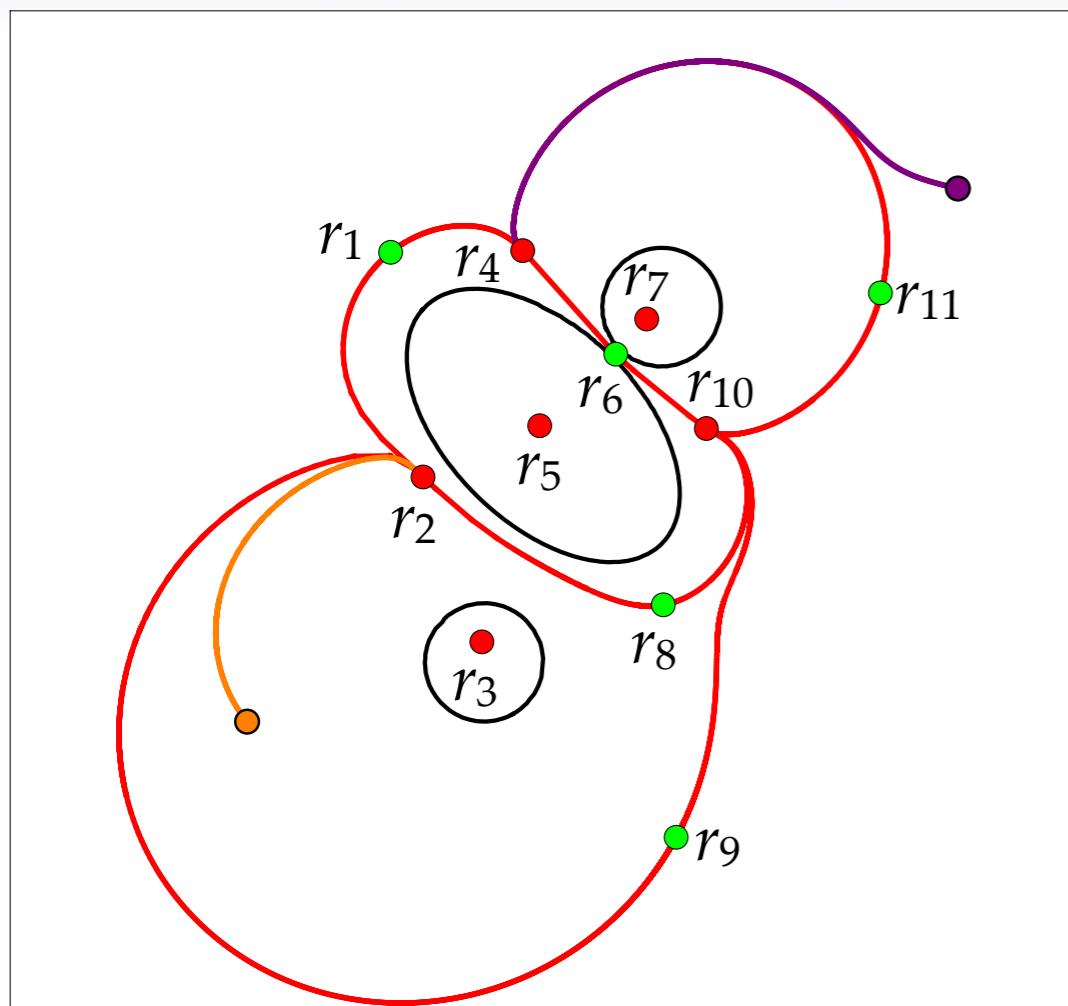


|          | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ | $r_6$ | $r_7$ | $r_8$ | $r_9$ | $r_{10}$ | $r_{11}$ |
|----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|----------|
| $r_1$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_2$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_3$    | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_4$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_5$    | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_6$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_7$    | 0     | 0     | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0        | 0        |
| $r_8$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_9$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{10}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{11}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |

**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors
- 5: Form adjacency matrix
- 6: Closure of adjacency matrix
- 7: Steepest ascent from
- 8: Steepest ascent from
- 9: Read matrix

# Method: Overview



|          | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ | $r_6$ | $r_7$ | $r_8$ | $r_9$ | $r_{10}$ | $r_{11}$ |
|----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|----------|
| $r_1$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_2$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_3$    | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_4$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_5$    | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0        | 0        |
| $r_6$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_7$    | 0     | 0     | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0        | 0        |
| $r_8$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_9$    | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{10}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |
| $r_{11}$ | 1     | 1     | 0     | 1     | 0     | 1     | 0     | 1     | 1     | 1        | 1        |

**Input:**  $f(x_1, x_2)$ , ,

- 1:  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$
- 2: Solve  $\nabla g(x) = 0 \wedge g(x) \neq 0$
- 3: Find eigenvectors of  $(\text{Hess } g)(x)$
- 4: Steepest ascent using positive eigenvectors
- 5: Form adjacency matrix
- 6: Closure of adjacency matrix
- 7: Steepest ascent from
- 8: Steepest ascent from
- 9: Read matrix

**Output:** True

# Research Challenges

# Research Challenges

## 1. Correctness

# Research Challenges

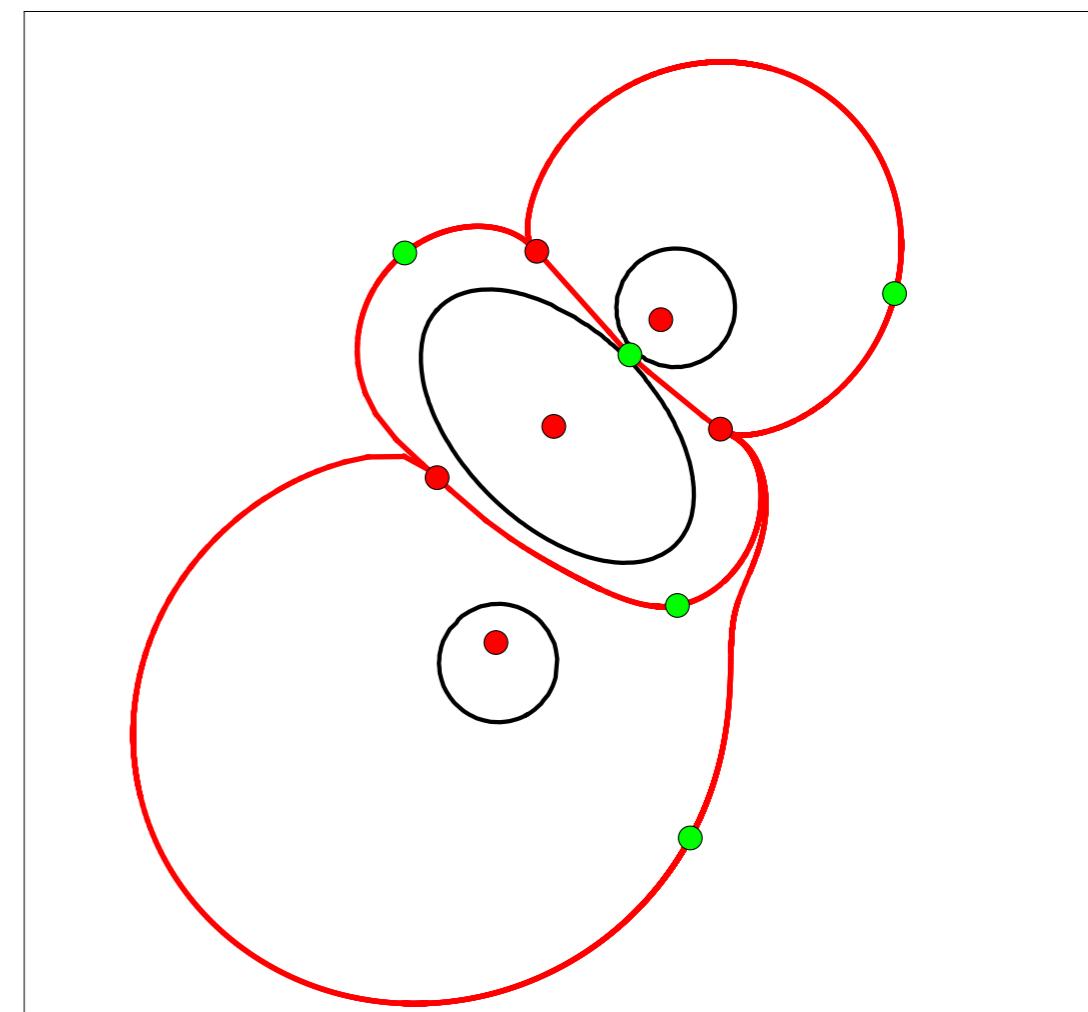
1. Correctness
2. Termination

# Research Challenges

1. Correctness
2. Termination
3. Complexity

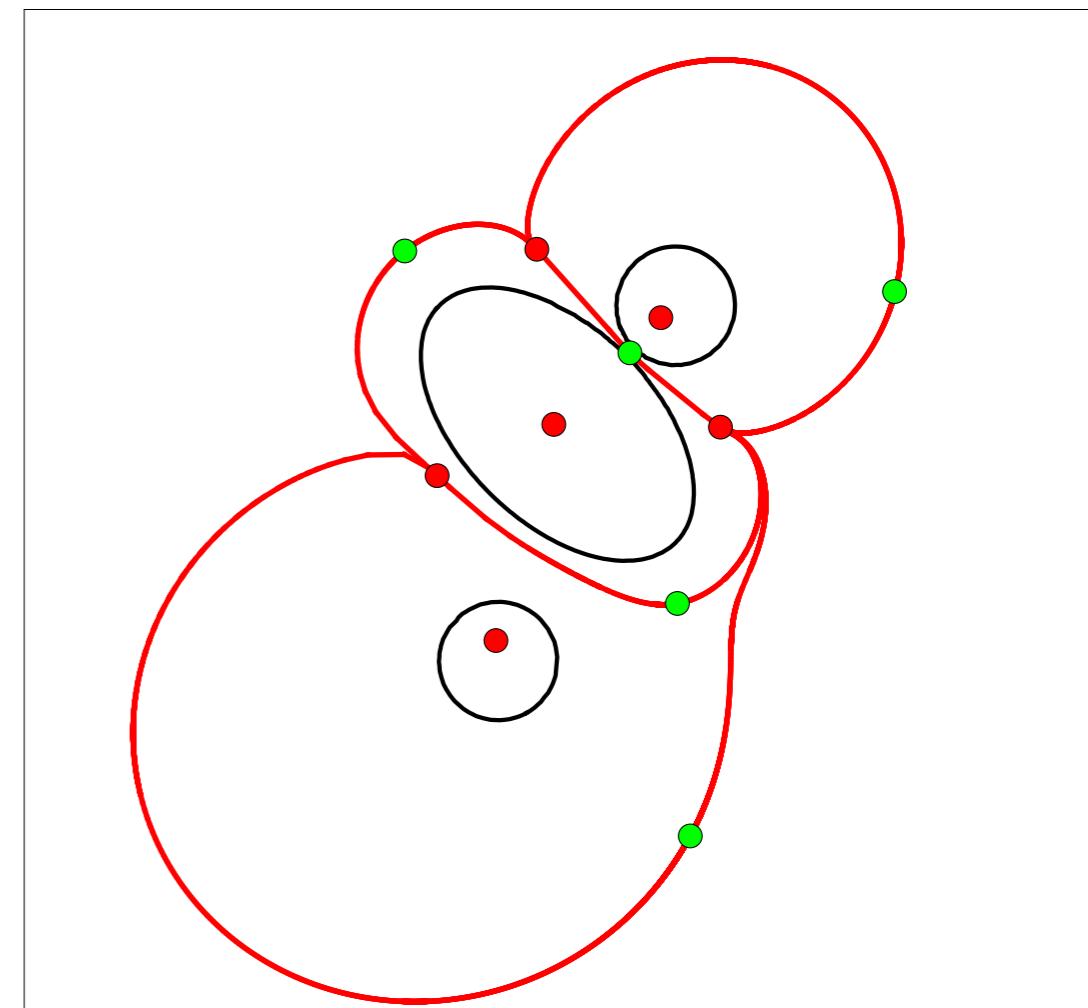
# 1. Correctness: Theorem

# 1. Correctness: Theorem



# 1. Correctness: Theorem

Let  $g$  be a routing function.

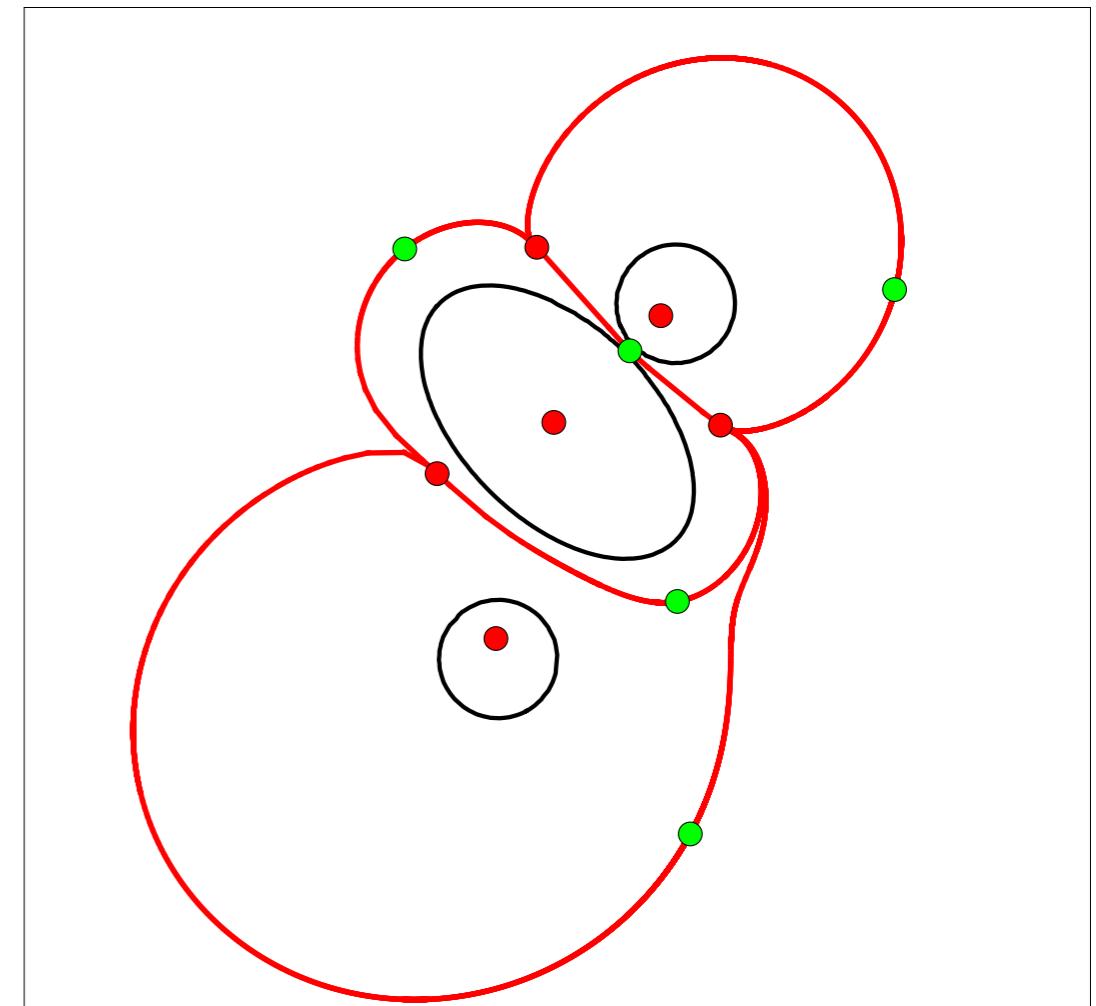


# 1. Correctness: Theorem

Let  $g$  be a routing function.

- finitely many routing points

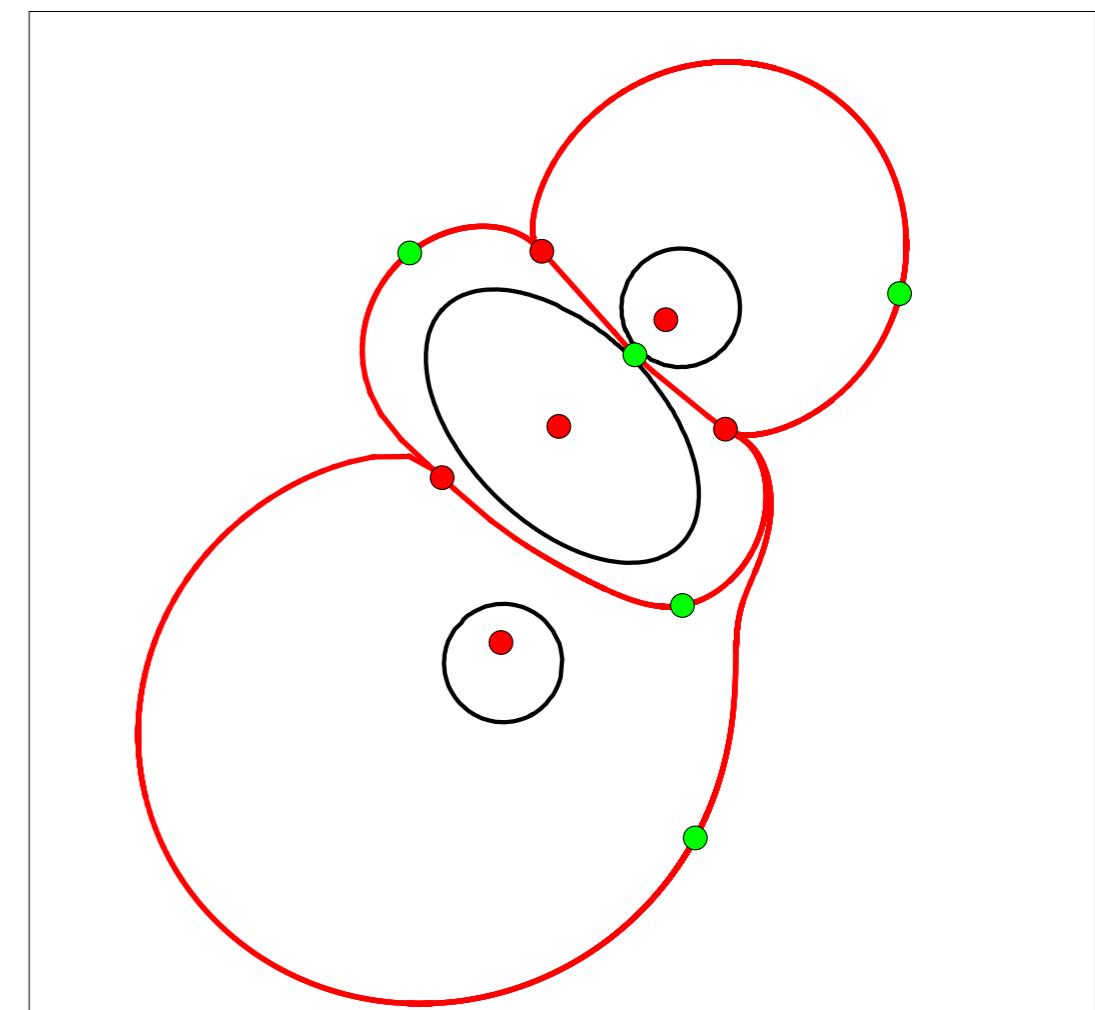
$$\nabla g(x) = 0 \wedge g(x) \neq 0$$



# 1. Correctness: Theorem

Let  $g$  be a routing function.

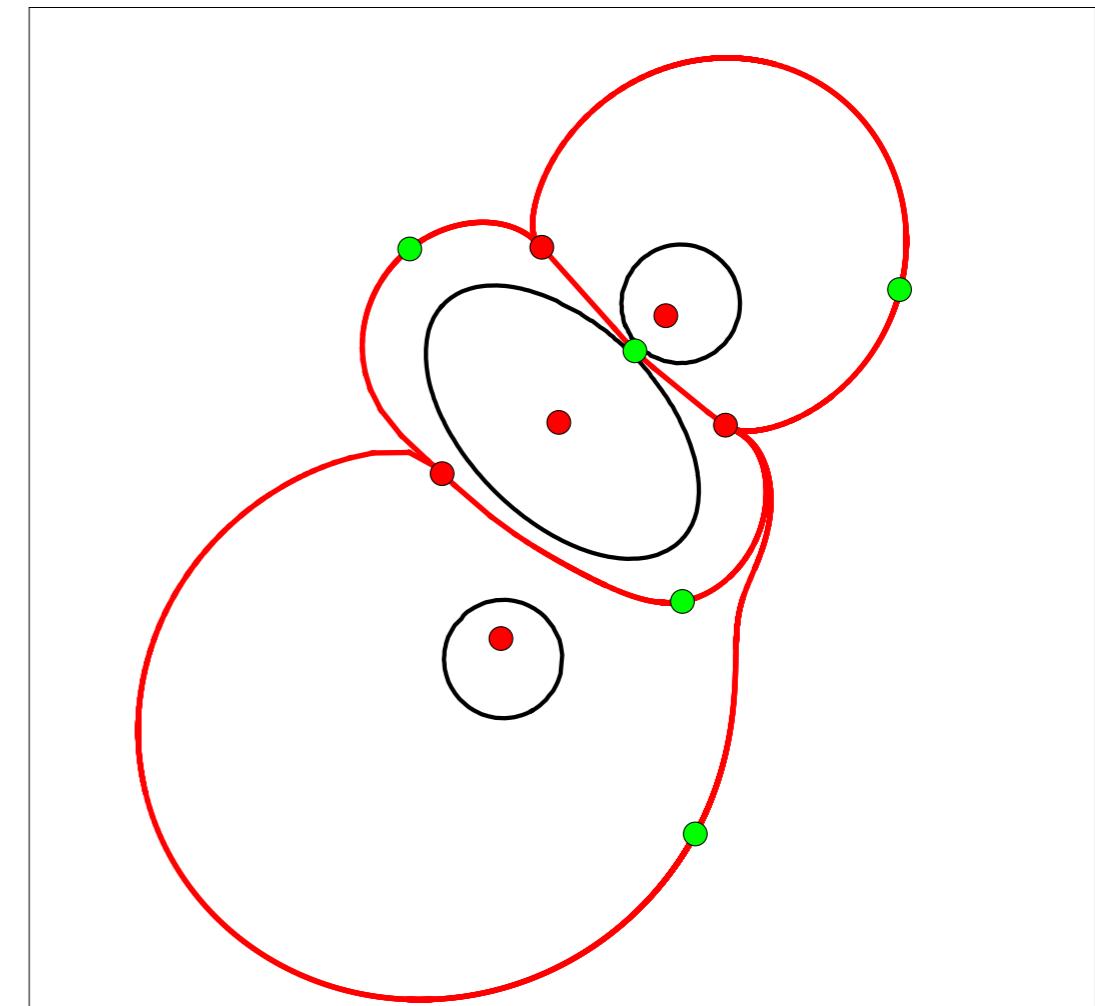
- finitely many routing points  
 $\nabla g(x) = 0 \wedge g(x) \neq 0$
- routing points are nondegenerate  
 $\det(\text{Hess } g)(x) \neq 0$



# 1. Correctness: Theorem

Let  $g$  be a routing function.

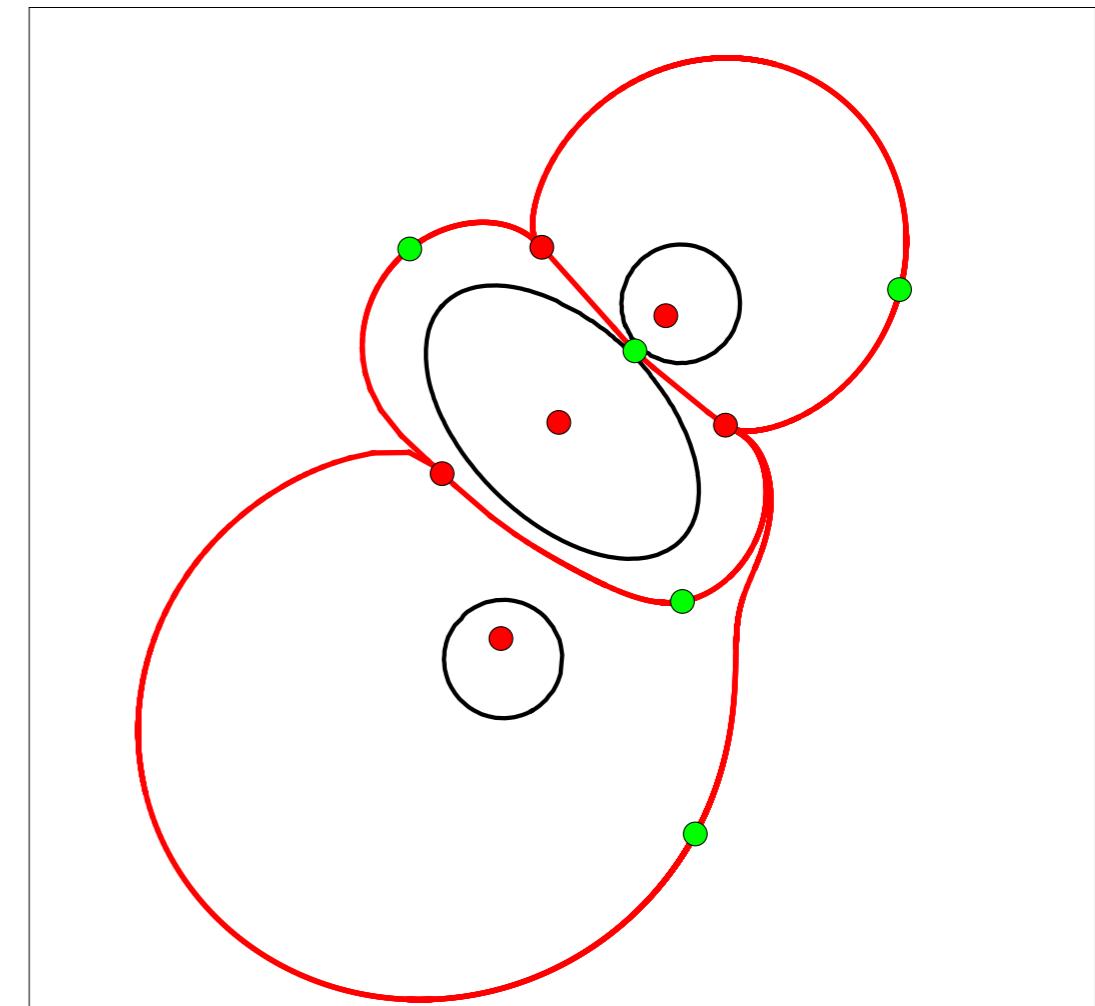
- finitely many routing points  
 $\nabla g(x) = 0 \wedge g(x) \neq 0$
- routing points are nondegenerate  
 $\det(\text{Hess } g)(x) \neq 0$
- $g(x) \rightarrow 0$  as  $\|x\| \rightarrow \infty$



# 1. Correctness: Theorem

Let  $g$  be a routing function.

- finitely many routing points  
 $\nabla g(x) = 0 \wedge g(x) \neq 0$
- routing points are nondegenerate  
 $\det(\text{Hess } g)(x) \neq 0$
- $g(x) \rightarrow 0$  as  $\|x\| \rightarrow \infty$
- $g(x) \geq 0$

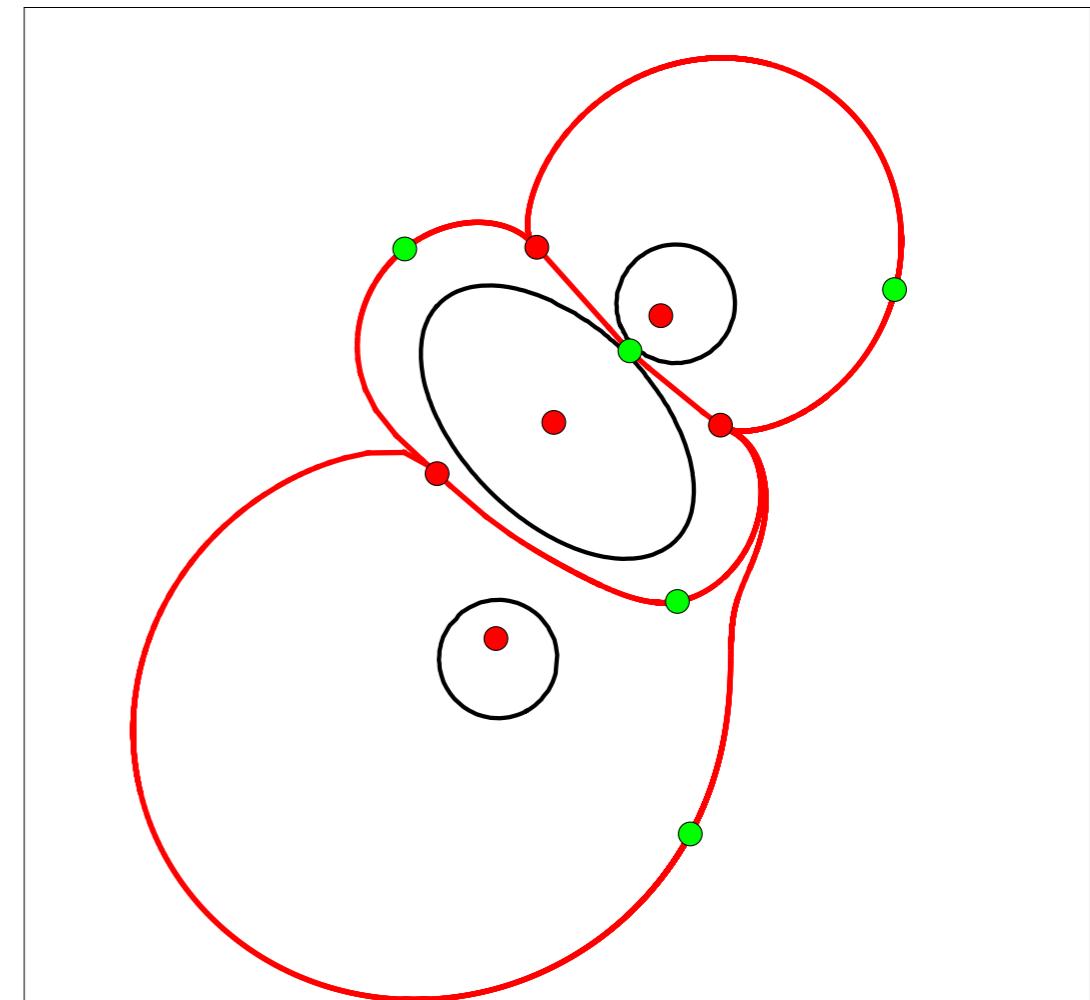


# 1. Correctness: Theorem

Let  $g$  be a routing function.

- finitely many routing points  
 $\nabla g(x) = 0 \wedge g(x) \neq 0$
- routing points are nondegenerate  
 $\det(\text{Hess } g)(x) \neq 0$
- $g(x) \rightarrow 0$  as  $\|x\| \rightarrow \infty$
- $g(x) \geq 0$

**Example:**  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$



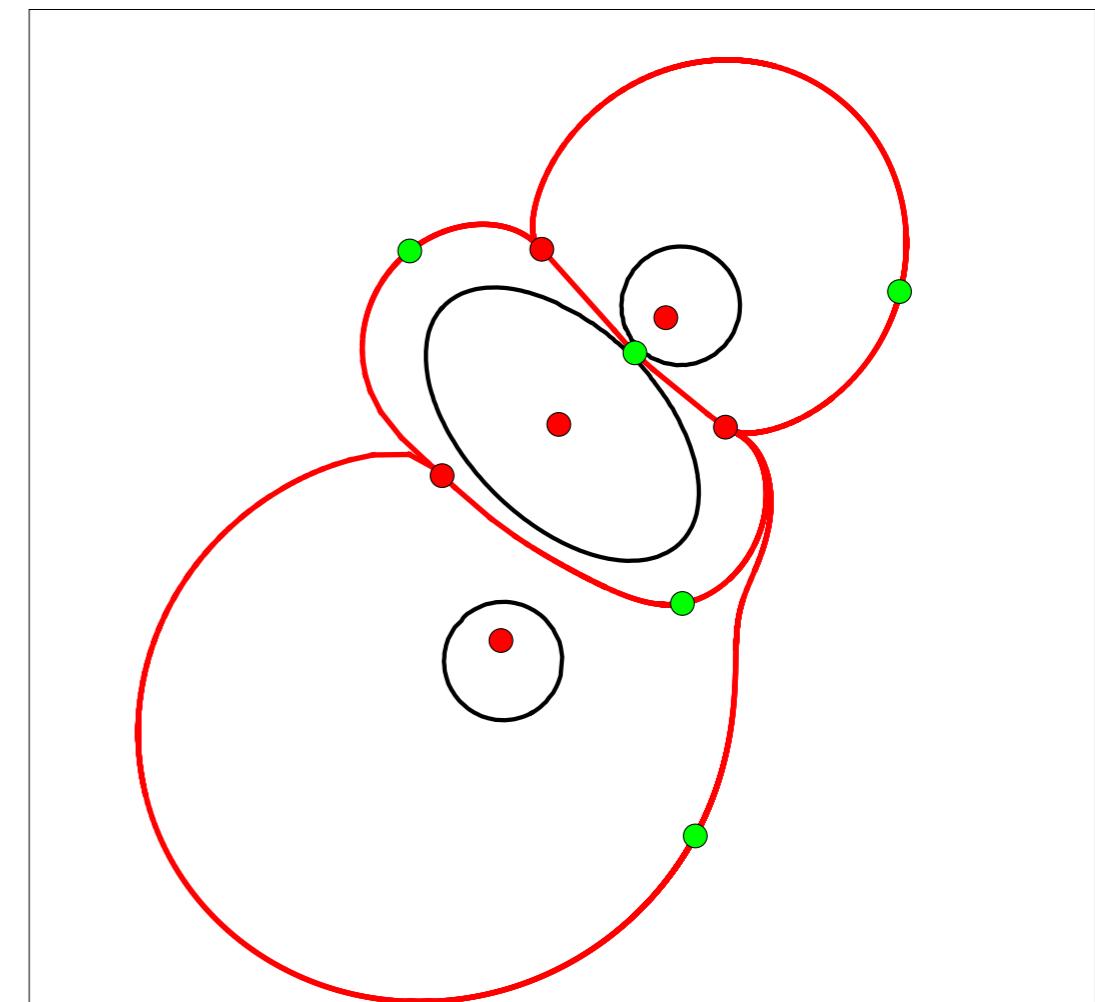
# 1. Correctness: Theorem

Let  $g$  be a routing function.

Any two routing points in the same connected component of  $\{g \neq 0\}$

- finitely many routing points  
$$\nabla g(x) = 0 \wedge g(x) \neq 0$$
- routing points are nondegenerate  
$$\det(\text{Hess } g)(x) \neq 0$$
- $g(x) \rightarrow 0$  as  $\|x\| \rightarrow \infty$
- $g(x) \geq 0$

**Example:** 
$$g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$$



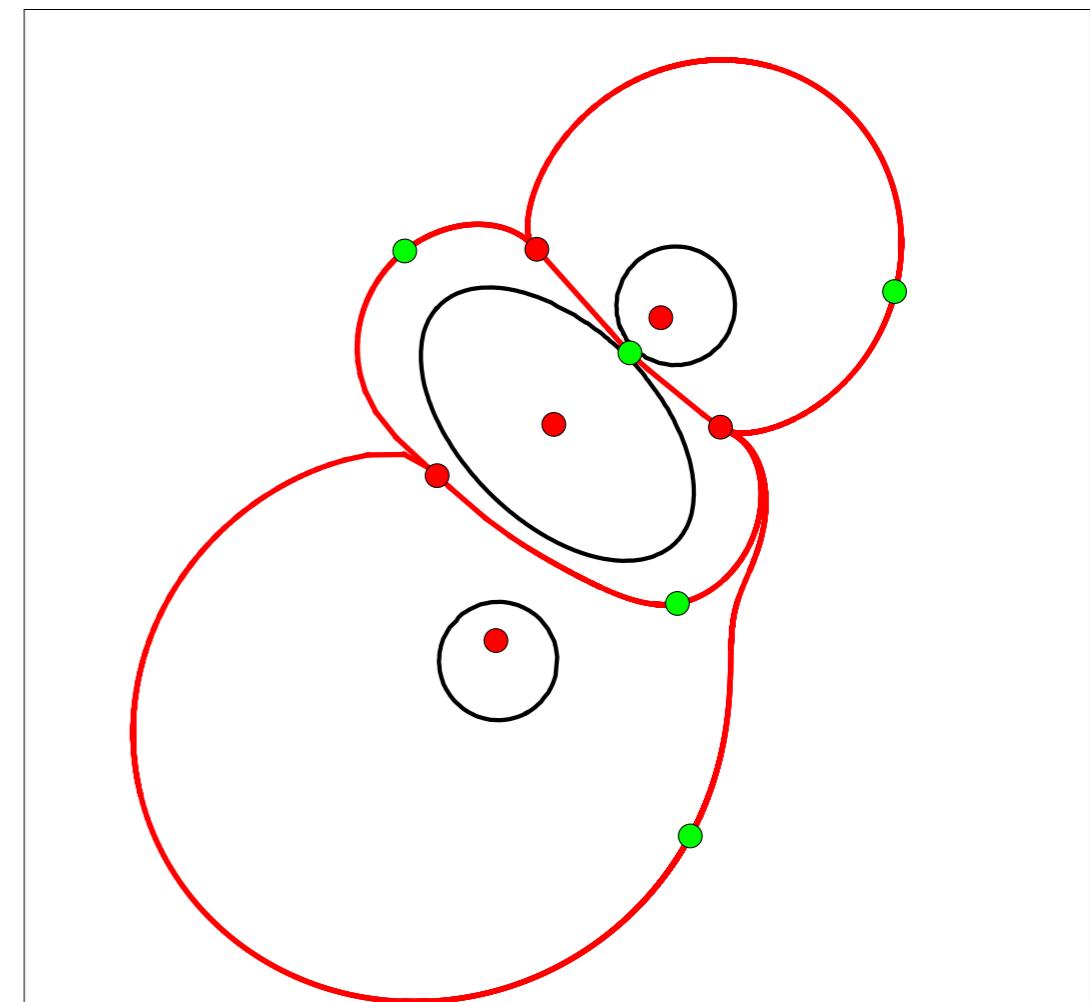
# 1. Correctness: Theorem

Let  $g$  be a routing function.

Any two routing points in the same connected component of  $\{g \neq 0\}$  are connected by steepest ascent paths

- finitely many routing points  
 $\nabla g(x) = 0 \wedge g(x) \neq 0$
- routing points are nondegenerate  
 $\det(\text{Hess } g)(x) \neq 0$
- $g(x) \rightarrow 0$  as  $\|x\| \rightarrow \infty$
- $g(x) \geq 0$

**Example:**  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$



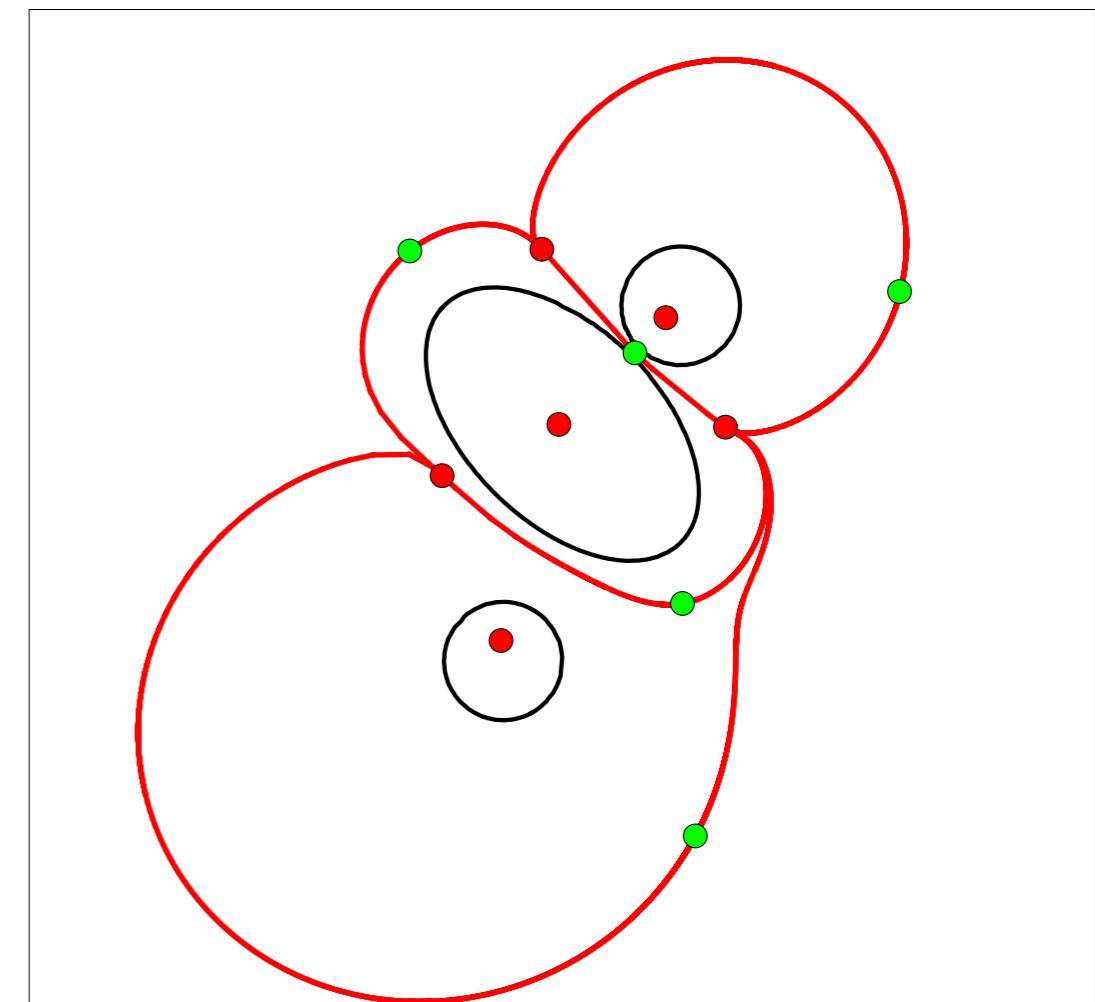
# 1. Correctness: Theorem

Let  $g$  be a routing function.

Any two routing points in the same connected component of  $\{g \neq 0\}$  are connected by steepest ascent paths using positive eigenvectors.

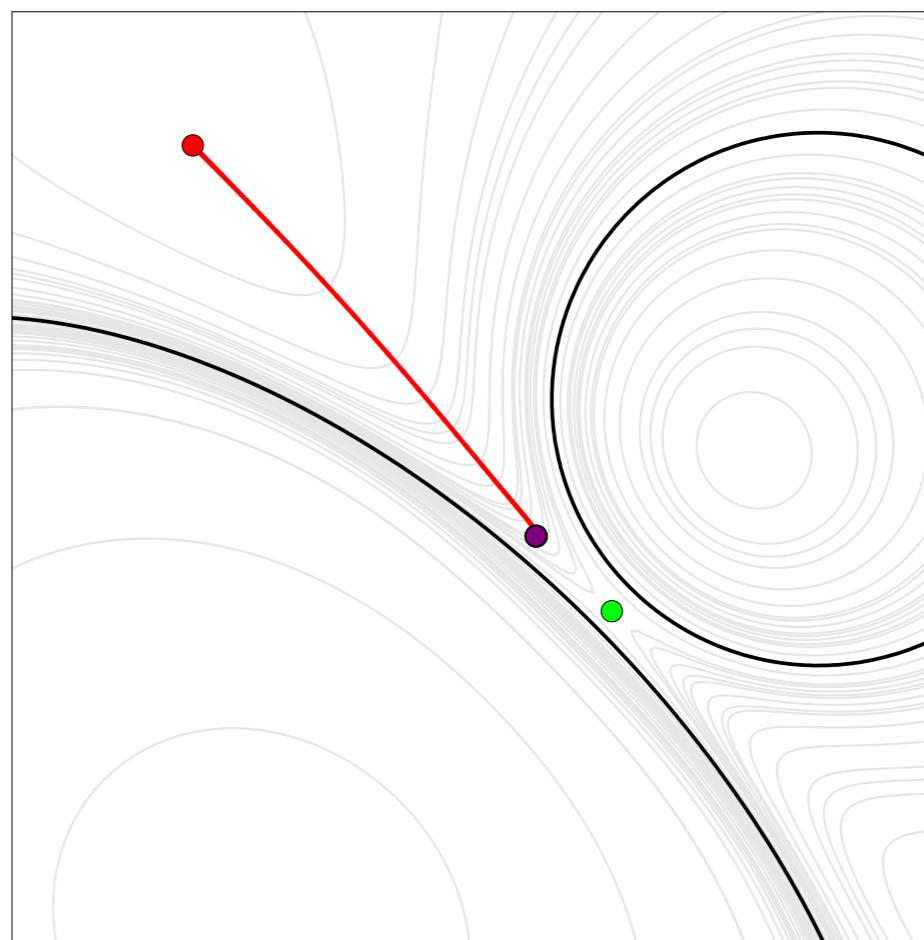
- finitely many routing points  
 $\nabla g(x) = 0 \wedge g(x) \neq 0$
- routing points are nondegenerate  
 $\det(\text{Hess } g)(x) \neq 0$
- $g(x) \rightarrow 0$  as  $\|x\| \rightarrow \infty$
- $g(x) \geq 0$

**Example:**  $g(x_1, x_2) = \frac{f(x_1, x_2)^2}{(x_1^2 + x_2^2 + 1)^{\deg(f)+1}}$



# 1. Correctness: Preliminaries

# 1. Correctness: Preliminaries

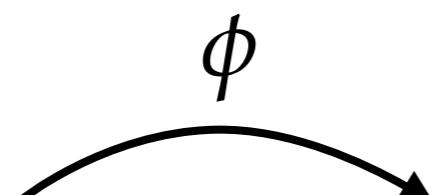


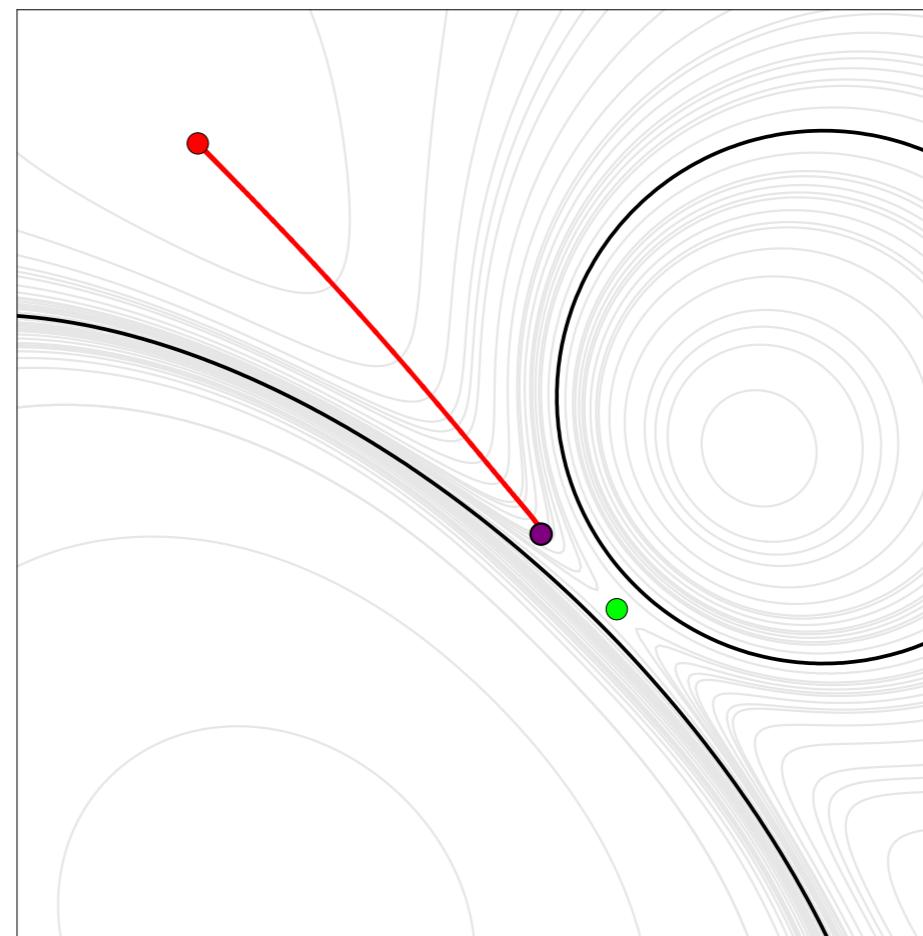
# 1. Correctness: Preliminaries

Trajectory of  $\nabla g$  through ●

$$\phi'(t) = \nabla g(\phi(t))$$

$$\phi(0) = \bullet$$

$[0, \infty)$    
 $\phi$



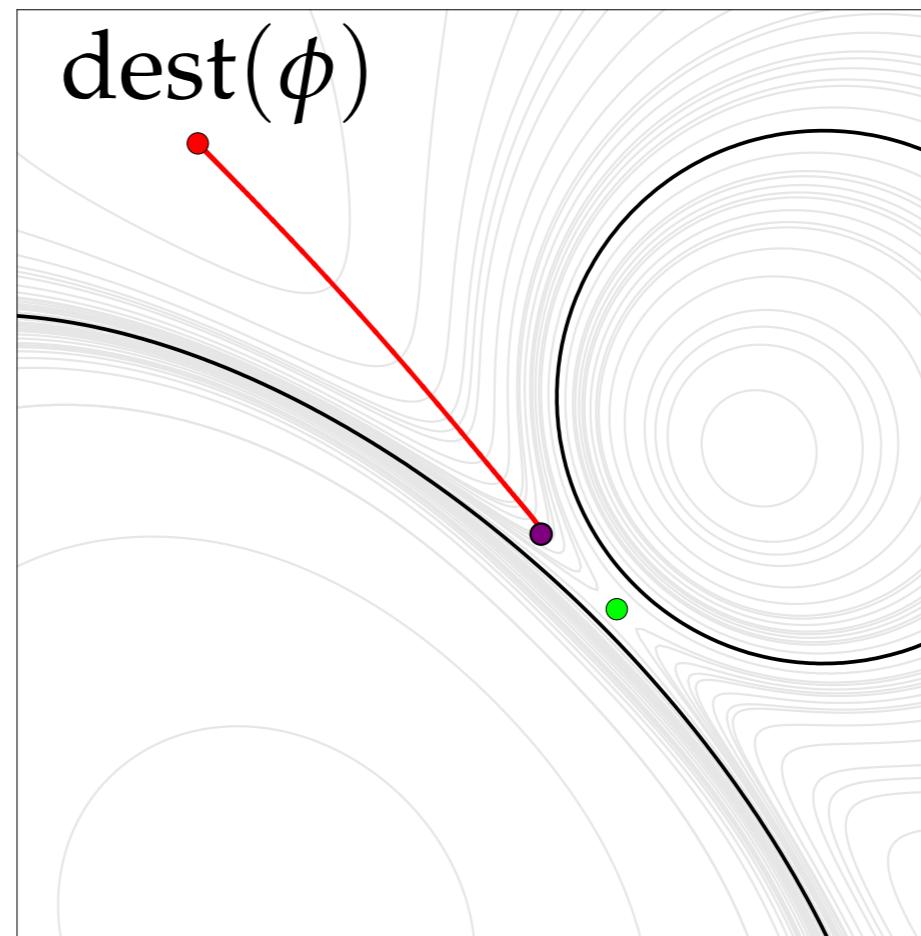
# 1. Correctness: Preliminaries

Trajectory of  $\nabla g$  through ●

$$\phi'(t) = \nabla g(\phi(t))$$

$$\phi(0) = \bullet$$

$$[0, \infty) \xrightarrow{\phi}$$



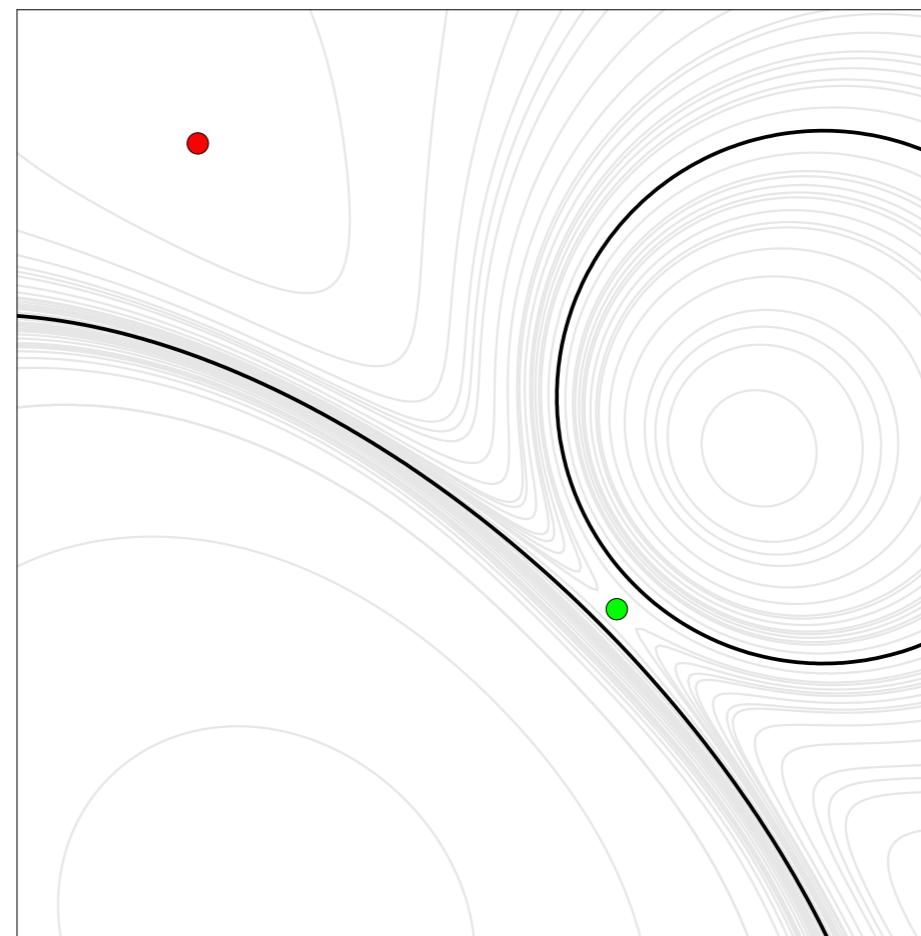
# 1. Correctness: Preliminaries

Trajectory of  $\nabla g$  through ●

$$\phi'(t) = \nabla g(\phi(t))$$

$$\phi(0) = \bullet$$

$$[0, \infty) \xrightarrow{\phi}$$



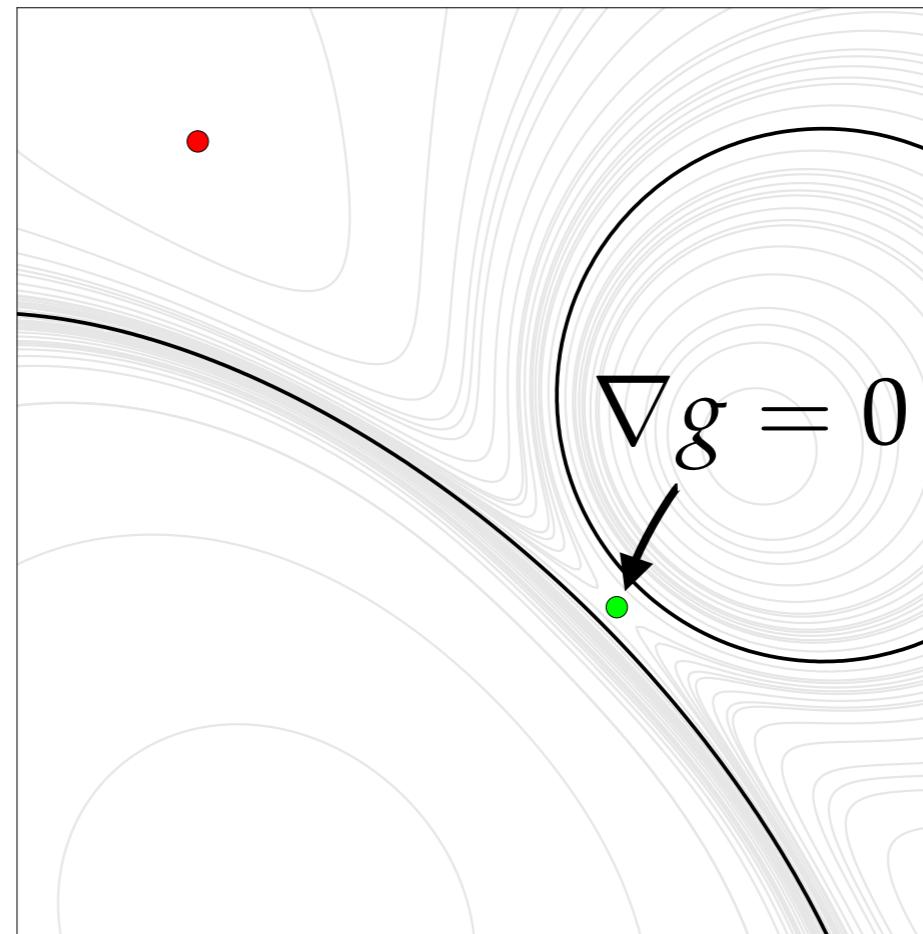
# 1. Correctness: Preliminaries

Trajectory of  $\nabla g$  through ●

$$\phi'(t) = \nabla g(\phi(t))$$

$$\phi(0) = \bullet$$

$$[0, \infty) \xrightarrow{\phi}$$



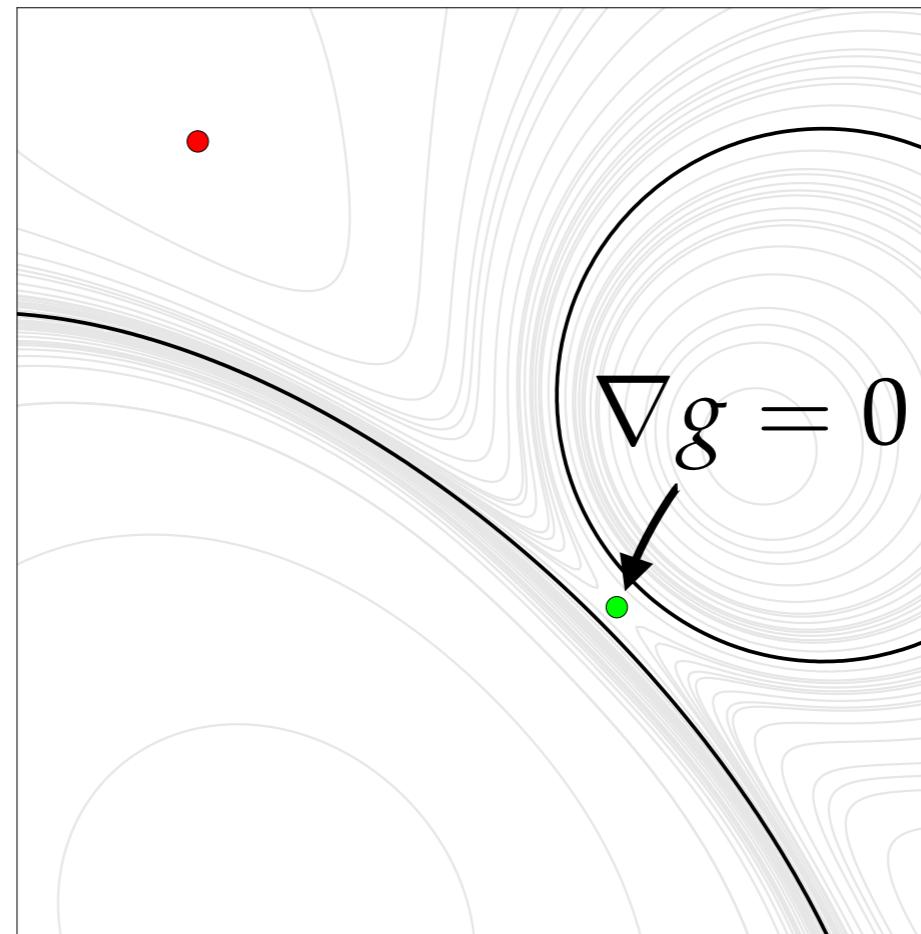
# 1. Correctness: Preliminaries

Trajectory of  $\nabla g$  through ●

$$\phi'(t) = 0$$

$$\phi(0) = \bullet$$

$$[0, \infty) \xrightarrow{\phi}$$



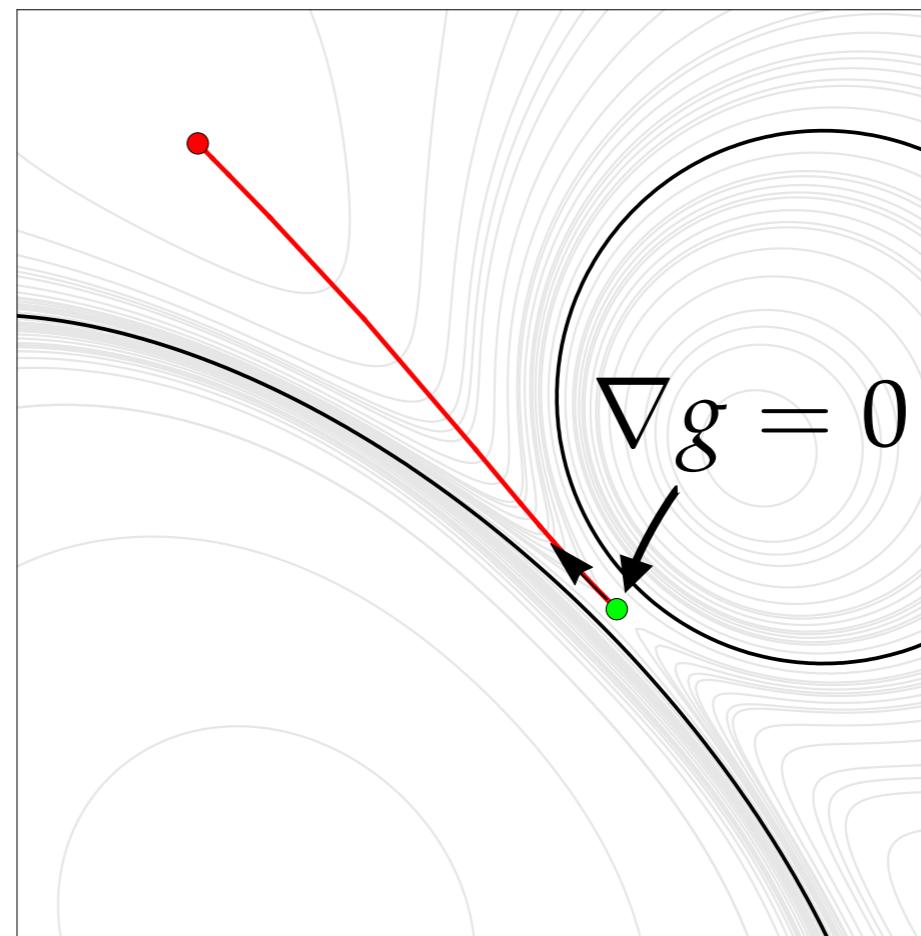
# 1. Correctness: Preliminaries

Trajectory of  $\nabla g$  through ●

$$\phi'(t) = 0$$

$$\phi(0) = \bullet$$

$$[0, \infty) \xrightarrow{\phi}$$



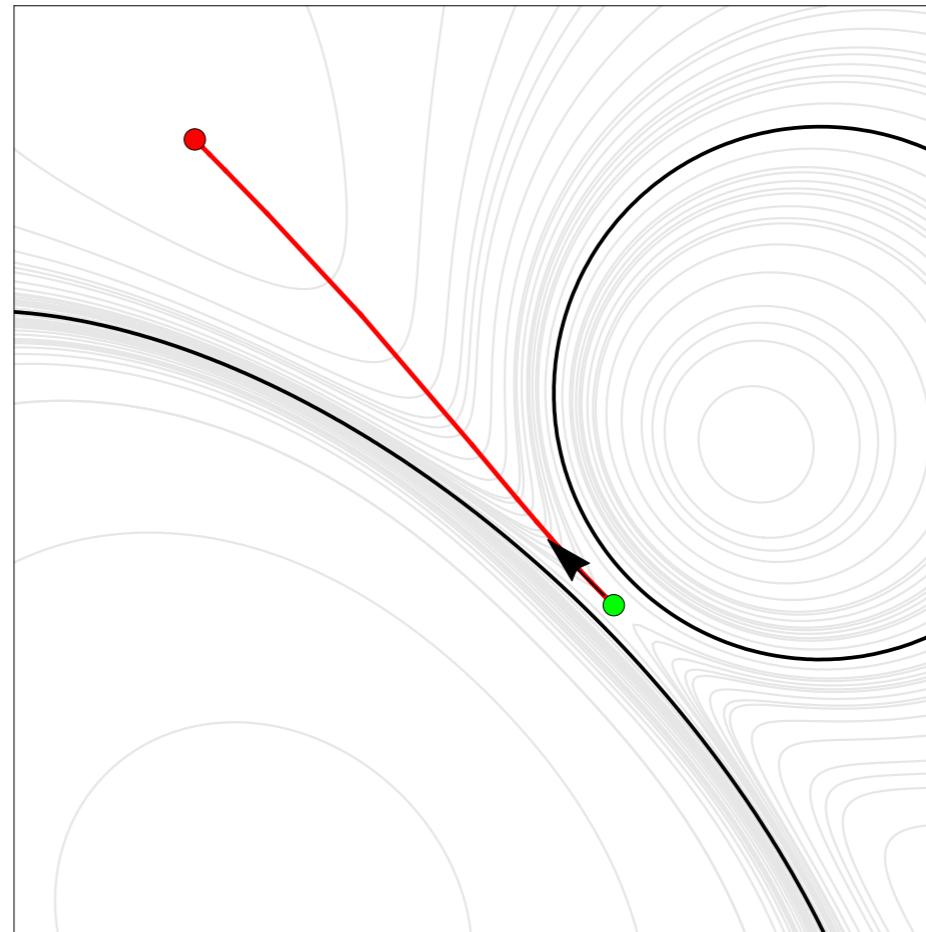
# 1. Correctness: Preliminaries

Trajectory of  $\nabla g$  through ● in the direction  $\uparrow$

$$\phi'(t) = \begin{cases} \nabla g(\phi(t)) & \text{if } t > 0 \\ \uparrow & \text{if } t = 0 \end{cases}$$

$$\phi(0) = \bullet$$

$$[0, \infty) \xrightarrow{\phi}$$



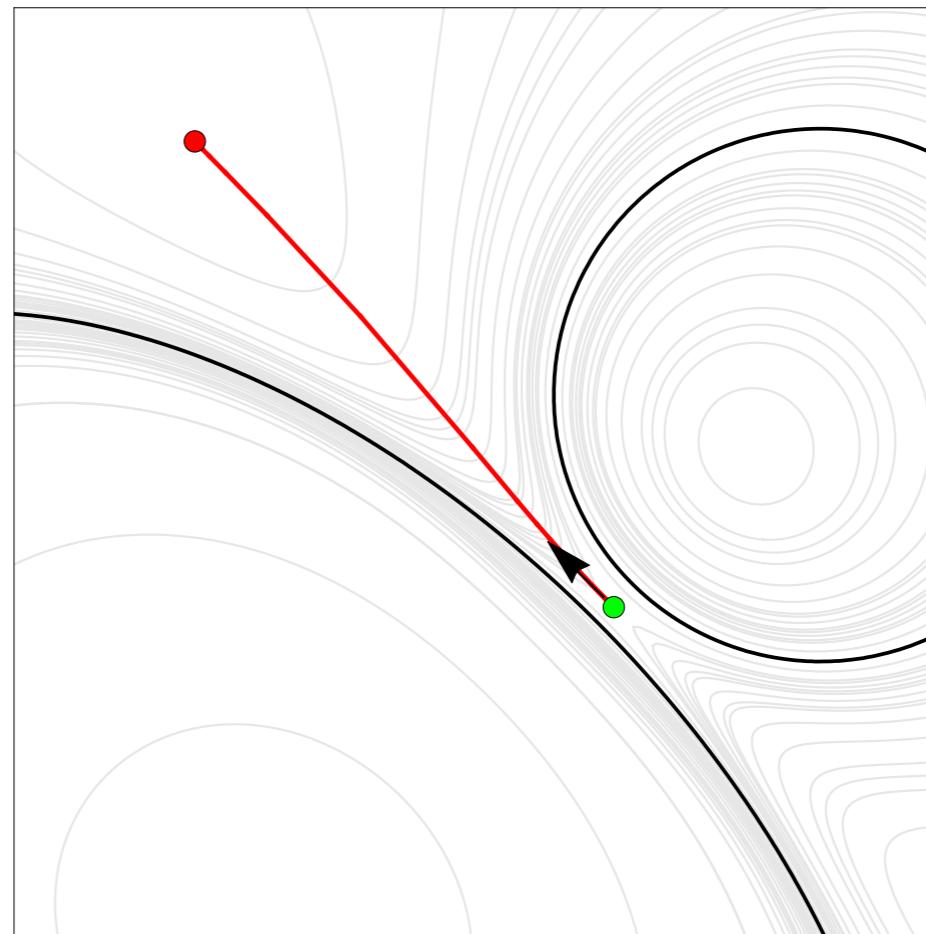
# 1. Correctness: Preliminaries

Trajectory of  $\nabla g$  through  $\bullet$  in the direction  $\uparrow$

$$\phi'(t) = \begin{cases} \nabla g(\phi(t)) & \text{if } t > 0 \\ \uparrow & \text{if } t = 0 \end{cases}$$

$$\phi(0) = \bullet$$

$$[0, \infty) \xrightarrow{\phi}$$



SA( $g, \bullet, \uparrow$ )

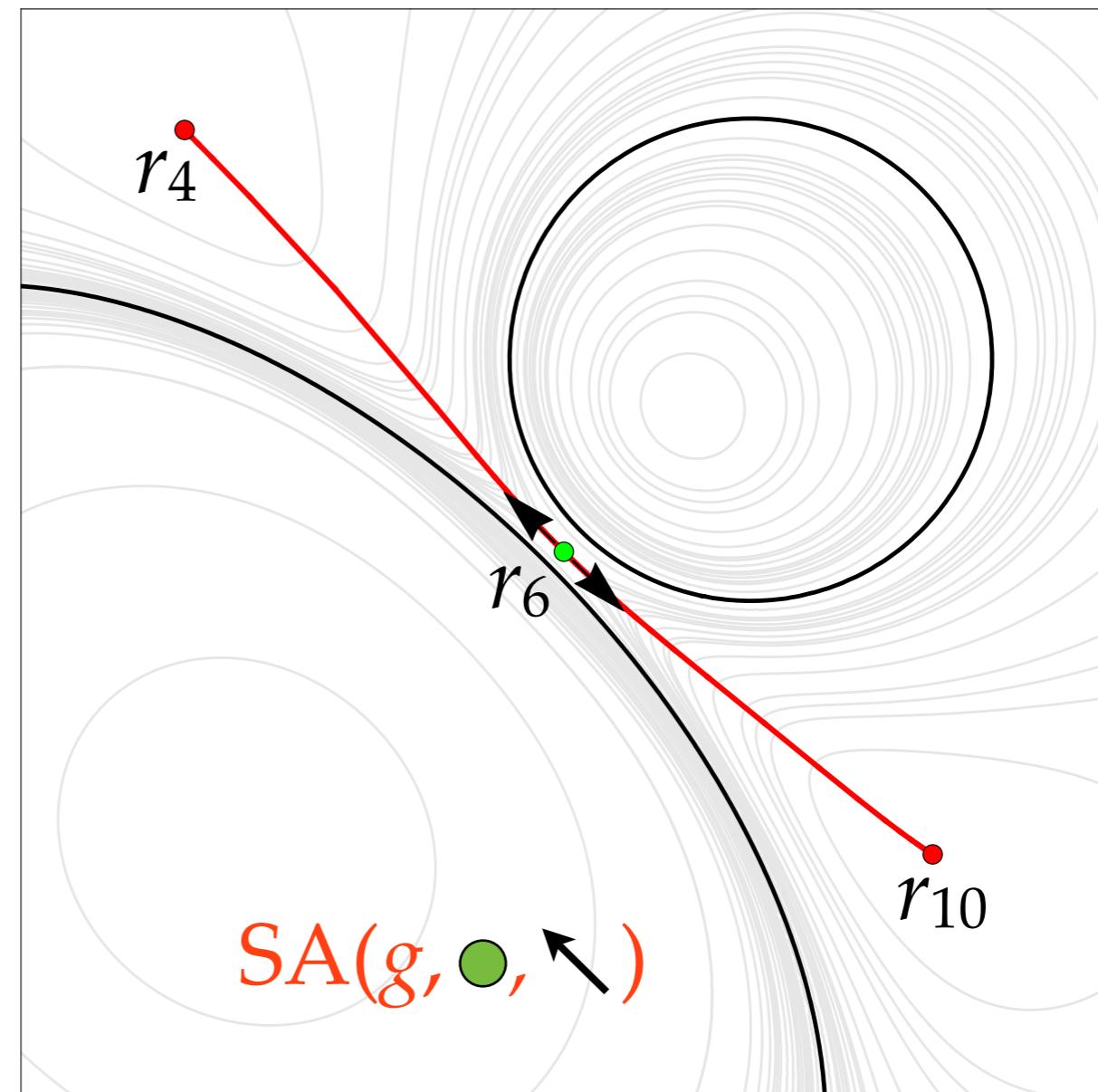
# 1. Correctness: Preliminaries

Trajectory of  $\nabla g$  through  $\bullet$  in the direction  $\uparrow$

$$\phi'(t) = \begin{cases} \nabla g(\phi(t)) & \text{if } t > 0 \\ \uparrow & \text{if } t = 0 \end{cases}$$

$$\phi(0) = \bullet$$

$$[0, \infty) \xrightarrow{\phi}$$



# 1. Correctness: Preliminaries

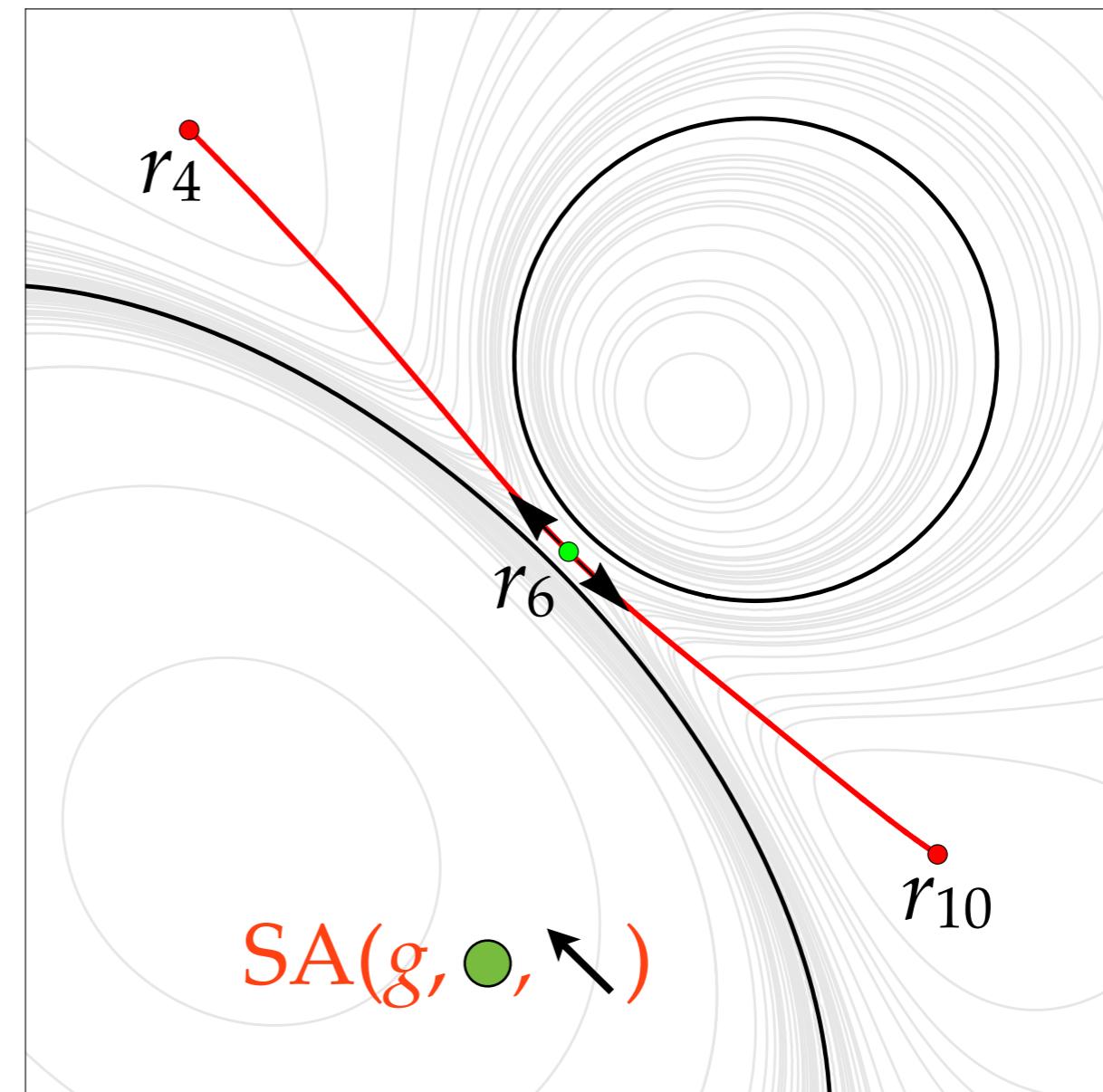
Trajectory of  $\nabla g$  through  $\bullet$  in the direction  $\uparrow$

$$\phi'(t) = \begin{cases} \nabla g(\phi(t)) & \text{if } t > 0 \\ \uparrow & \text{if } t = 0 \end{cases}$$

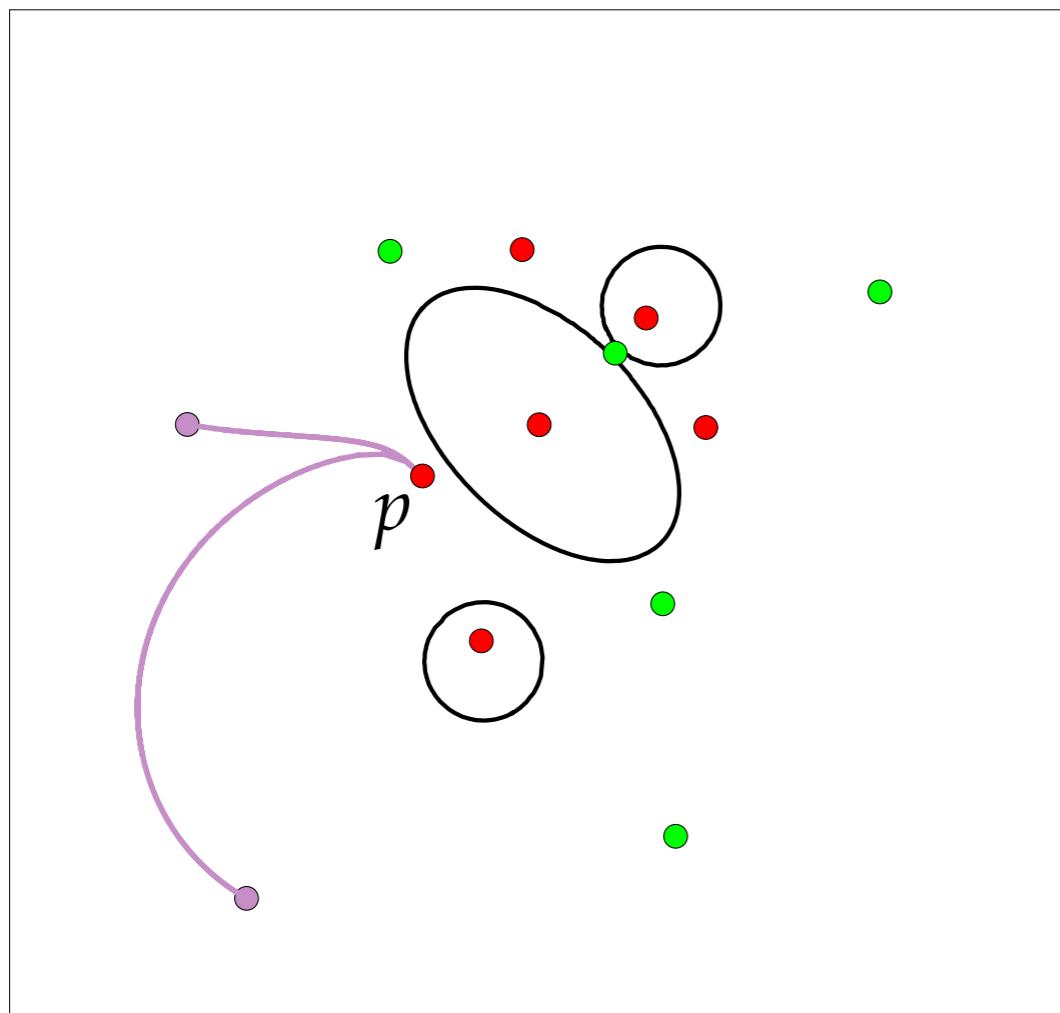
$$\phi(0) = \bullet$$

$$[0, \infty) \xrightarrow{\phi}$$

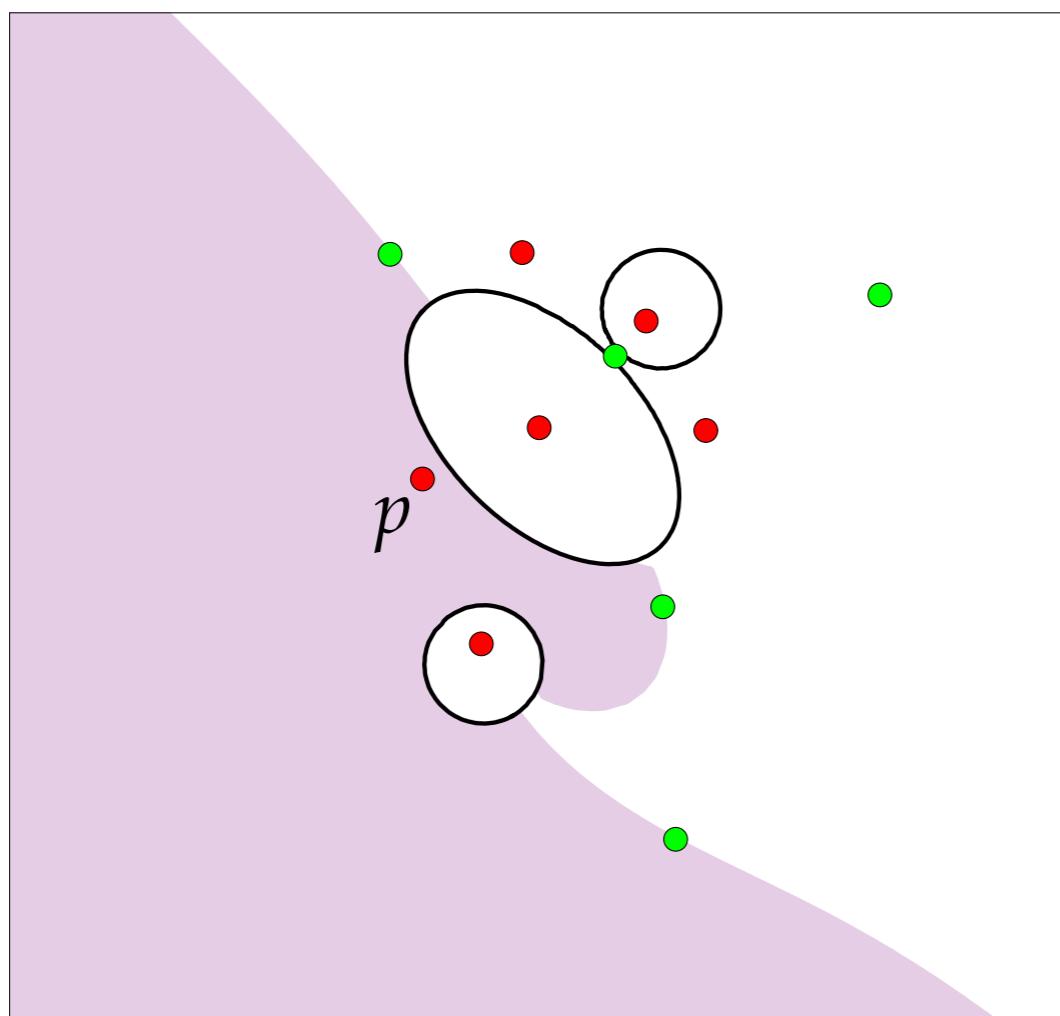
$r_4, r_6, r_{10}$  are connected  
by steepest ascent  
paths using positive  
eigenvectors



# 1. Correctness: Preliminaries



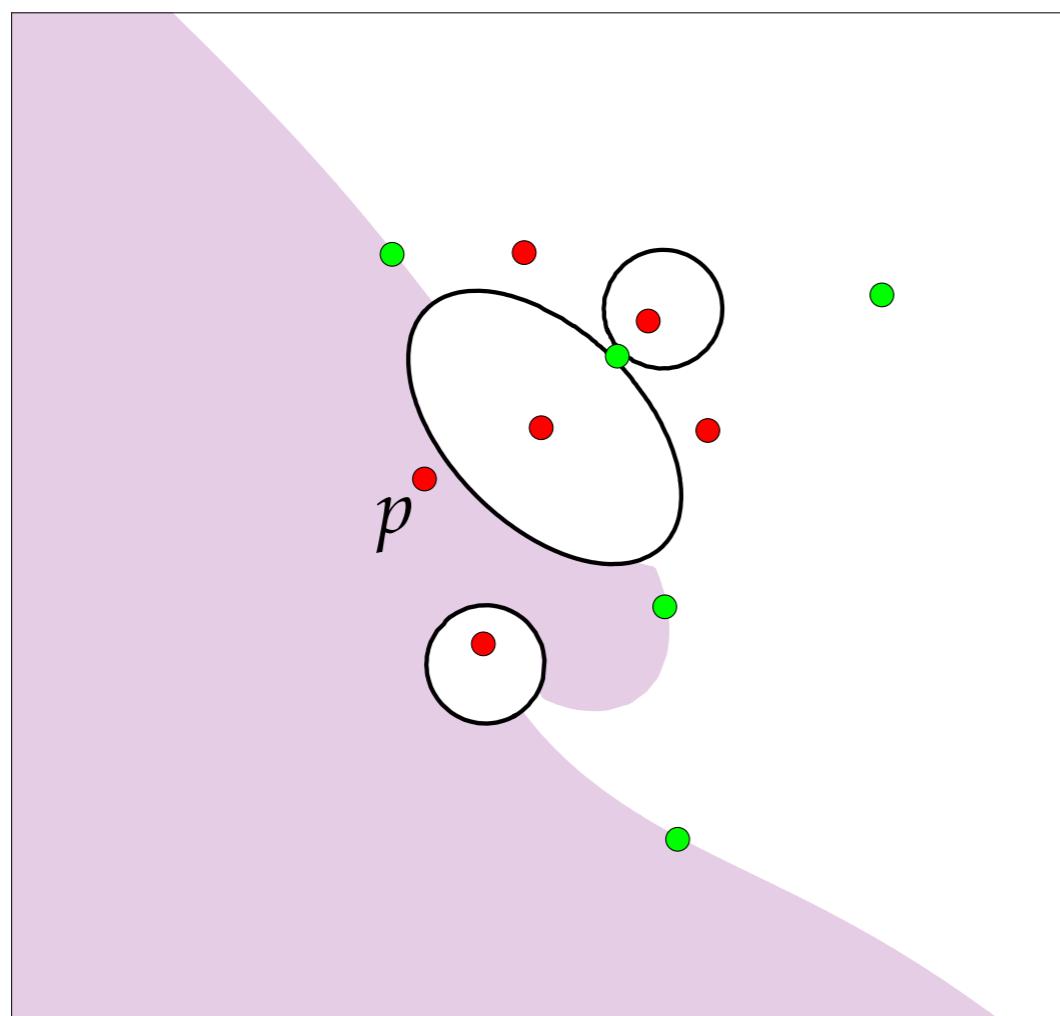
# 1. Correctness: Preliminaries



# 1. Correctness: Preliminaries

**stable manifold for  $p$**

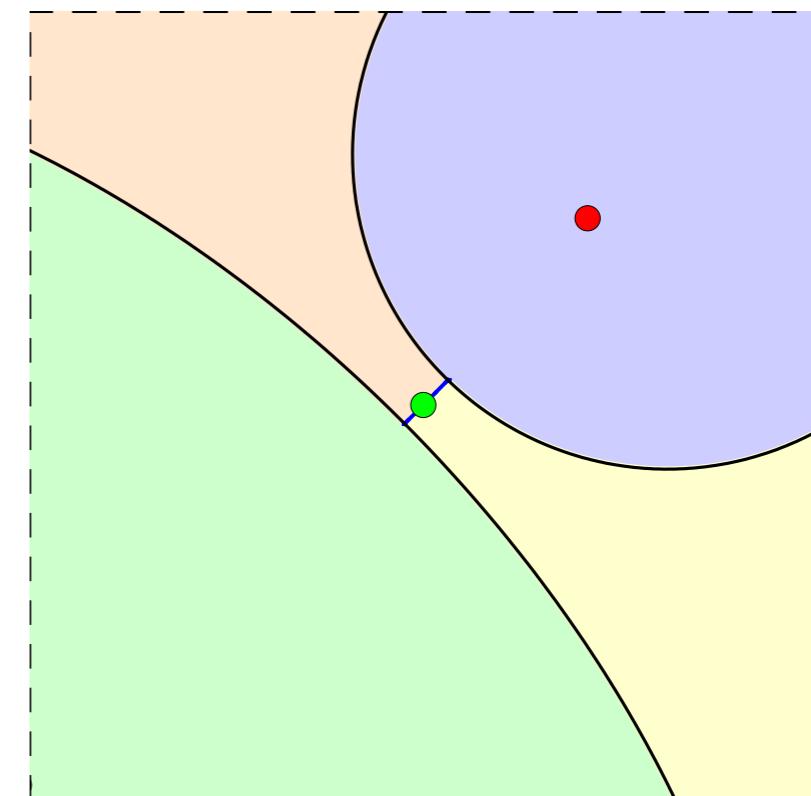
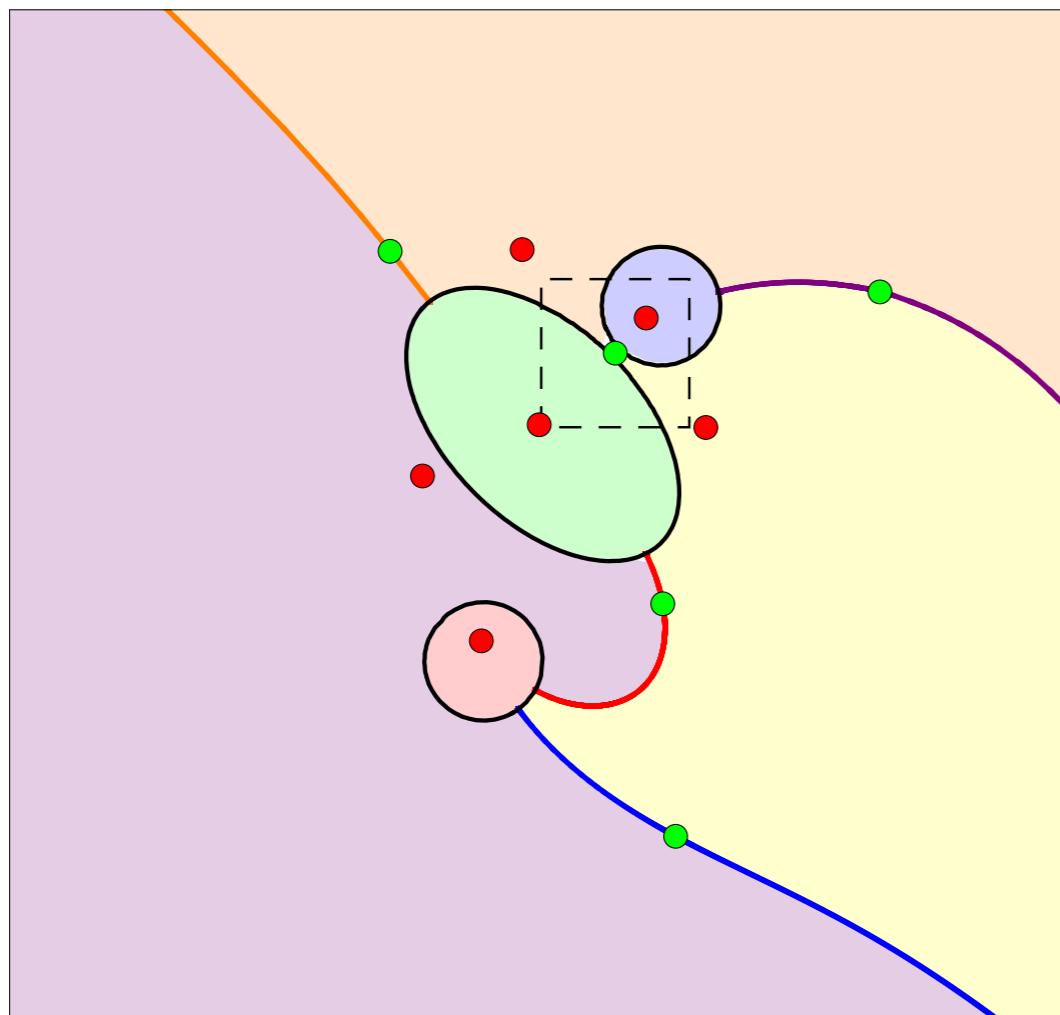
$W^s(p) = \{x \in \mathbb{R}^n \mid \text{dest}(\phi_x) = p\}$  where  $\phi_x$  is the  $\nabla g$  trajectory through  $x$



# 1. Correctness: Preliminaries

**stable manifold for  $p$**

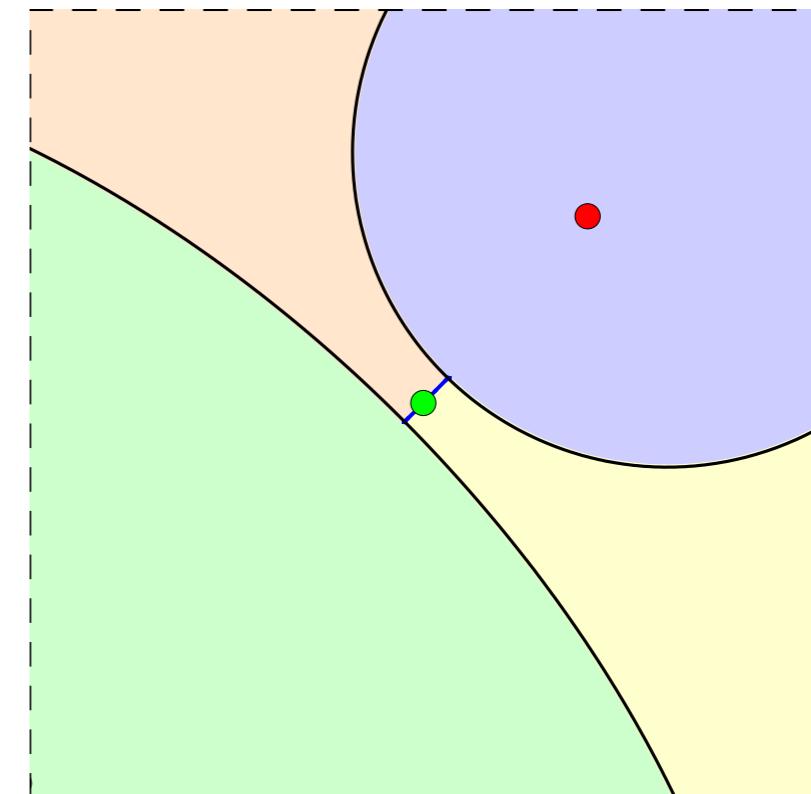
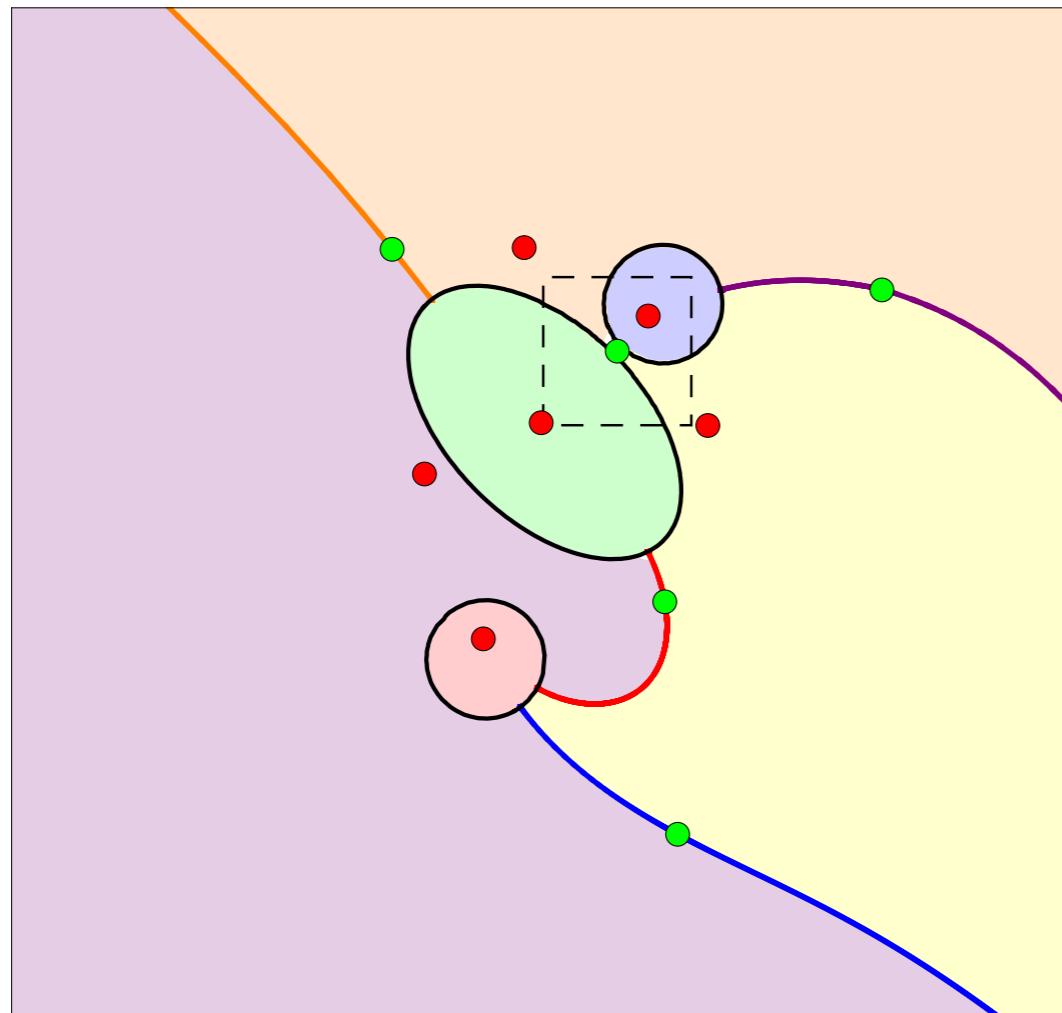
$W^s(p) = \{x \in \mathbb{R}^n \mid \text{dest}(\phi_x) = p\}$  where  $\phi_x$  is the  $\nabla g$  trajectory through  $x$



# 1. Correctness: Preliminaries

stable manifold for  $p$

$W^s(p) = \{x \in \mathbb{R}^n \mid \text{dest}(\phi_x) = p\}$  where  $\phi_x$  is the  $\nabla g$  trajectory through  $x$

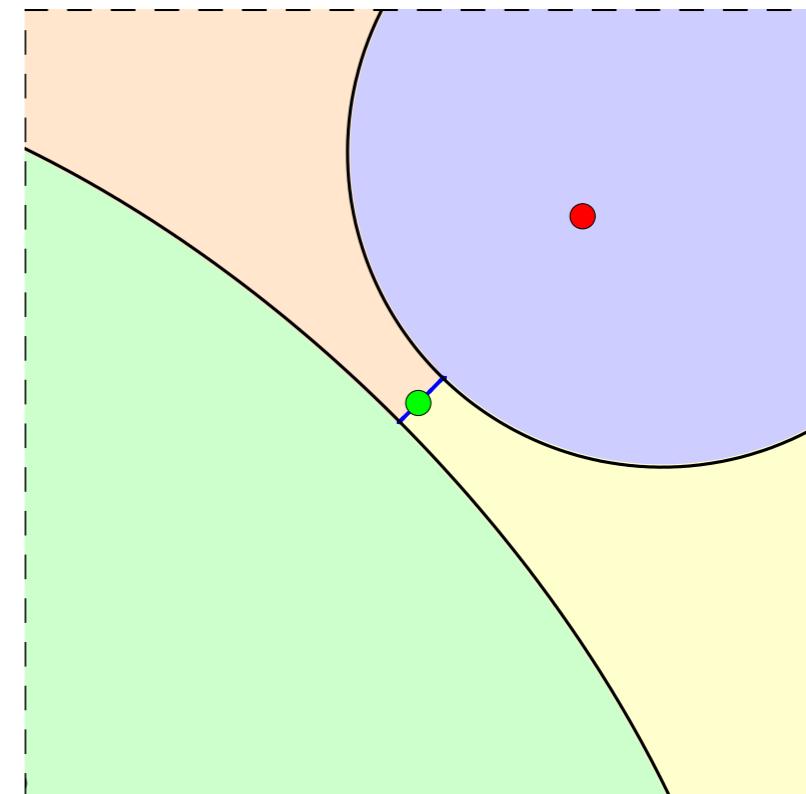
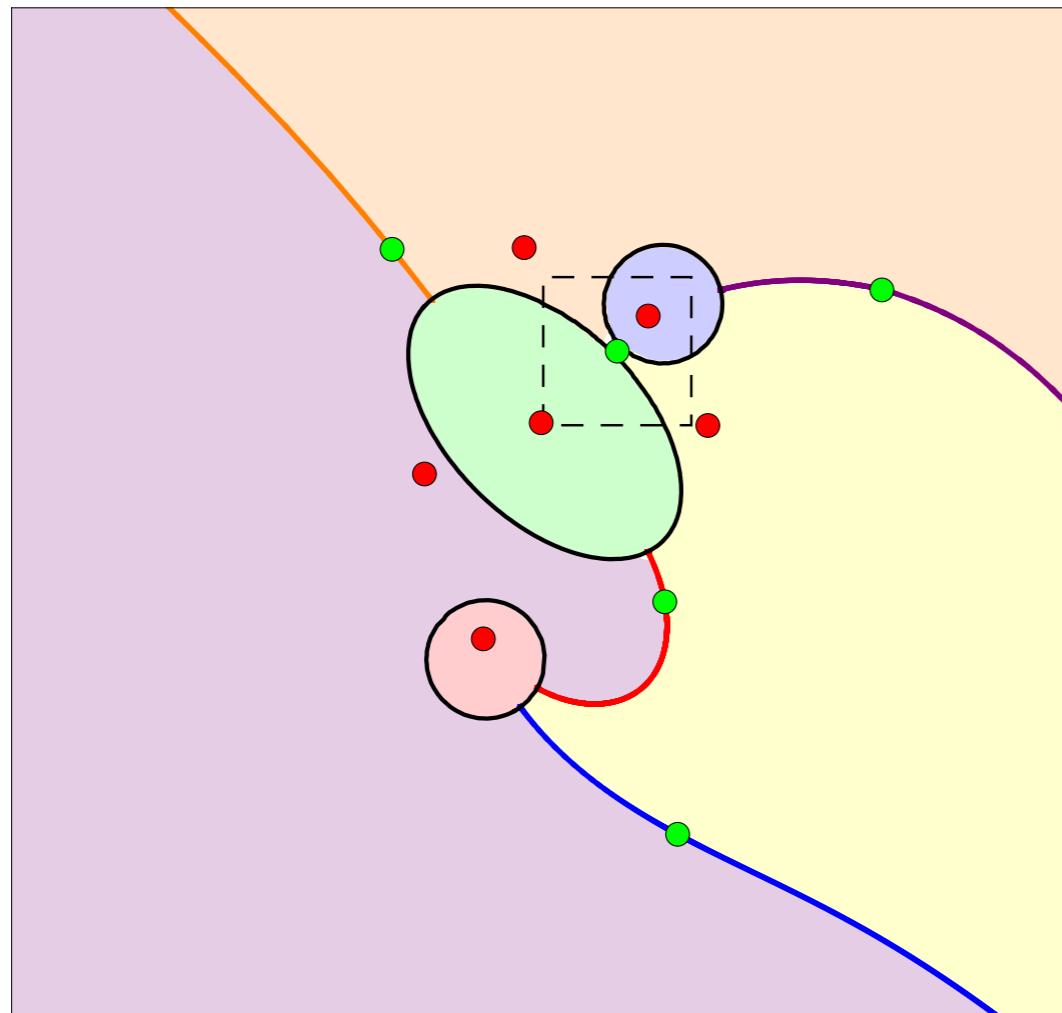


$\forall x, \text{dest}(\phi_x)$  exists and  
is a routing point of  $g$

# 1. Correctness: Preliminaries

**stable manifold for  $p$**

$W^s(p) = \{x \in \mathbb{R}^n \mid \text{dest}(\phi_x) = p\}$  where  $\phi_x$  is the  $\nabla g$  trajectory through  $x$



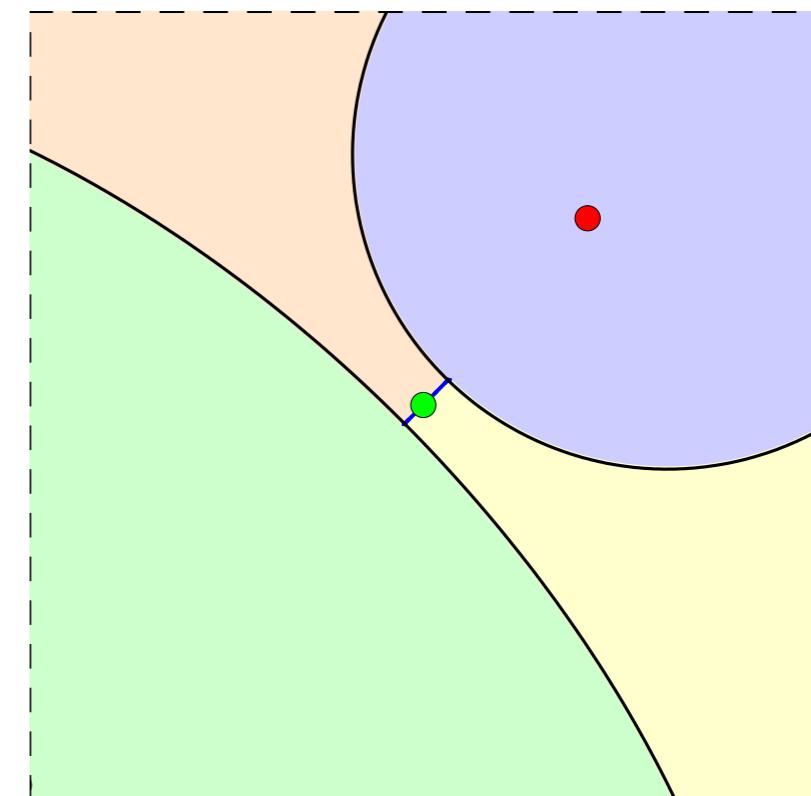
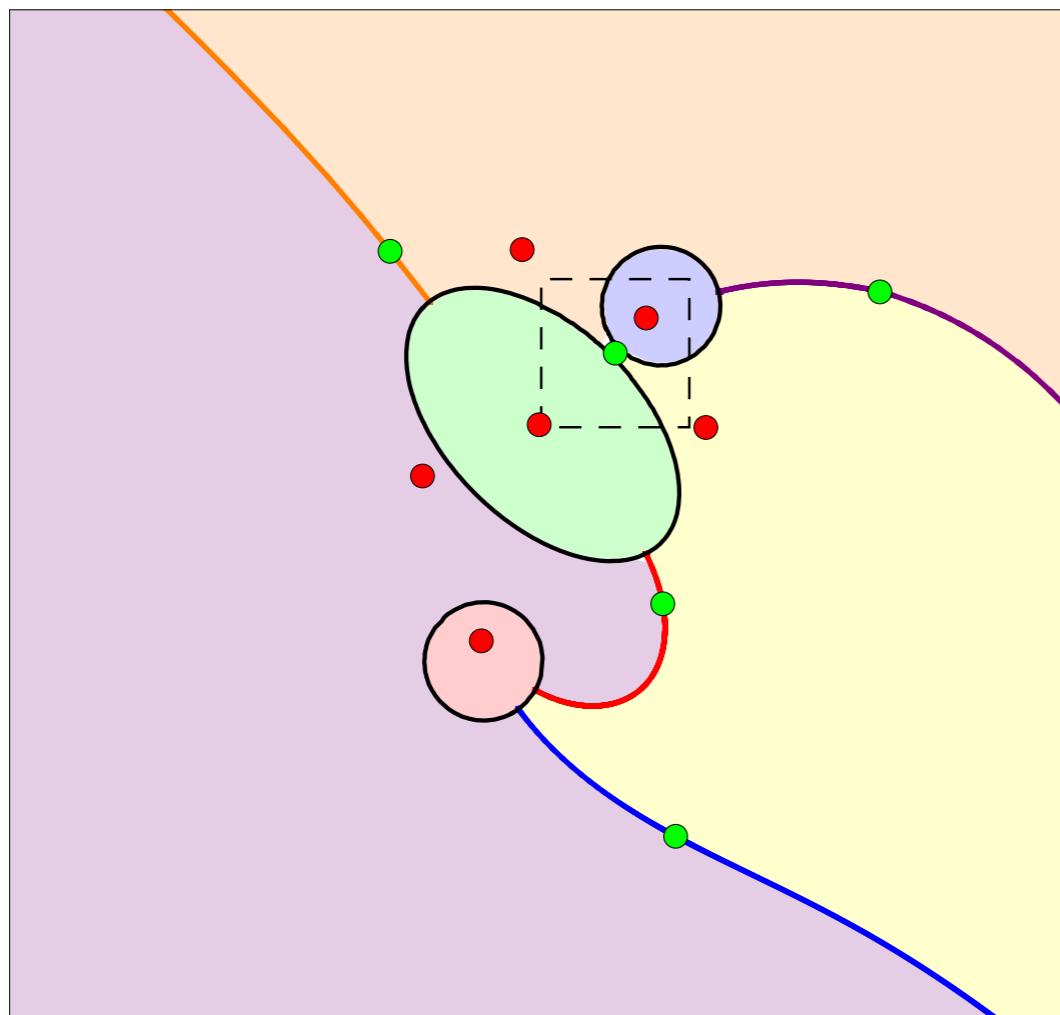
$\forall x, \text{dest}(\phi_x)$  exists and  
is a routing point of  $g$

$\implies$  stable manifolds form a disjoint partition  
of each connected component of  $\{g \neq 0\}$

# 1. Correctness: Preliminaries

**stable manifold for  $p$**

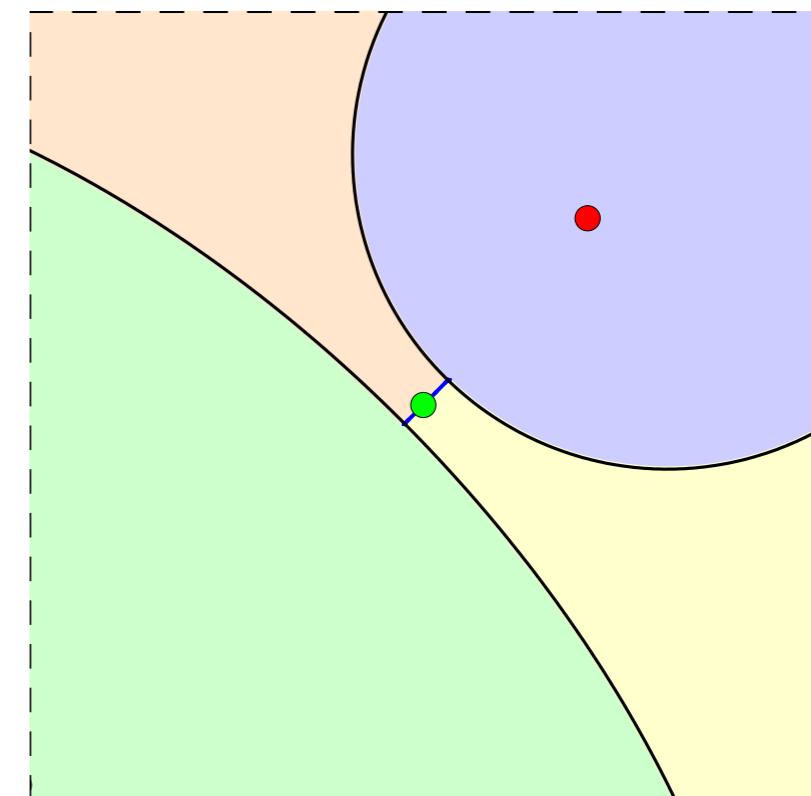
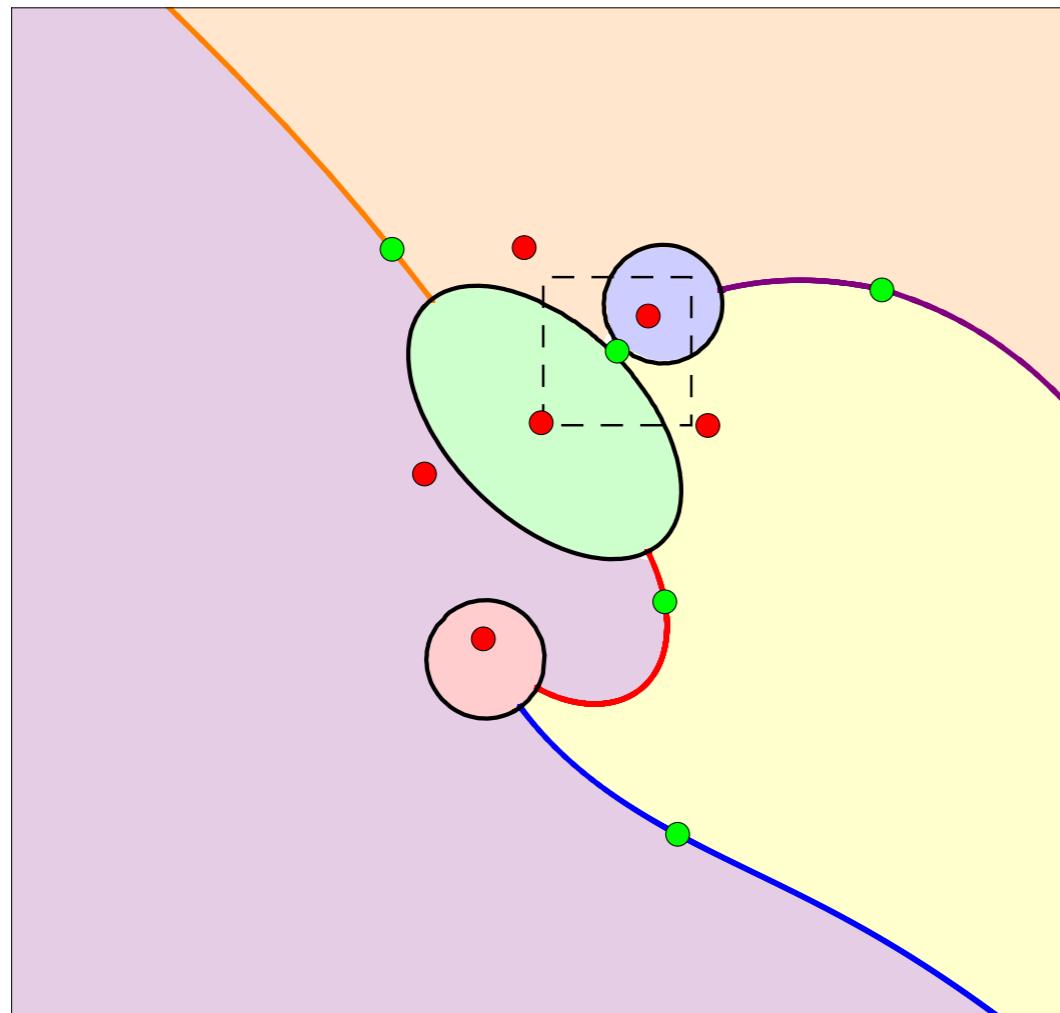
$W^s(p) = \{x \in \mathbb{R}^n \mid \text{dest}(\phi_x) = p\}$  where  $\phi_x$  is the  $\nabla g$  trajectory through  $x$



# 1. Correctness: Preliminaries

**stable manifold for  $p$**

$W^s(p) = \{x \in \mathbb{R}^n \mid \text{dest}(\phi_x) = p\}$  where  $\phi_x$  is the  $\nabla g$  trajectory through  $x$



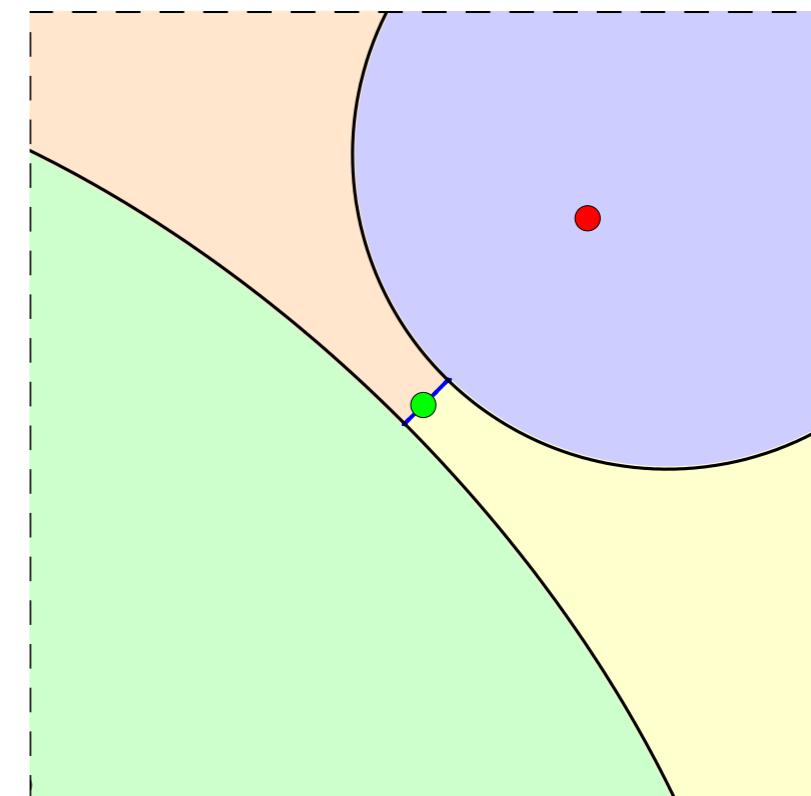
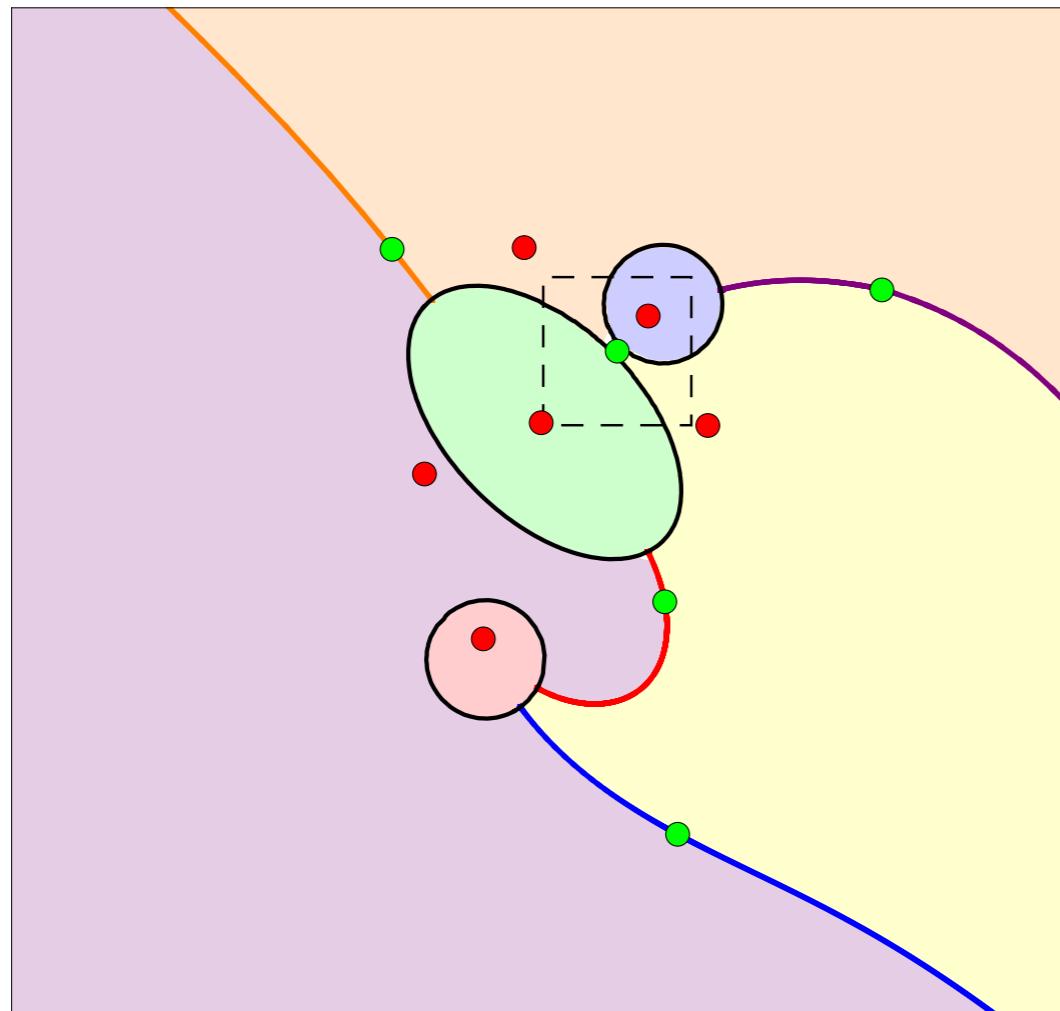
Stable Manifold Theorem  $\implies$

$\dim W^s(p) = \# \text{ of negative eigenvectors}$   
 $\text{of } (\text{Hess } g)(p)$

# 1. Correctness: Preliminaries

stable manifold for  $p$

$W^s(p) = \{x \in \mathbb{R}^n \mid \text{dest}(\phi_x) = p\}$  where  $\phi_x$  is the  $\nabla g$  trajectory through  $x$



Stable Manifold Theorem  $\implies$

$\dim W^s(p) = \# \text{ of negative eigenvectors}$   
 $\text{of } (\text{Hess } g)(p)$   
 $= \text{index}(p)$

# 1. Correctness: Proof Sketch

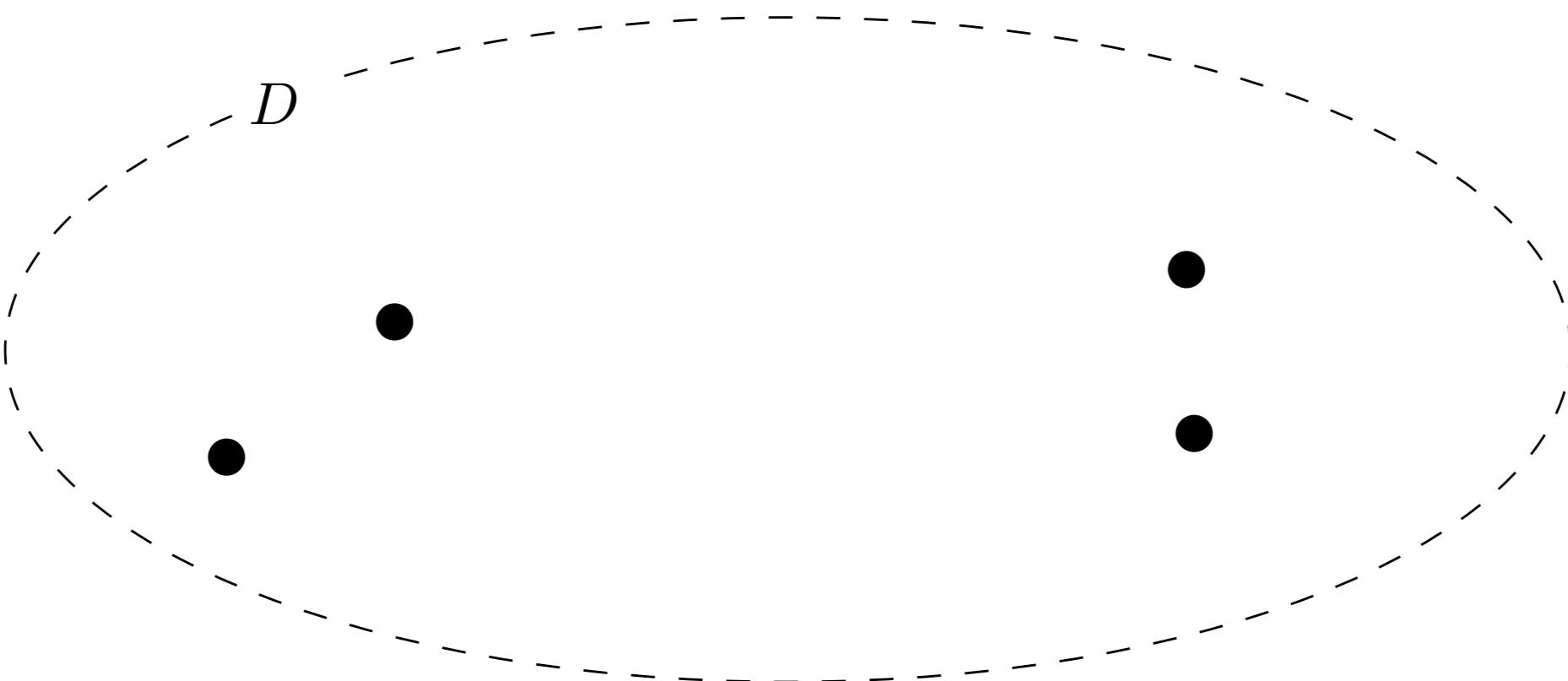
Let  $g$  be a routing function.

Any two routing points in the same connected component of  $\{g \neq 0\}$  are connected by steepest ascent paths using positive eigenvectors.

# 1. Correctness: Proof Sketch

Let  $g$  be a routing function.

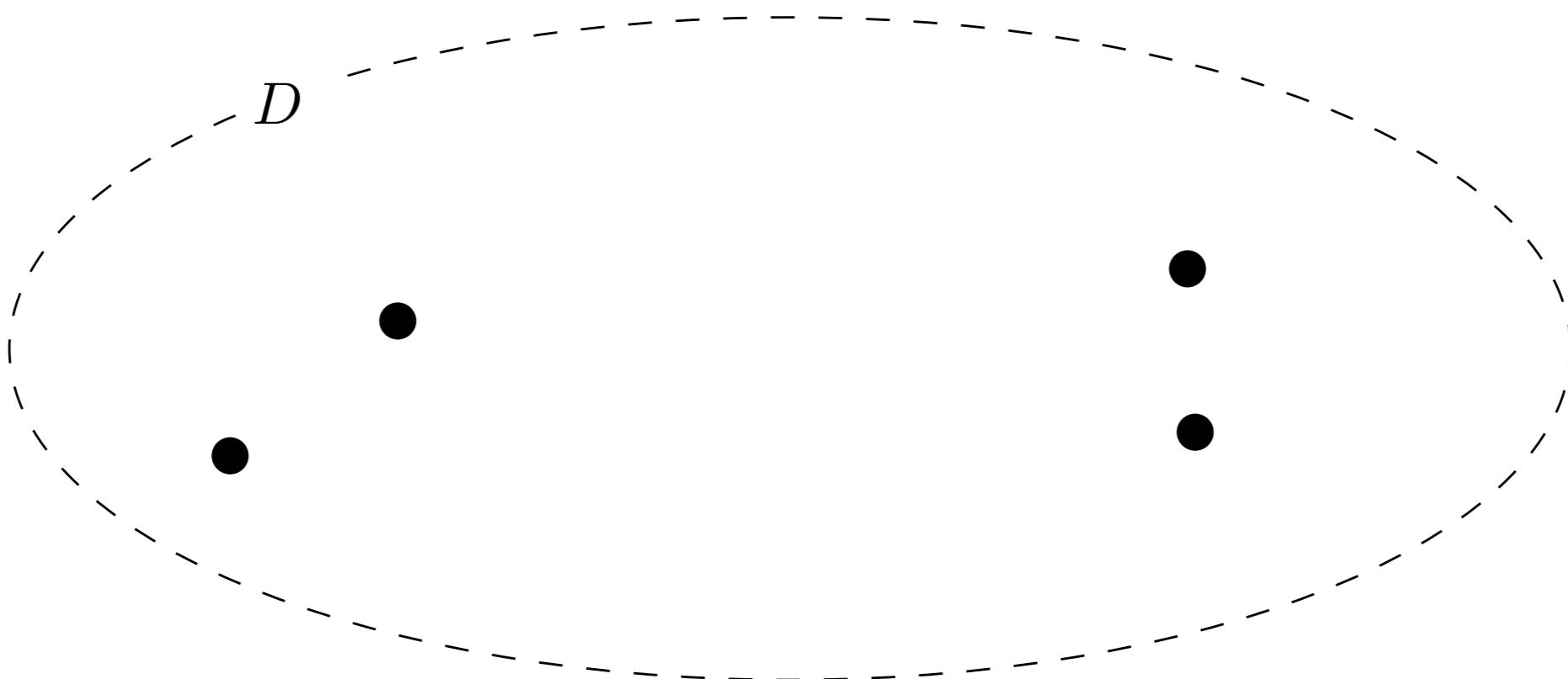
Any two routing points in the same connected component of  $\{g \neq 0\}$  are connected by steepest ascent paths using positive eigenvectors.



# 1. Correctness: Proof Sketch

Let  $g$  be a routing function.

Any two routing points in the same connected component of  $\{g \neq 0\}$  are connected by steepest ascent paths using positive eigenvectors.



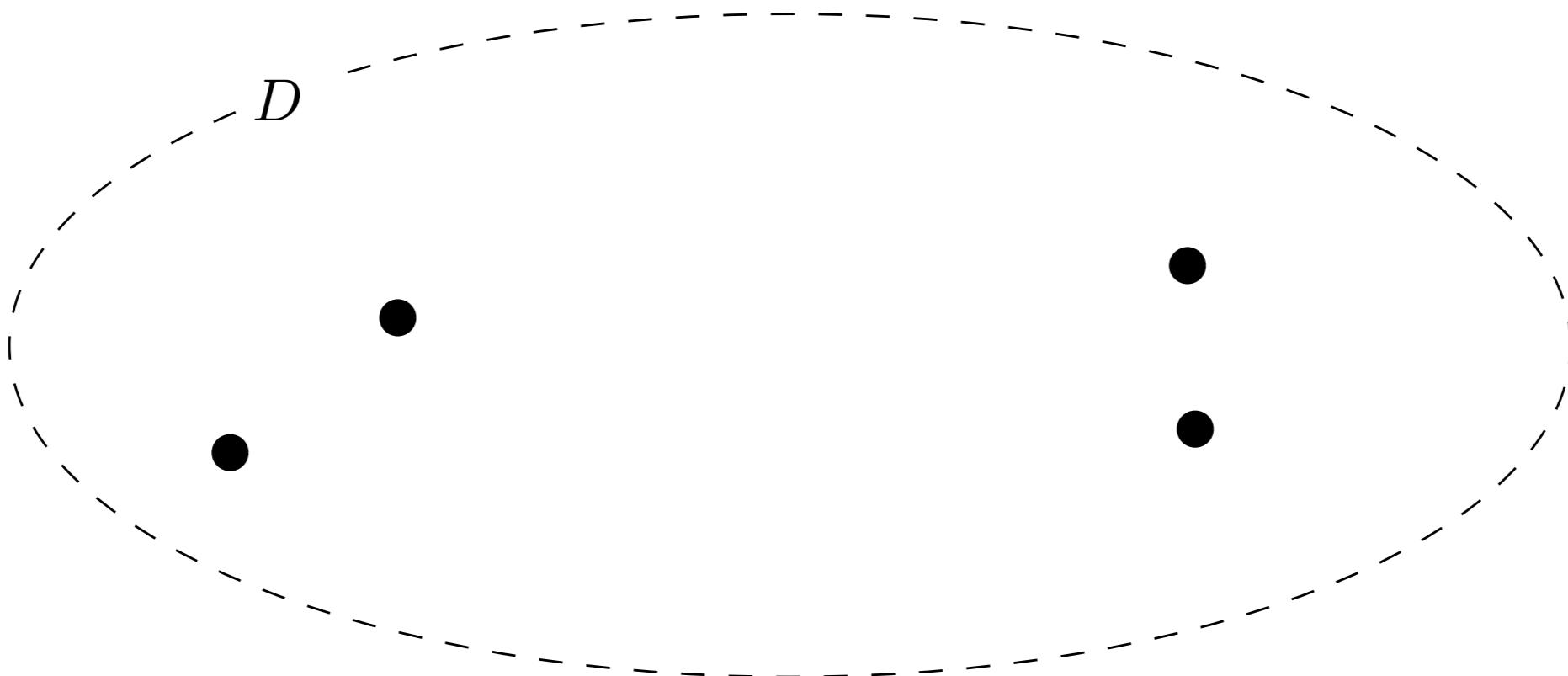
$n = 2$

# 1. Correctness: Proof Sketch

Let  $g$  be a routing function.

Any two routing points in the same connected component of  $\{g \neq 0\}$  are connected by steepest ascent paths using positive eigenvectors.

**Suppose the claim is false.**

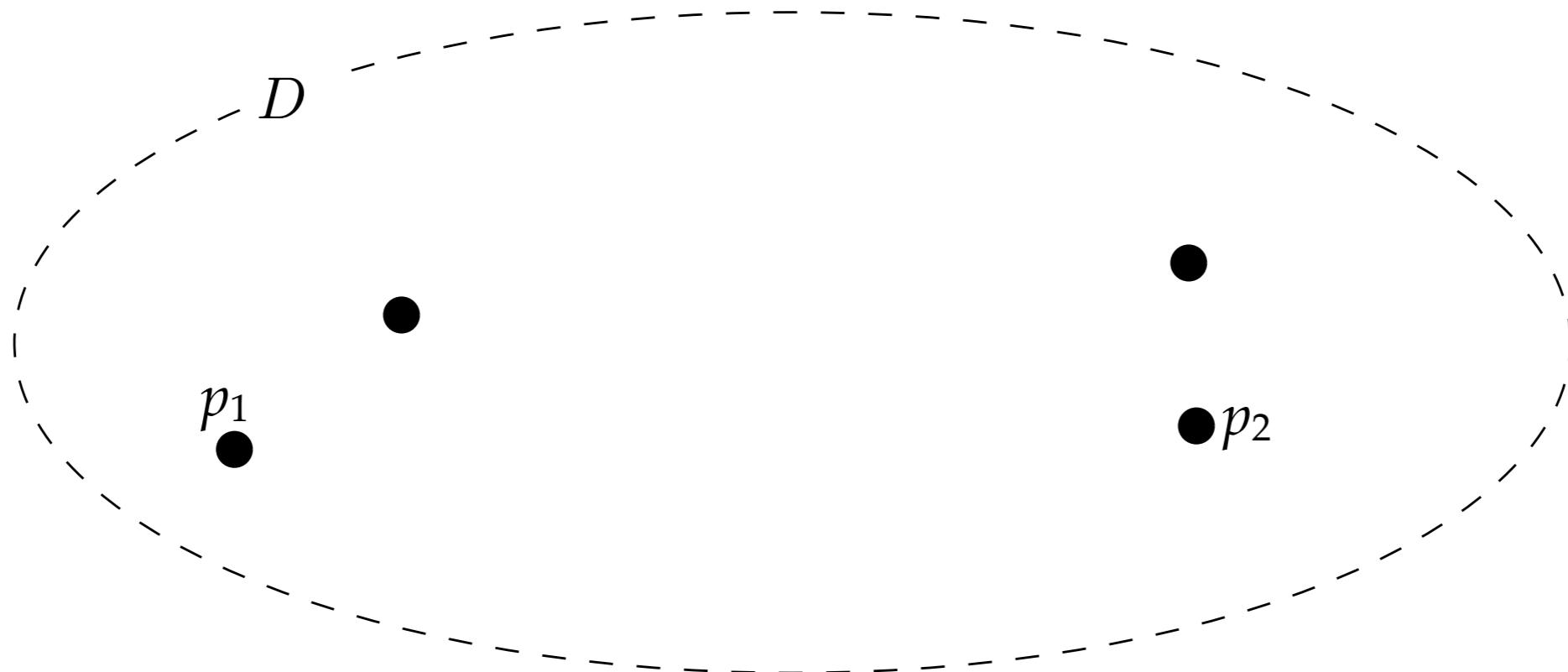


# 1. Correctness: Proof Sketch

Let  $g$  be a routing function.

Any two routing points in the same connected component of  $\{g \neq 0\}$  are connected by steepest ascent paths using positive eigenvectors.

**$p_1$  is not connected to  $p_2$  by steepest ascent paths using positive eigenvectors.**

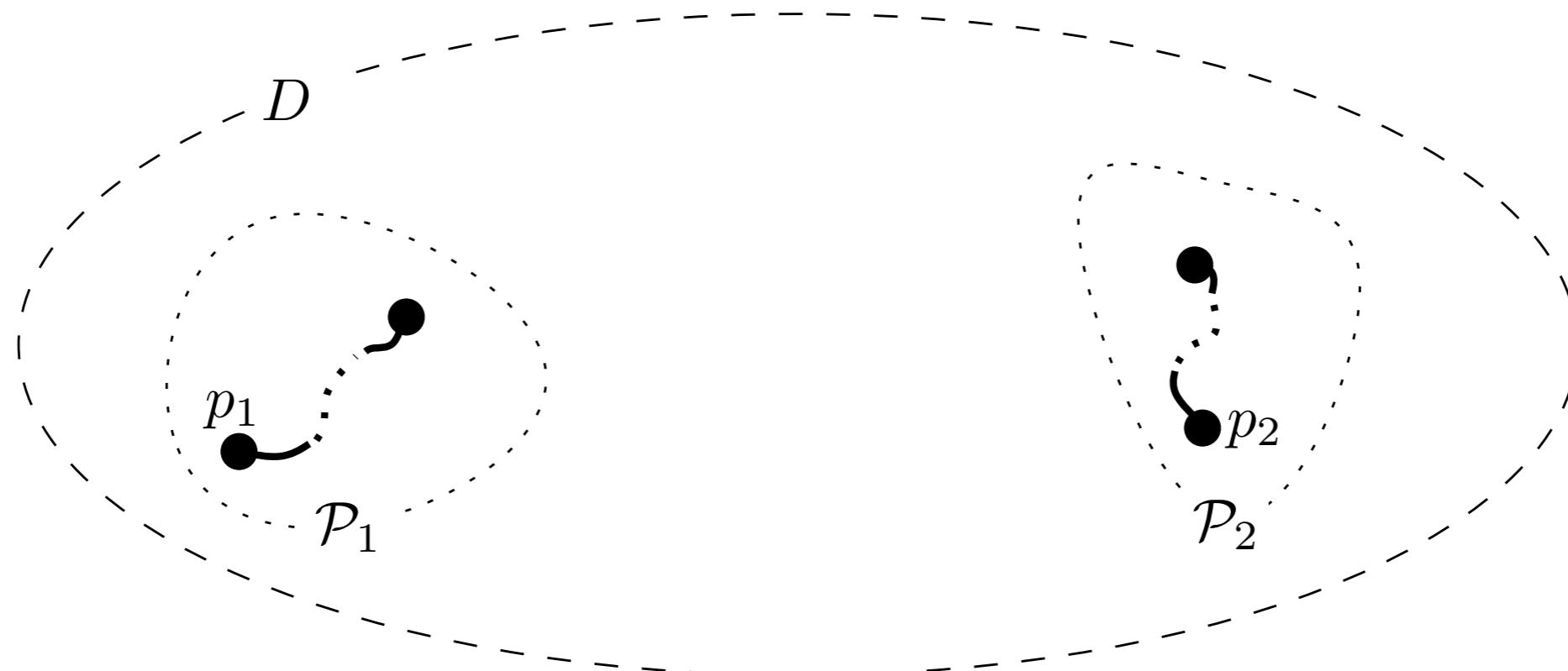


# 1. Correctness: Proof Sketch

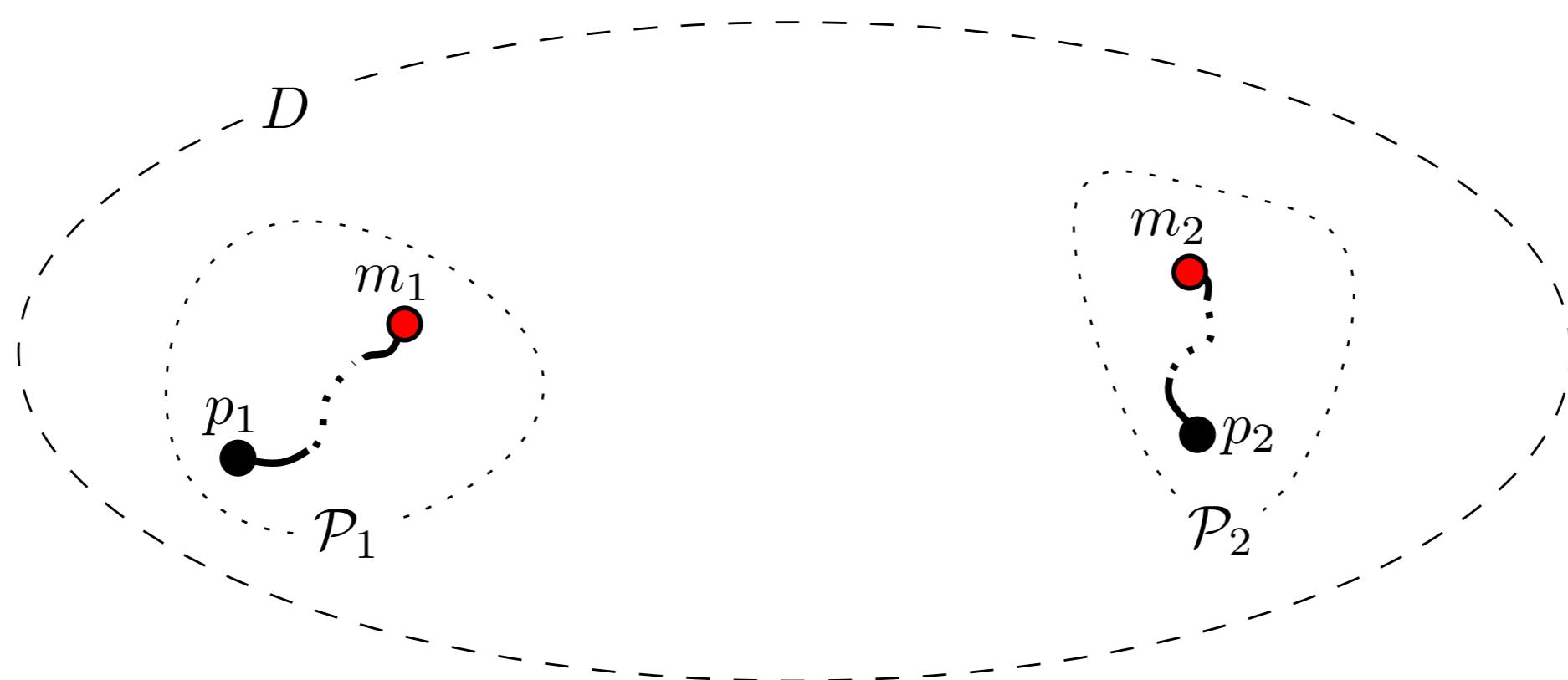
Let  $g$  be a routing function.

Any two routing points in the same connected component of  $\{g \neq 0\}$  are connected by steepest ascent paths using positive eigenvectors.

**$p_1$  is not connected to  $p_2$  by steepest ascent paths using positive eigenvectors.**

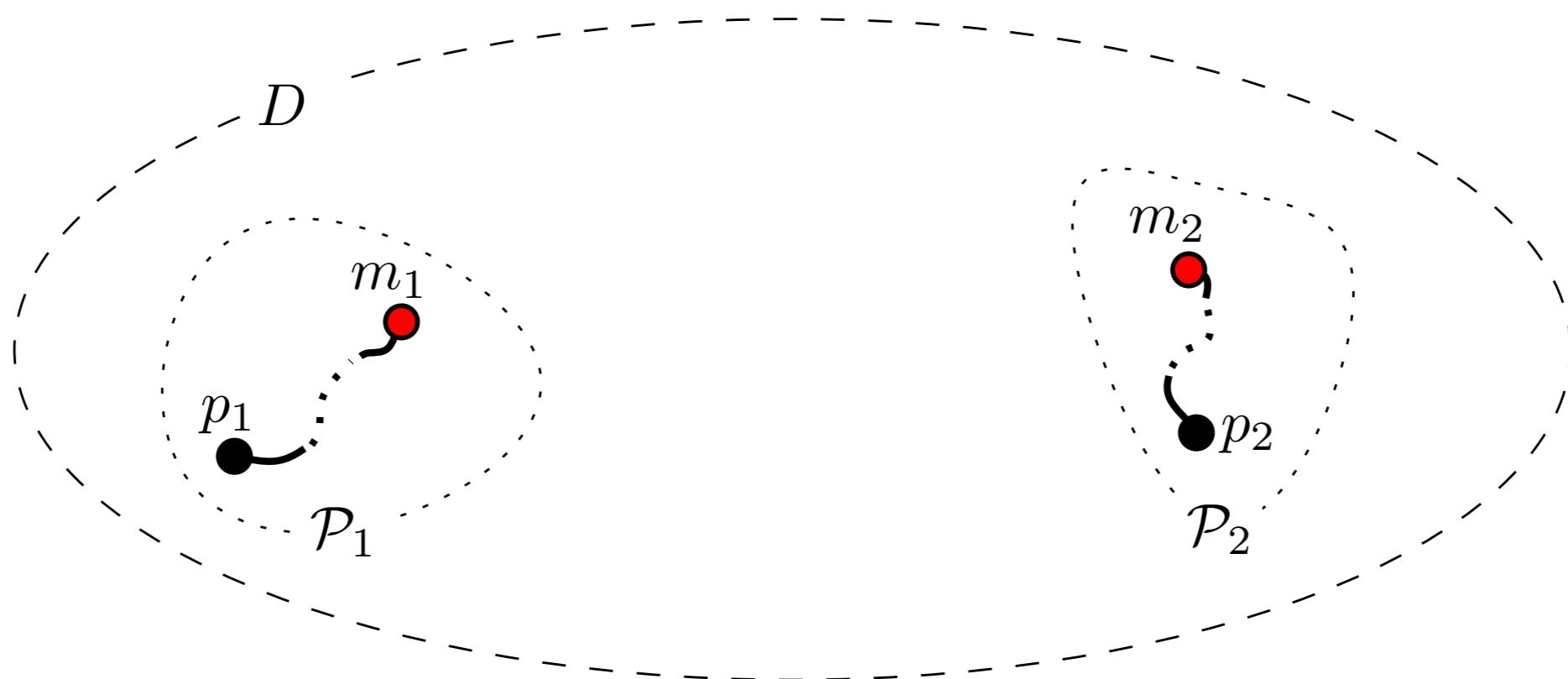


# 1. Correctness: Proof Sketch

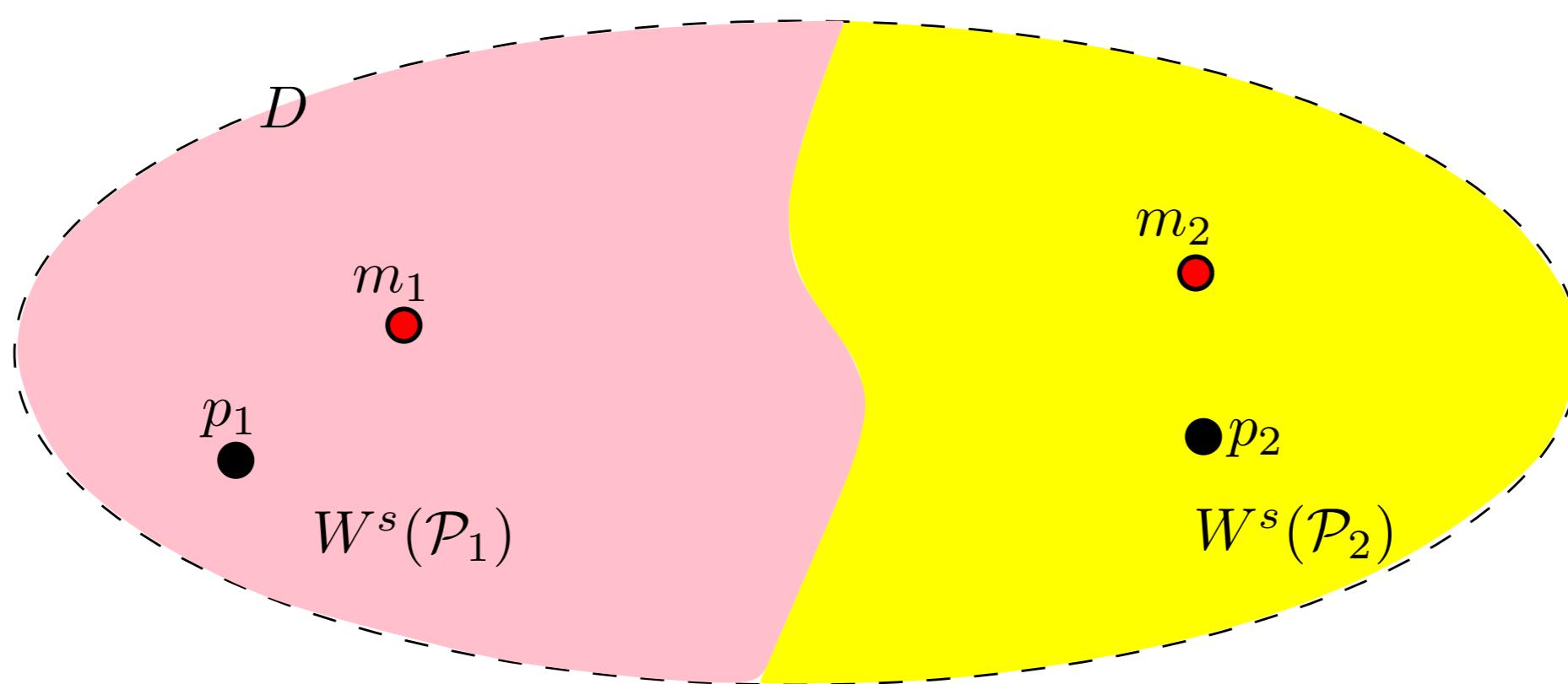


# 1. Correctness: Proof Sketch

Each  $\mathcal{P}_i$  contains at least one routing point of index  $n$ .



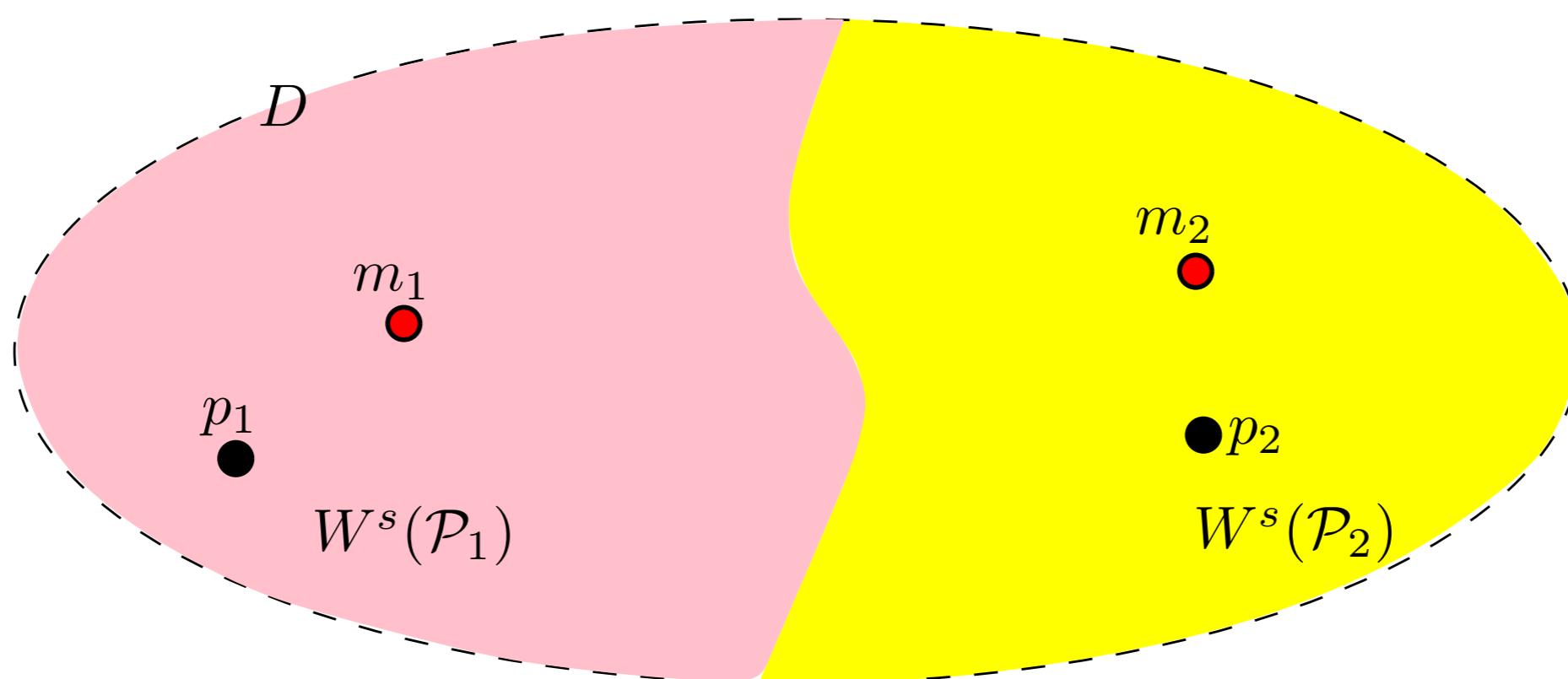
# 1. Correctness: Proof Sketch



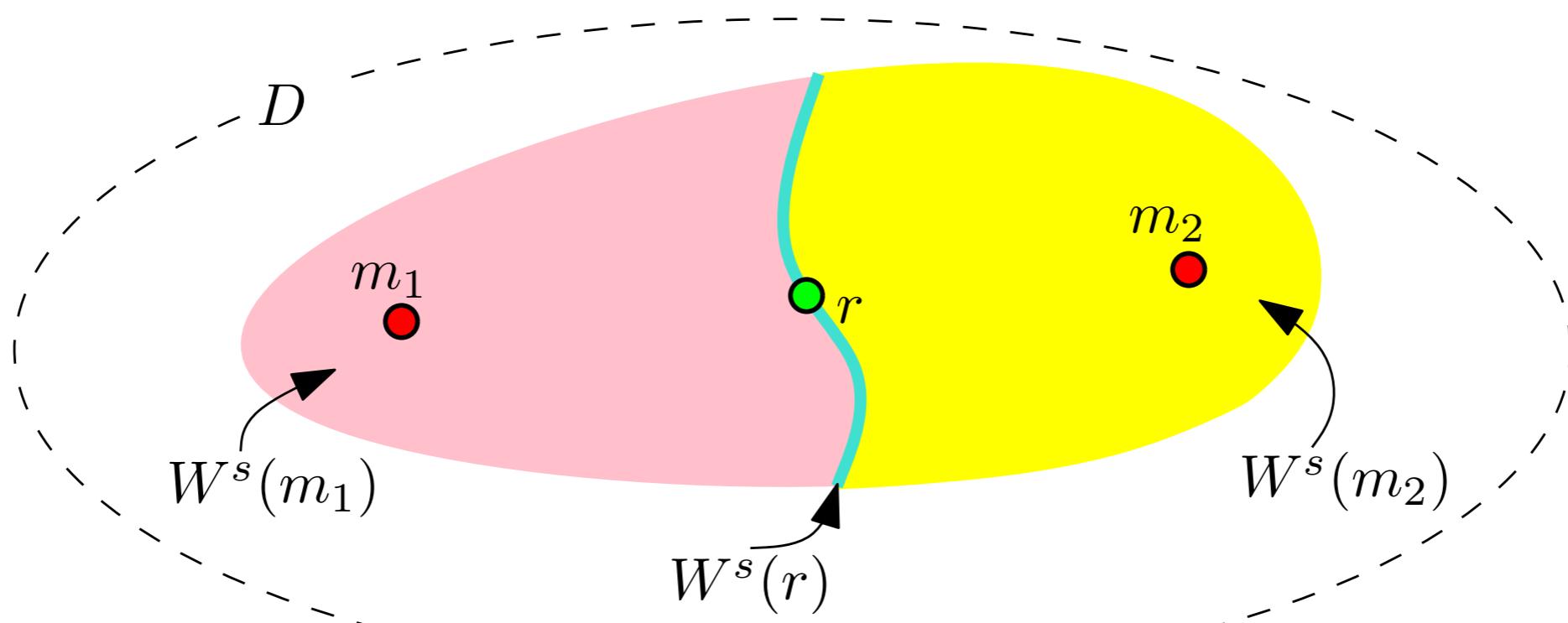
$n = 2$

# 1. Correctness: Proof Sketch

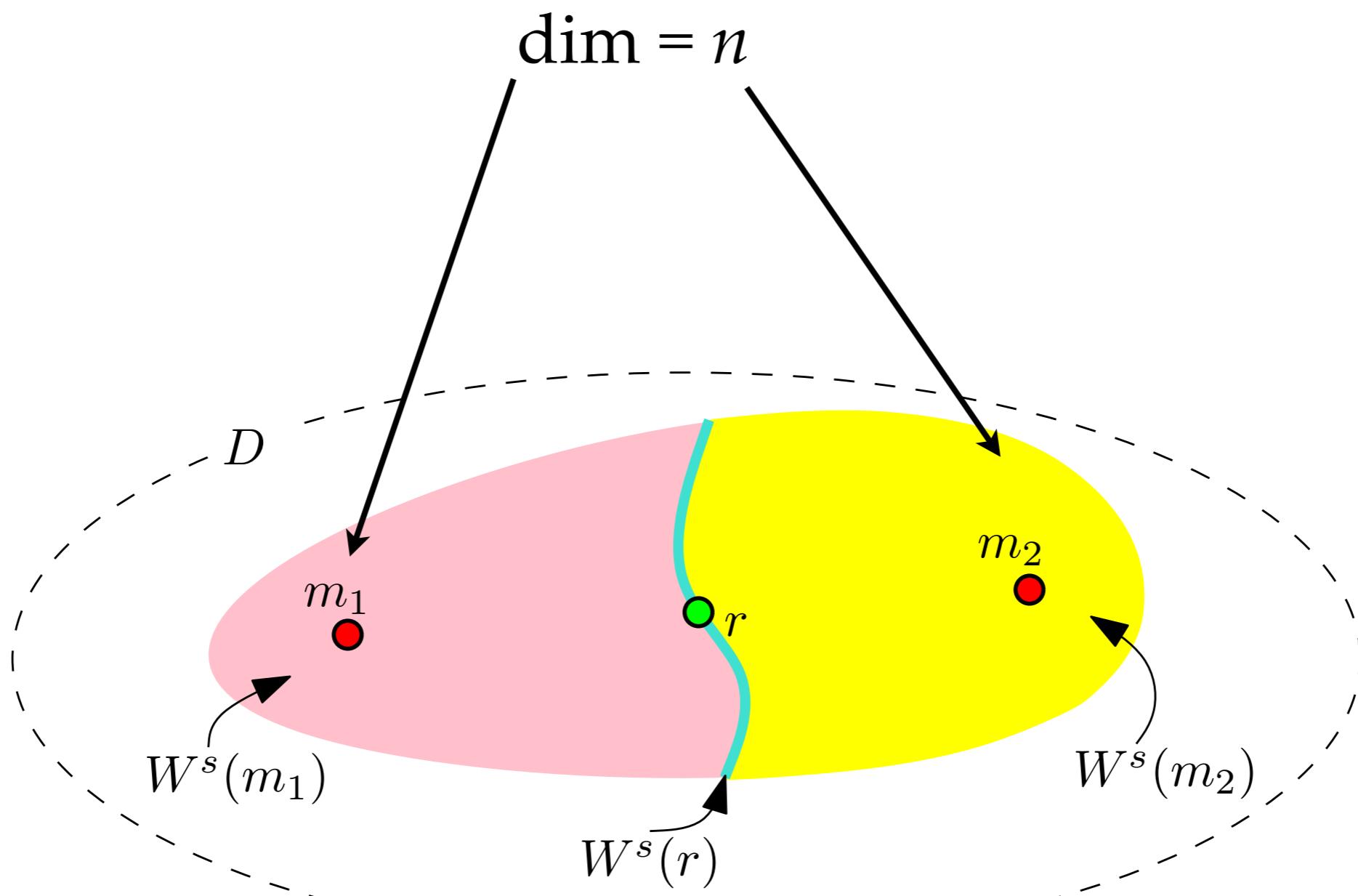
The component  $D$  is a disjoint union of  $W^s(\mathcal{P}_i)$ .



# 1. Correctness: Proof Sketch

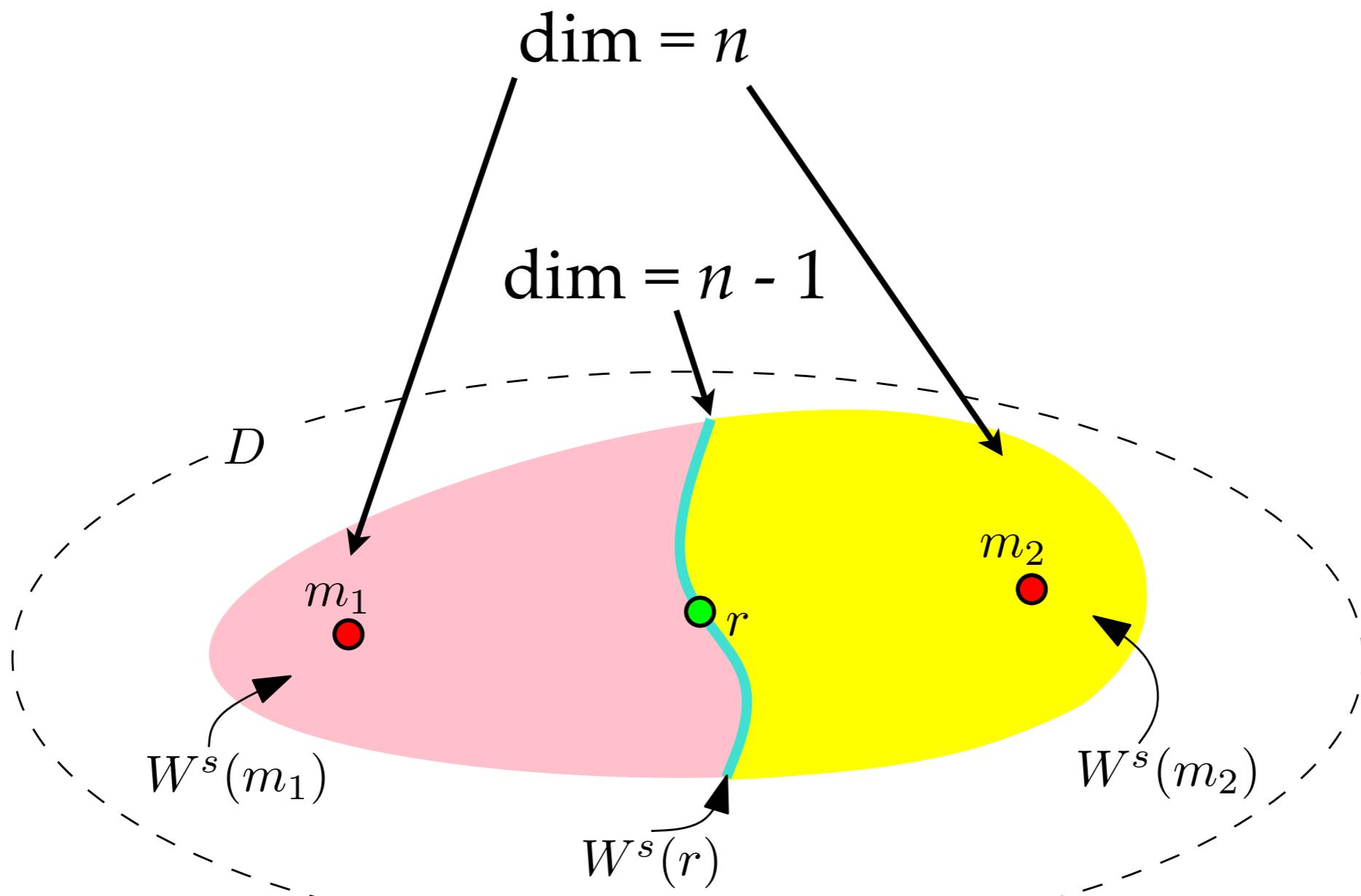


# 1. Correctness: Proof Sketch



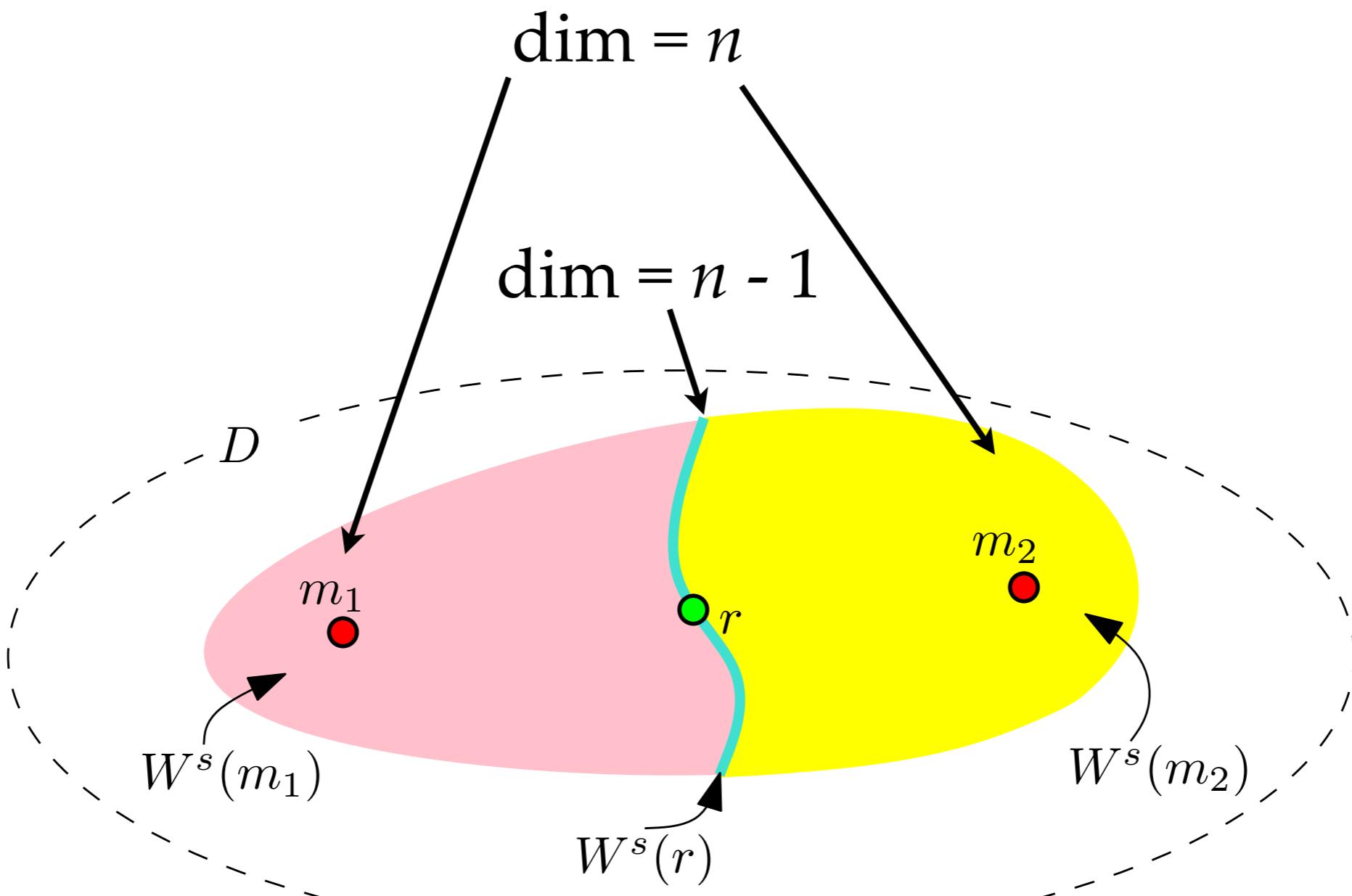
$n = 2$

# 1. Correctness: Proof Sketch

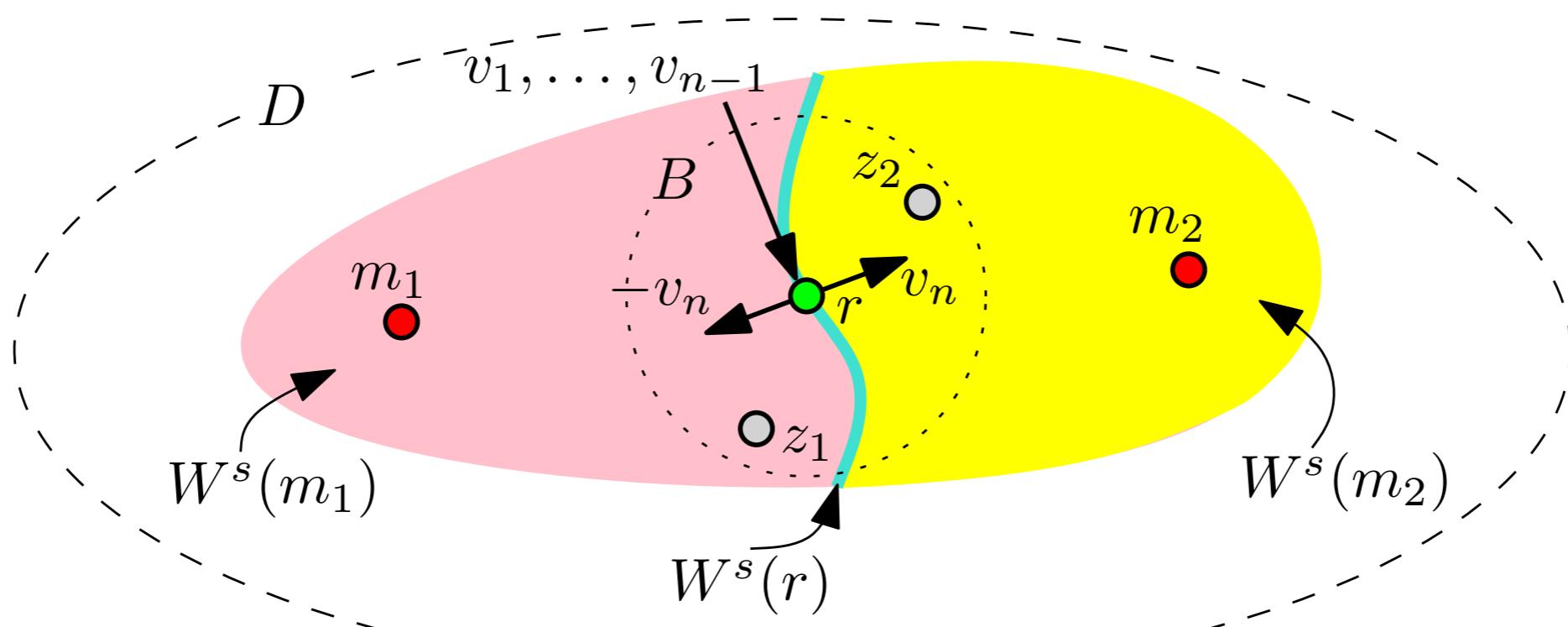


# 1. Correctness: Proof Sketch

There exists a routing point  $r$  with index  $n - 1$  in  $D \cap \partial W^s(m_1) \cap \partial W^s(m_2)$ .



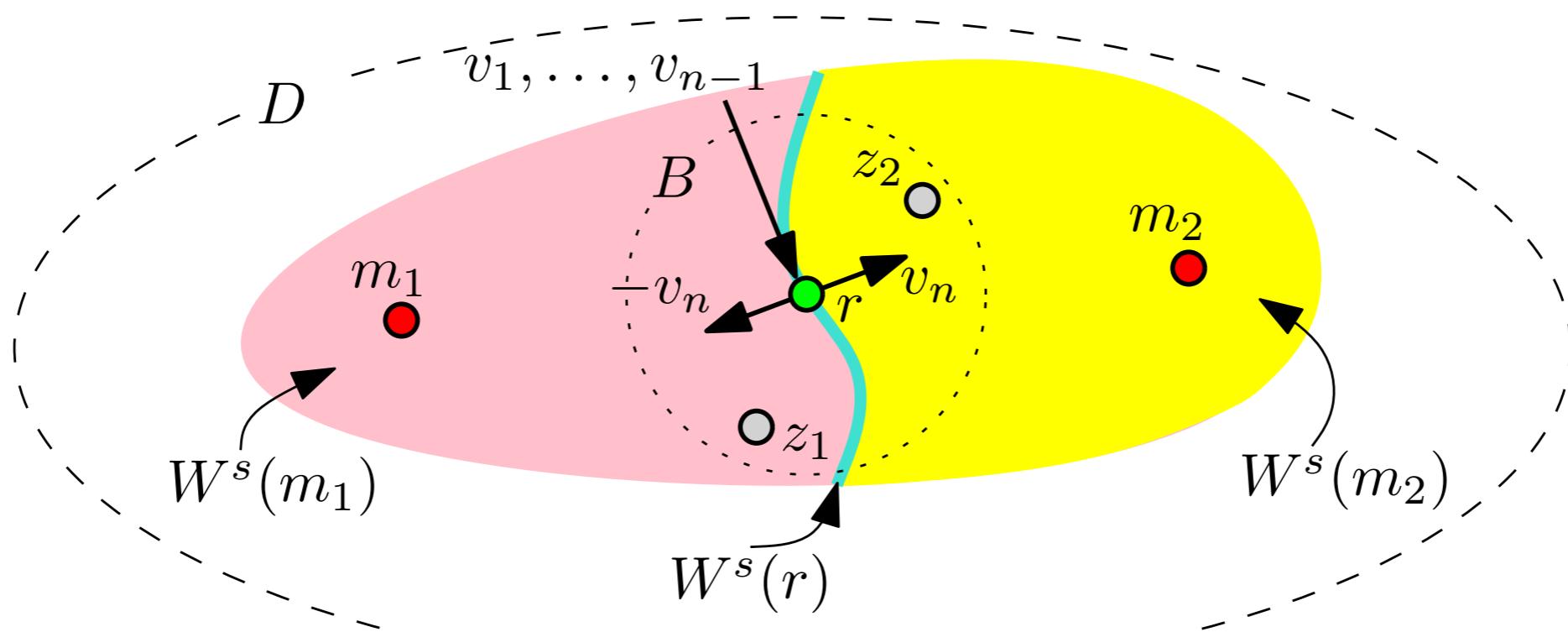
# 1. Correctness: Proof Sketch



$n = 2$

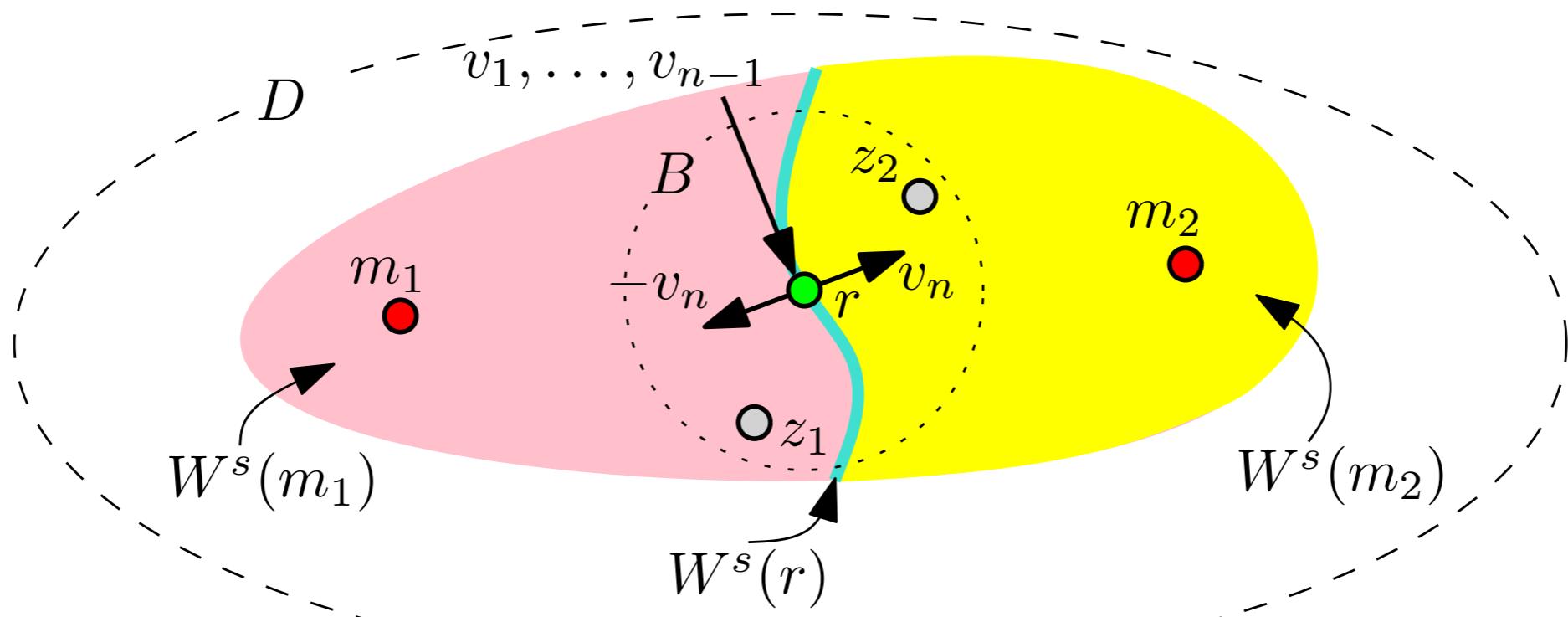
# 1. Correctness: Proof Sketch

There exist points  $z_i$  in  $W^s(m_i)$

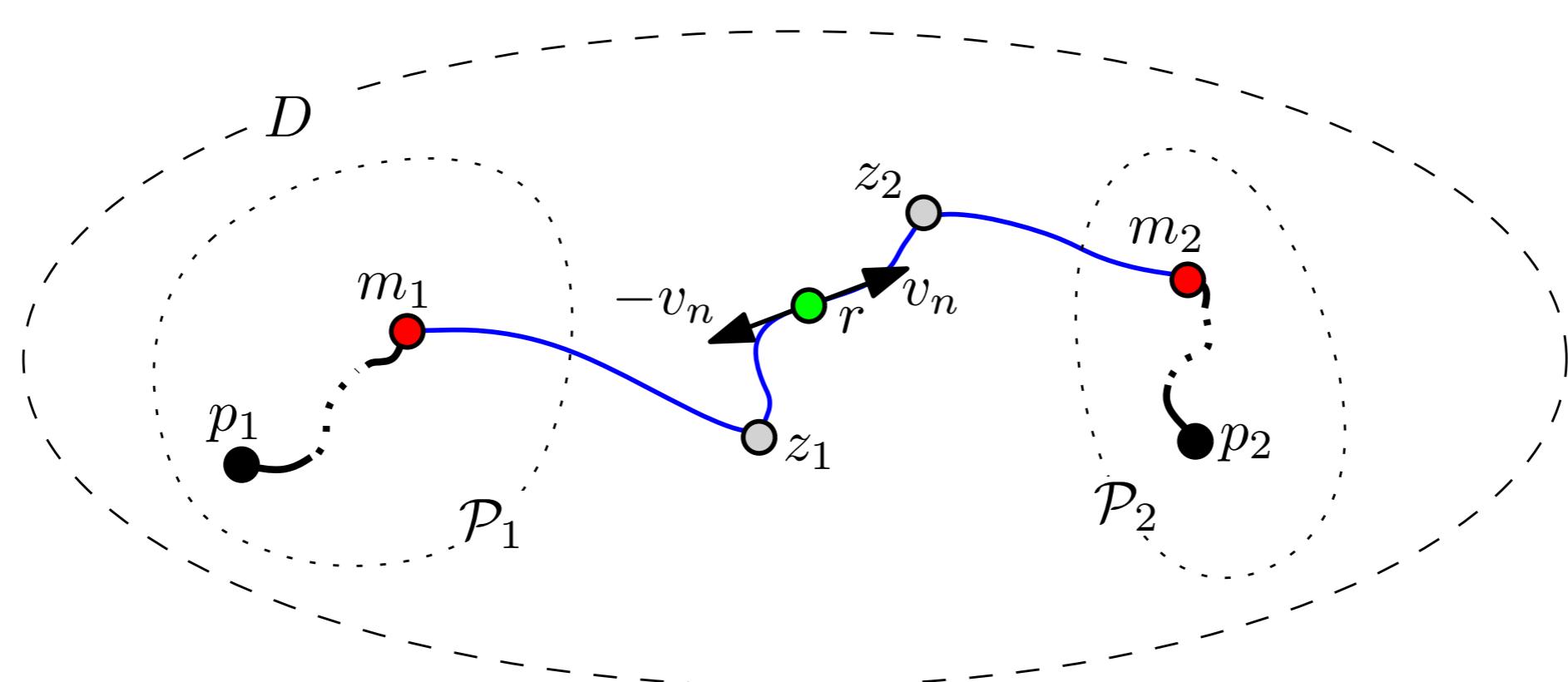


# 1. Correctness: Proof Sketch

There exist points  $z_i$  in  $W^s(m_i)$  such that  $z_1 \in \text{SA}(g, r, -v_n)$   
 $z_2 \in \text{SA}(g, r, v_n)$



# 1. Correctness: Proof Sketch

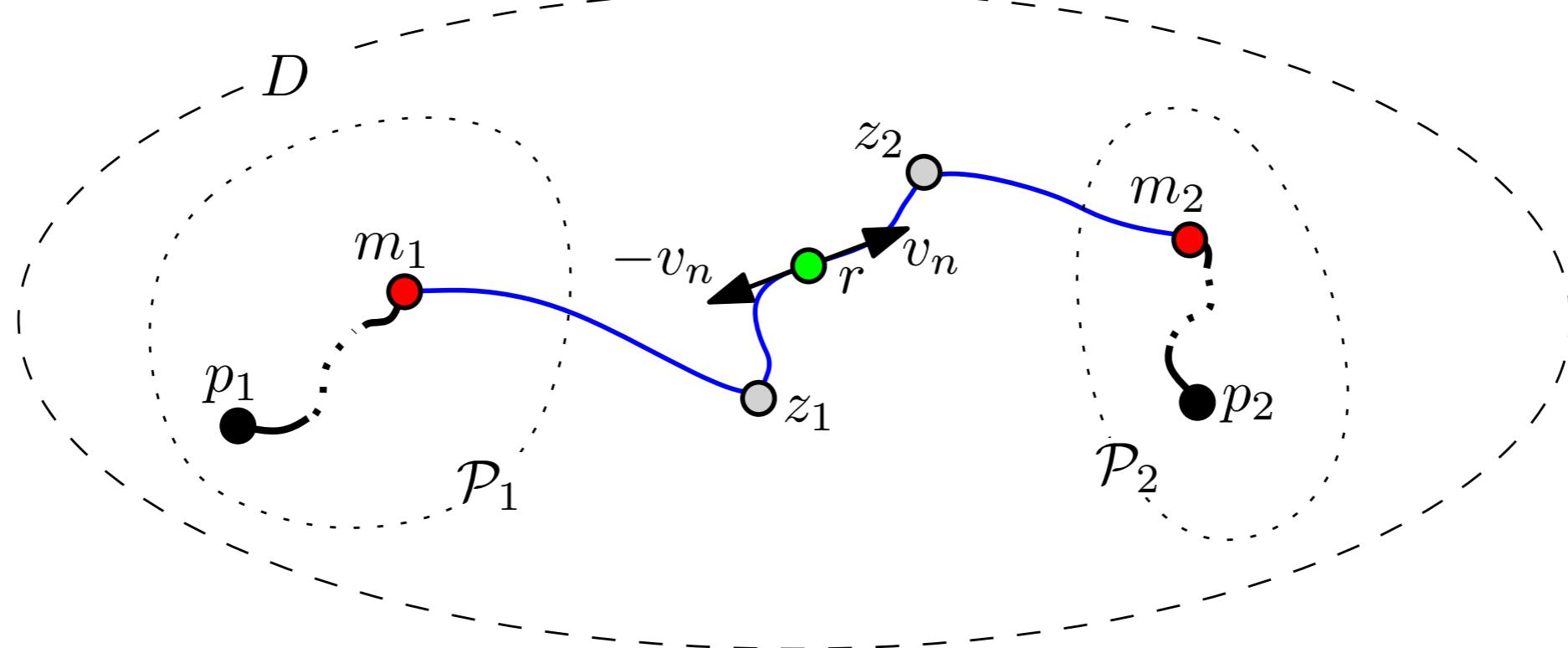


# 1. Correctness: Proof Sketch

$p_1$  **is** connected to  $p_2$

by steepest ascent paths using positive eigenvectors.

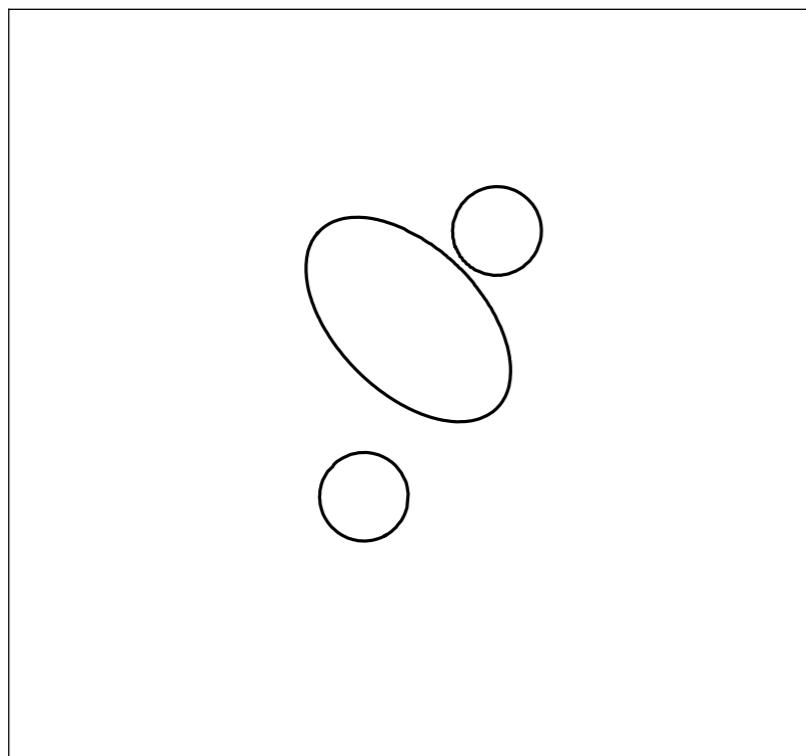
## Contradiction



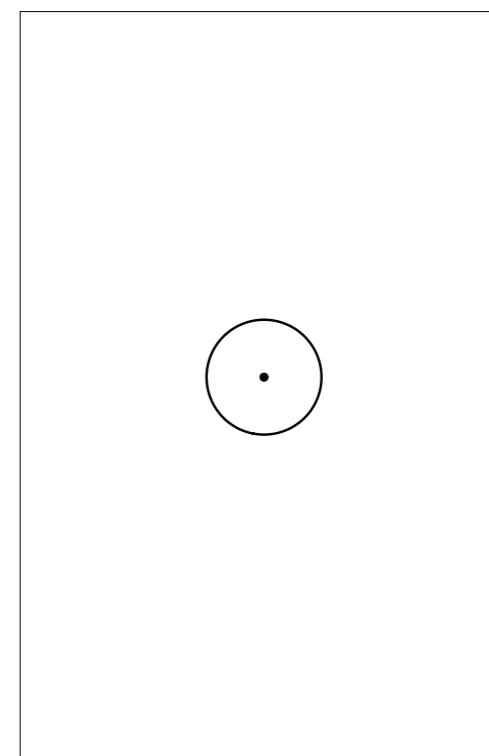
# 1. Correctness: What if $g$ is not a routing function?

# 1. Correctness: What if $g$ is not a routing function?

$f_1$

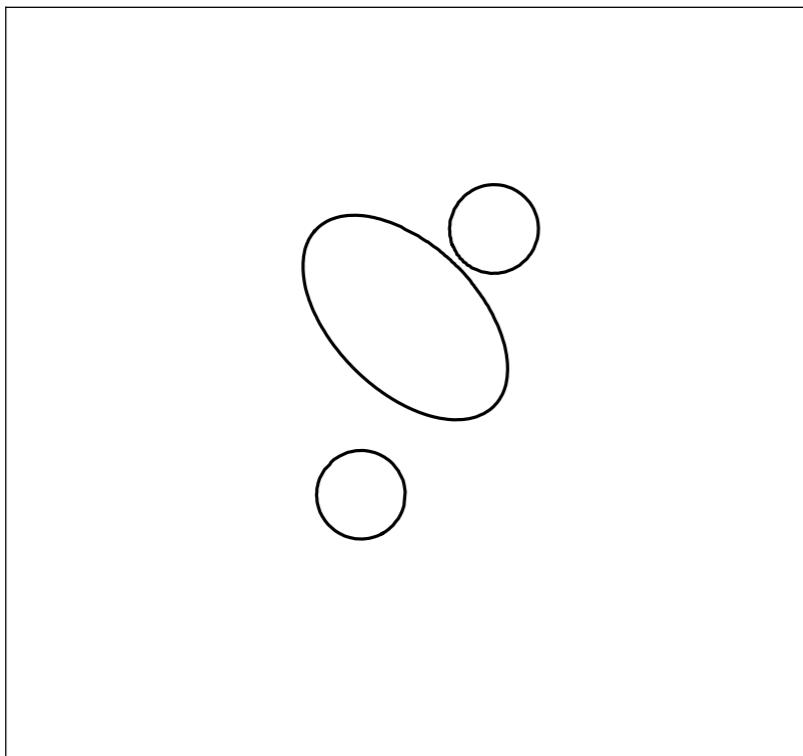


$f_2$

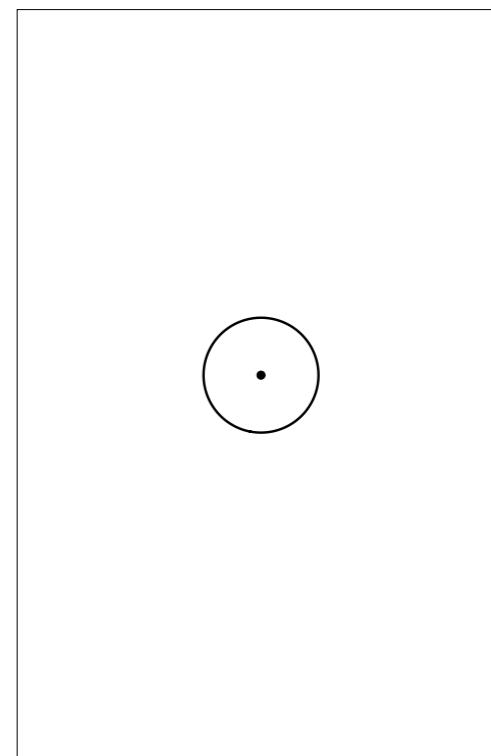


# 1. Correctness: What if $g$ is not a routing function?

$$g_1 = \frac{f_1^2}{(x_1^2 + x_2^2 + 1)^{\deg(f_1)+1}}$$

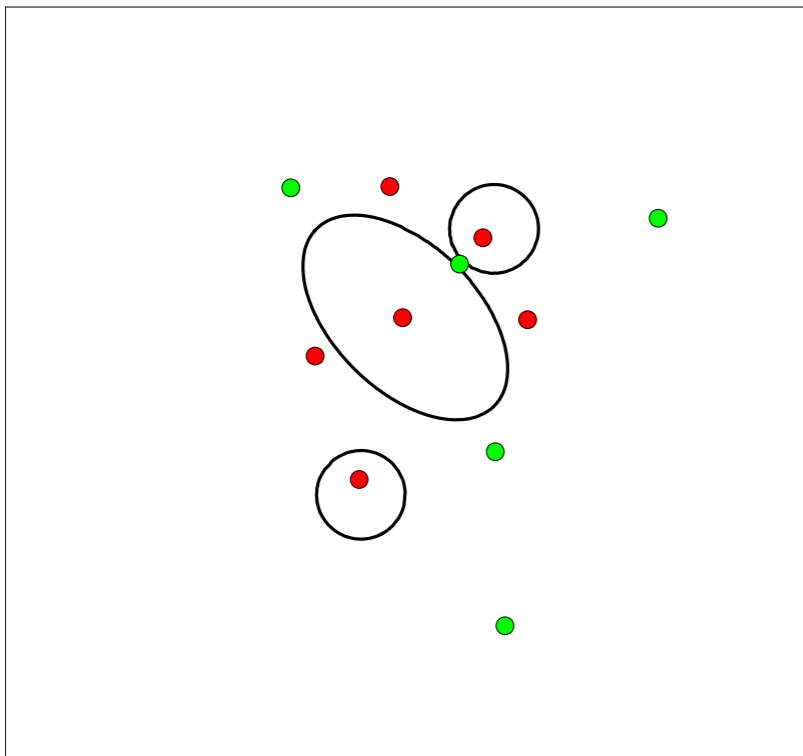


$$g_2 = \frac{f_2^2}{(x_1^2 + x_2^2 + 1)^{\deg(f_2)+1}}$$

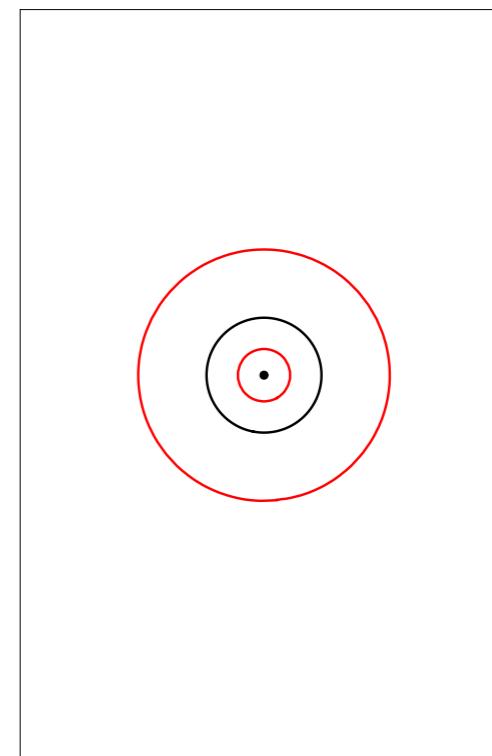


# 1. Correctness: What if $g$ is not a routing function?

$$g_1 = \frac{f_1^2}{(x_1^2 + x_2^2 + 1)^{\deg(f_1)+1}}$$

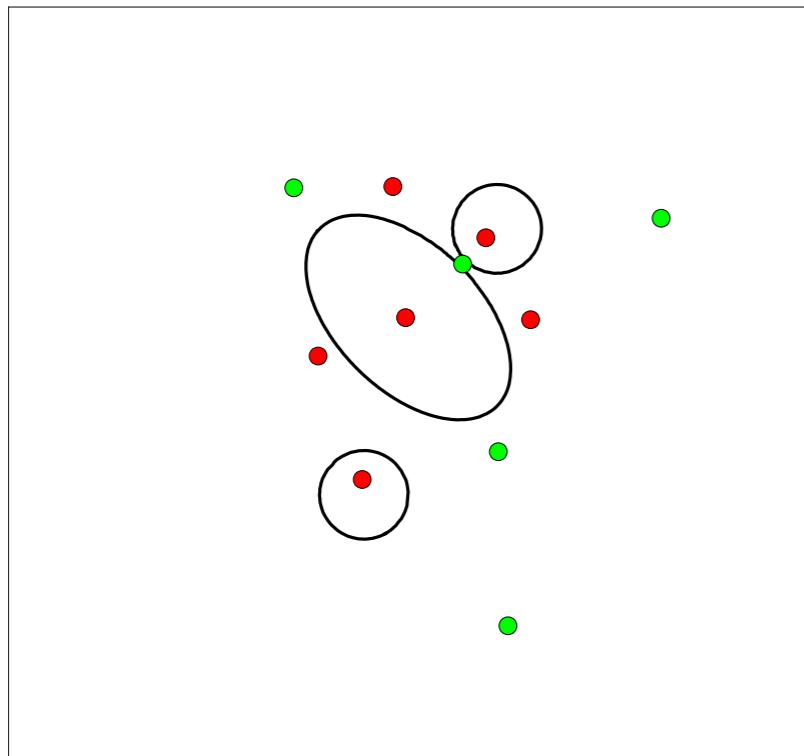


$$g_2 = \frac{f_2^2}{(x_1^2 + x_2^2 + 1)^{\deg(f_2)+1}}$$

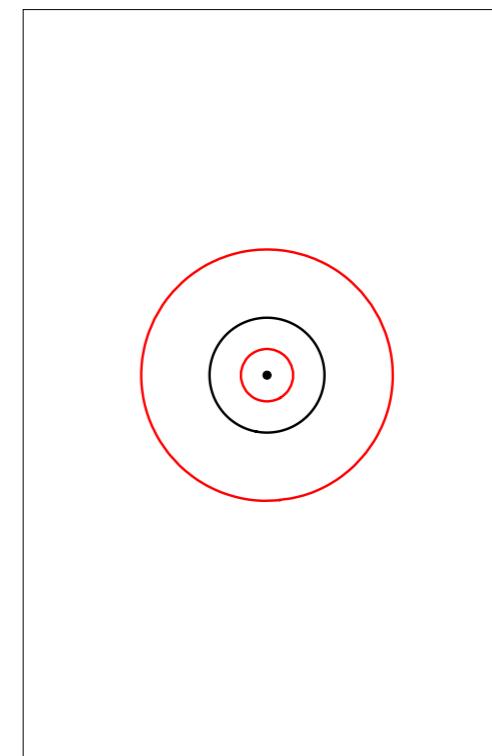


# 1. Correctness: What if $g$ is not a routing function?

$$g_1 = \frac{f_1^2}{(x_1^2 + x_2^2 + 1)^{\deg(f_1)+1}}$$



$$g_2 = \frac{f_2^2}{(x_1^2 + x_2^2 + 1)^{\deg(f_2)+1}}$$

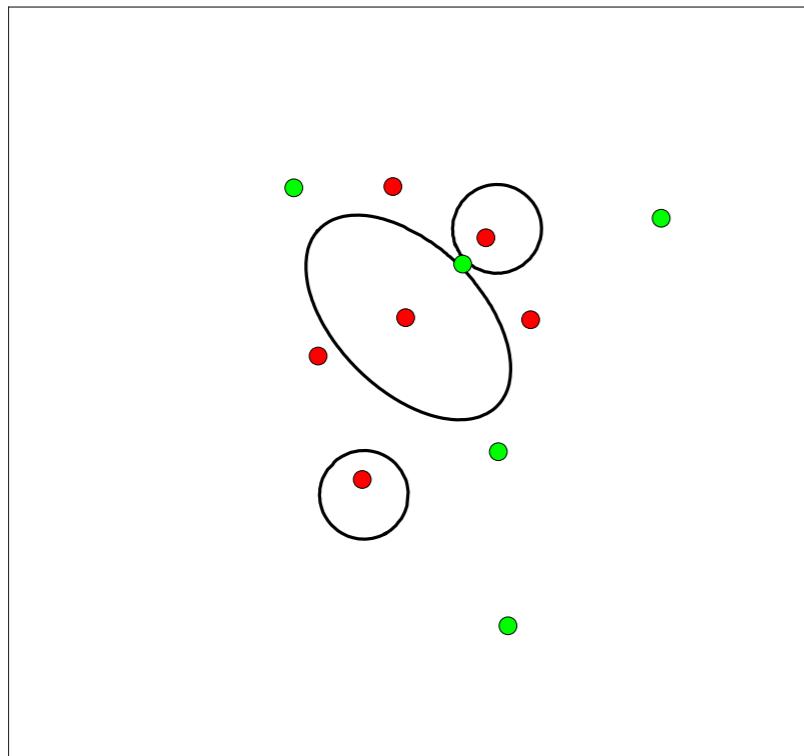


finitely many routing points

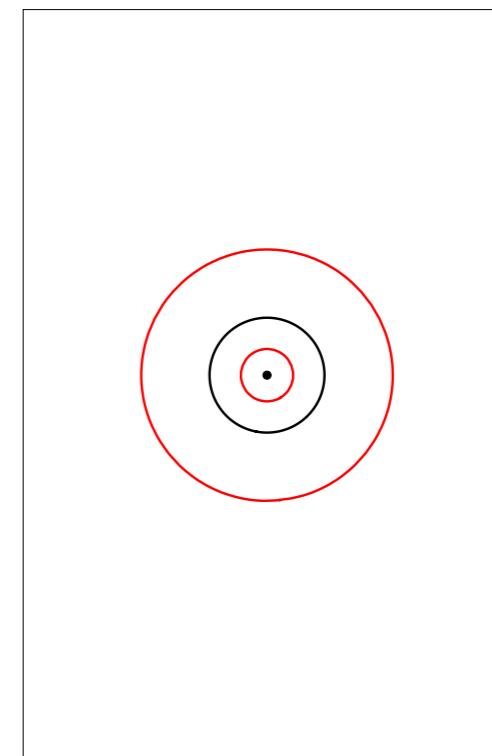
infinitely many routing points

# 1. Correctness: What if $g$ is not a routing function?

$$g_1 = \frac{f_1^2}{(x_1^2 + x_2^2 + 1)^{\deg(f_1)+1}}$$

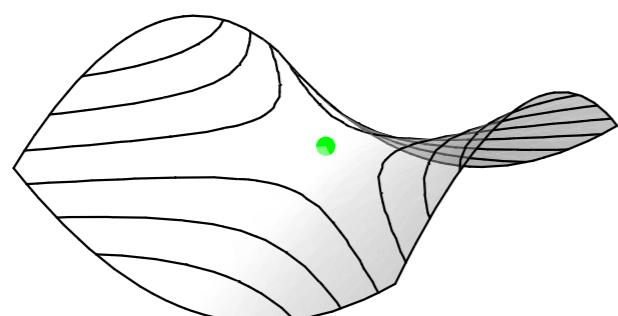


$$g_2 = \frac{f_2^2}{(x_1^2 + x_2^2 + 1)^{\deg(f_2)+1}}$$



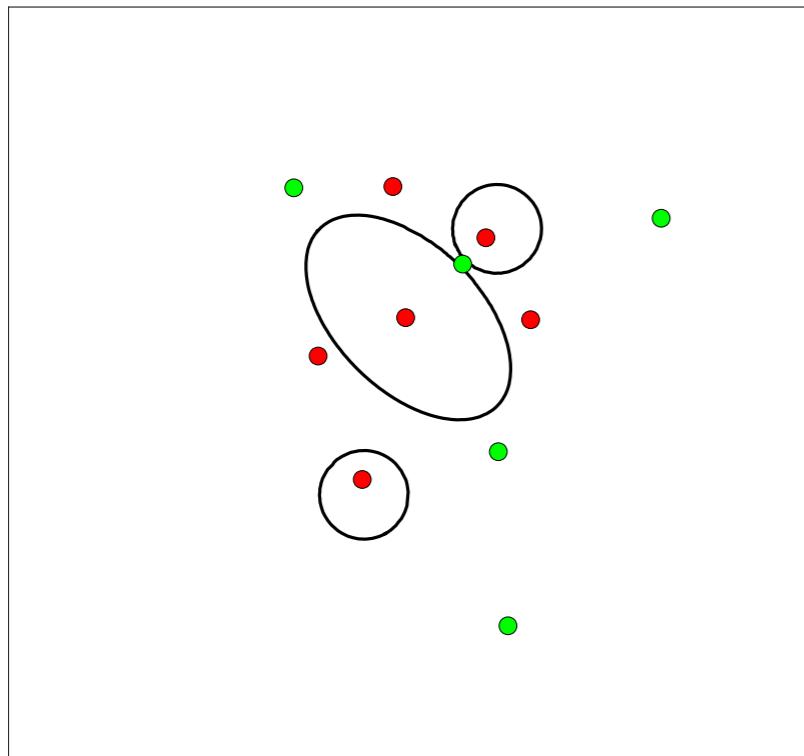
finitely many routing points

infinitely many routing points

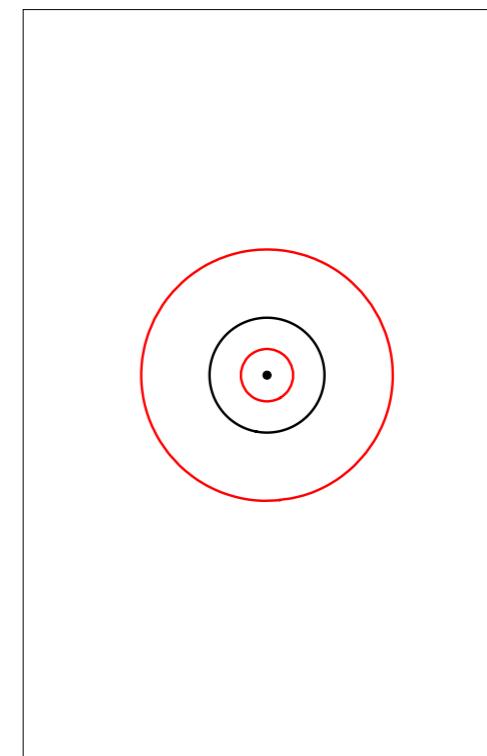


# 1. Correctness: What if $g$ is not a routing function?

$$g_1 = \frac{f_1^2}{(x_1^2 + x_2^2 + 1)^{\deg(f_1)+1}}$$

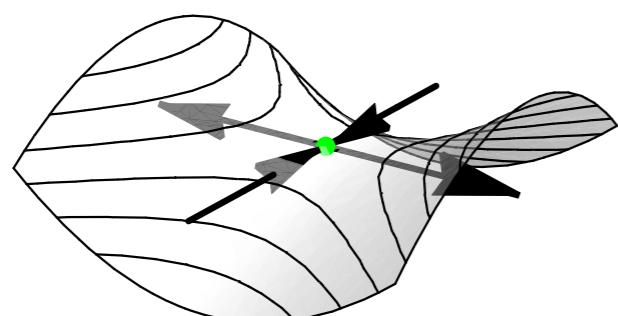


$$g_2 = \frac{f_2^2}{(x_1^2 + x_2^2 + 1)^{\deg(f_2)+1}}$$



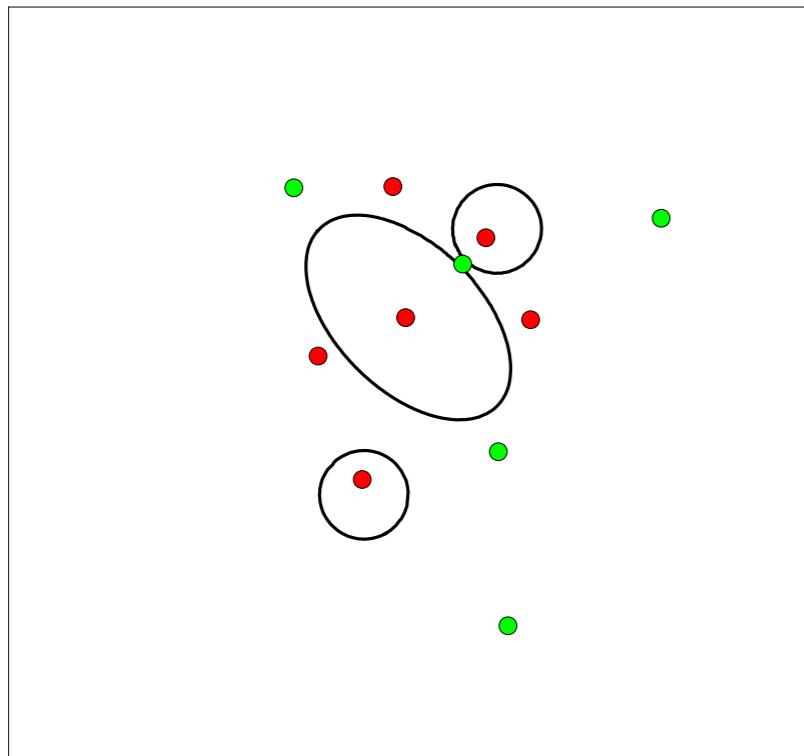
finitely many routing points

infinitely many routing points

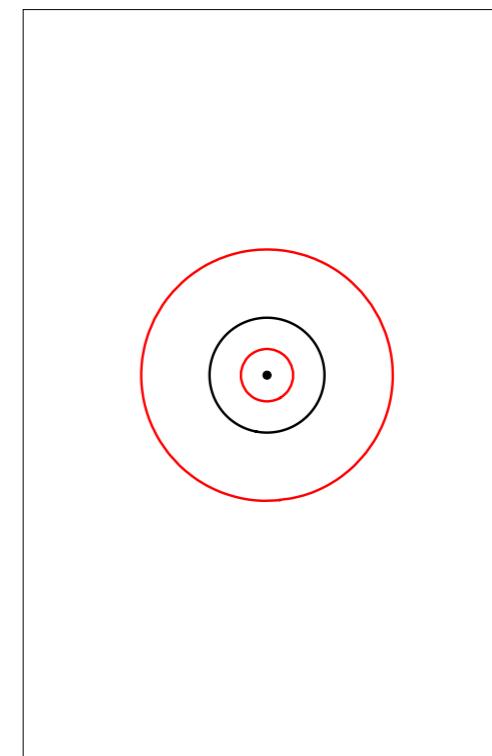


# 1. Correctness: What if $g$ is not a routing function?

$$g_1 = \frac{f_1^2}{(x_1^2 + x_2^2 + 1)^{\deg(f_1)+1}}$$

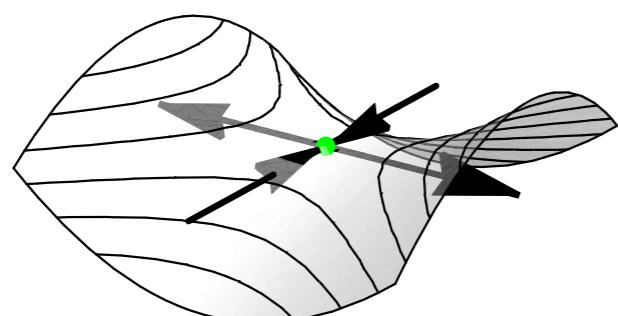


$$g_2 = \frac{f_2^2}{(x_1^2 + x_2^2 + 1)^{\deg(f_2)+1}}$$



finitely many routing points

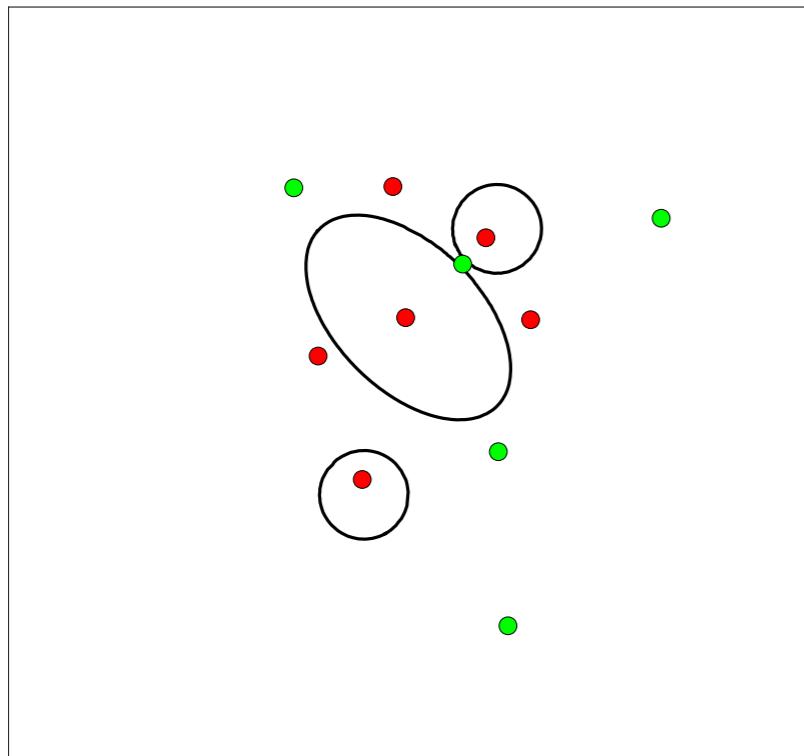
infinitely many routing points



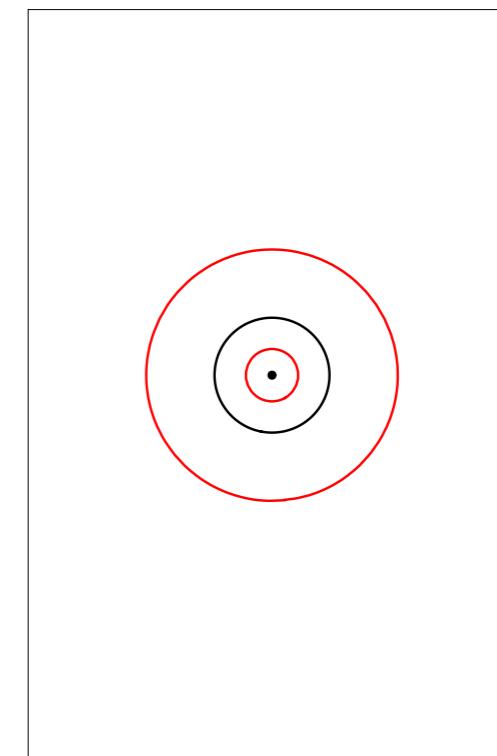
$$\det(\text{Hess } g)(\bullet) \neq 0$$

# 1. Correctness: What if $g$ is not a routing function?

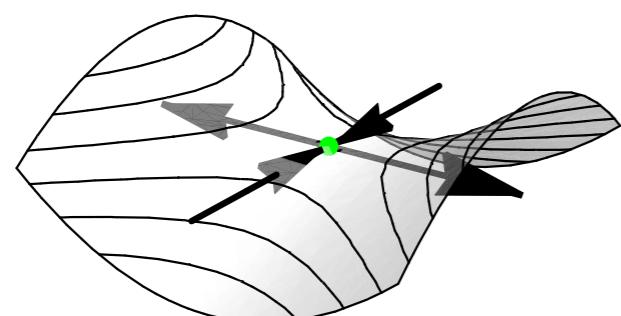
$$g_1 = \frac{f_1^2}{(x_1^2 + x_2^2 + 1)^{\deg(f_1)+1}}$$



$$g_2 = \frac{f_2^2}{(x_1^2 + x_2^2 + 1)^{\deg(f_2)+1}}$$

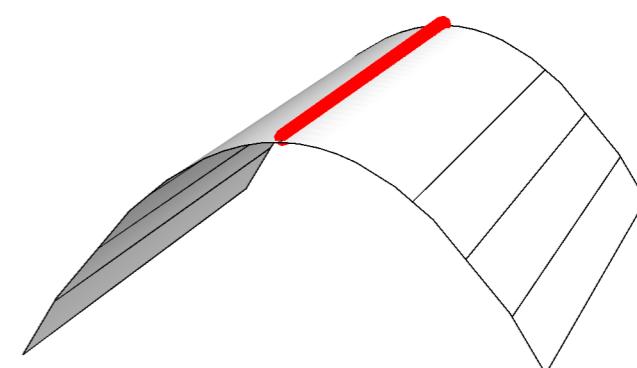


finitely many routing points



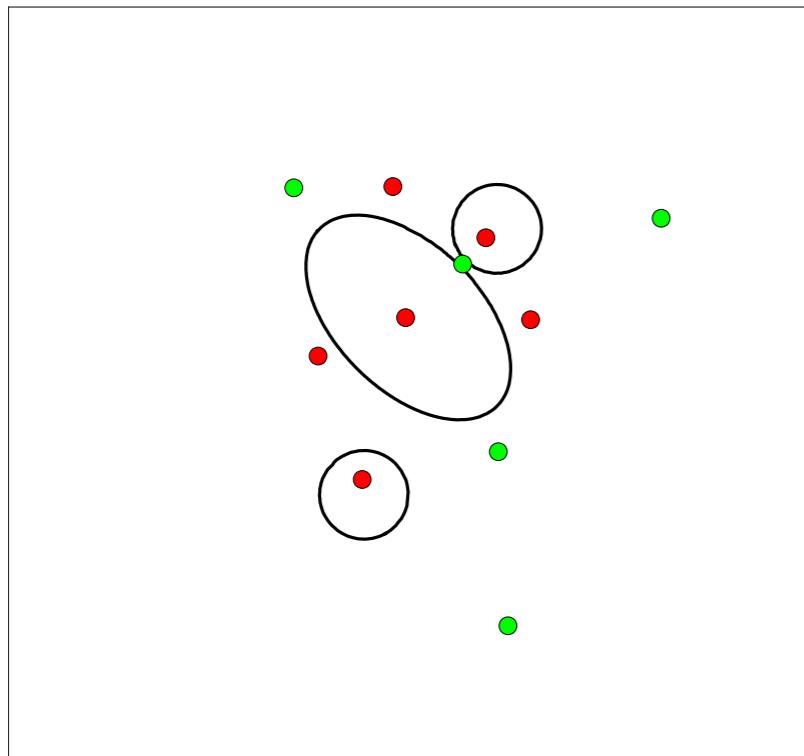
$$\det(\text{Hess } g)(\bullet) \neq 0$$

infinitely many routing points

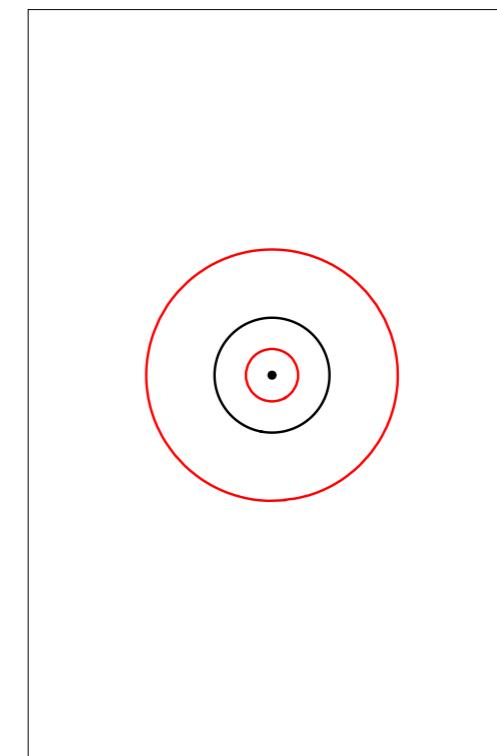


# 1. Correctness: What if $g$ is not a routing function?

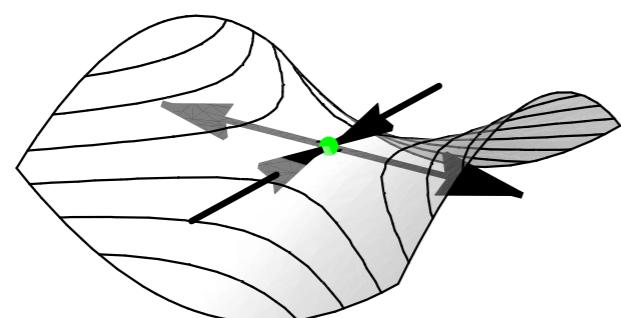
$$g_1 = \frac{f_1^2}{(x_1^2 + x_2^2 + 1)^{\deg(f_1)+1}}$$



$$g_2 = \frac{f_2^2}{(x_1^2 + x_2^2 + 1)^{\deg(f_2)+1}}$$

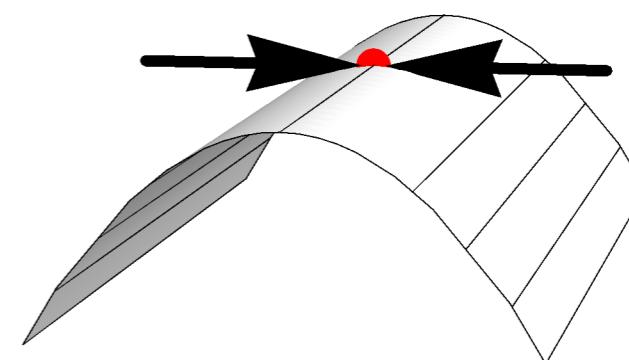


finitely many routing points



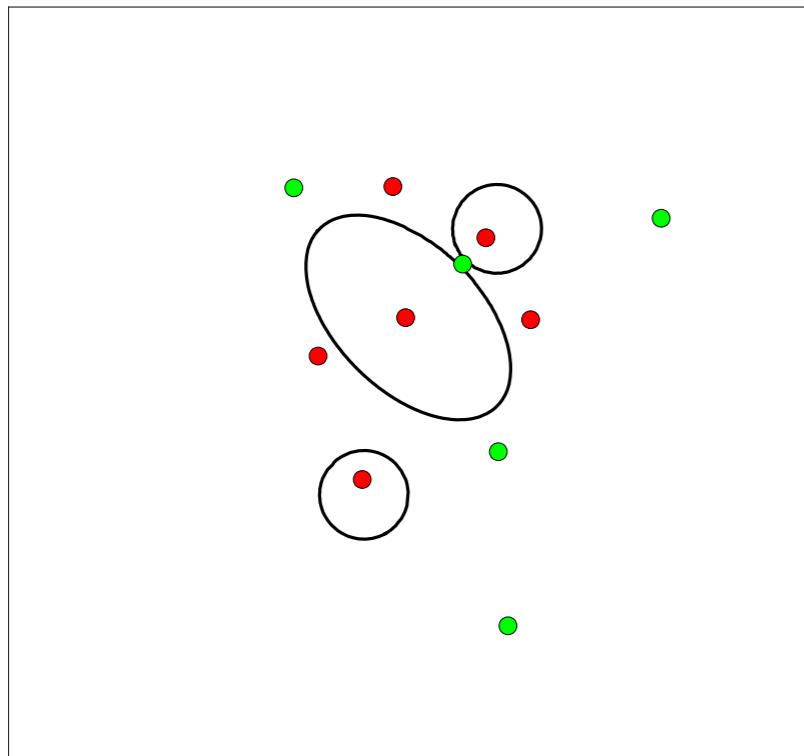
$$\det(\text{Hess } g)(\bullet) \neq 0$$

infinitely many routing points

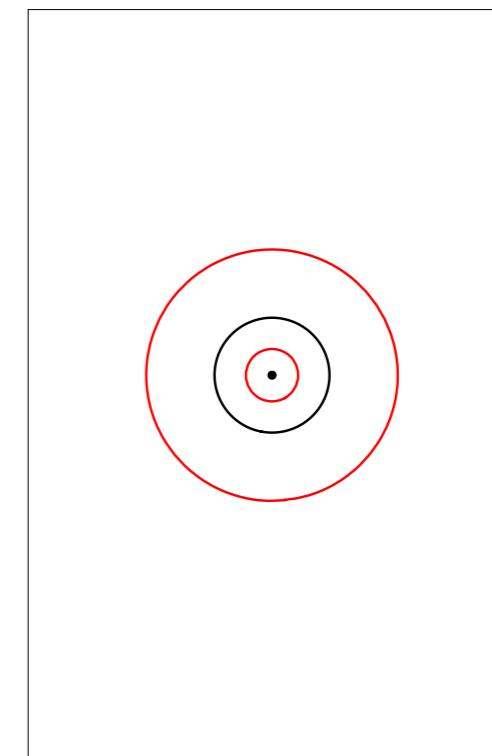


# 1. Correctness: What if $g$ is not a routing function?

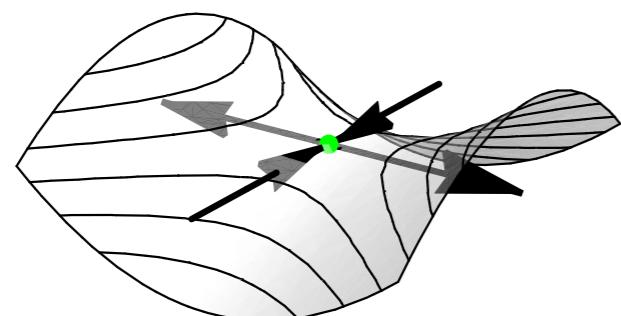
$$g_1 = \frac{f_1^2}{(x_1^2 + x_2^2 + 1)^{\deg(f_1)+1}}$$



$$g_2 = \frac{f_2^2}{(x_1^2 + x_2^2 + 1)^{\deg(f_2)+1}}$$

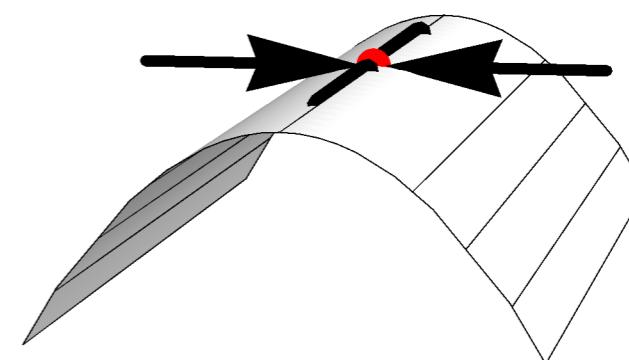


finitely many routing points



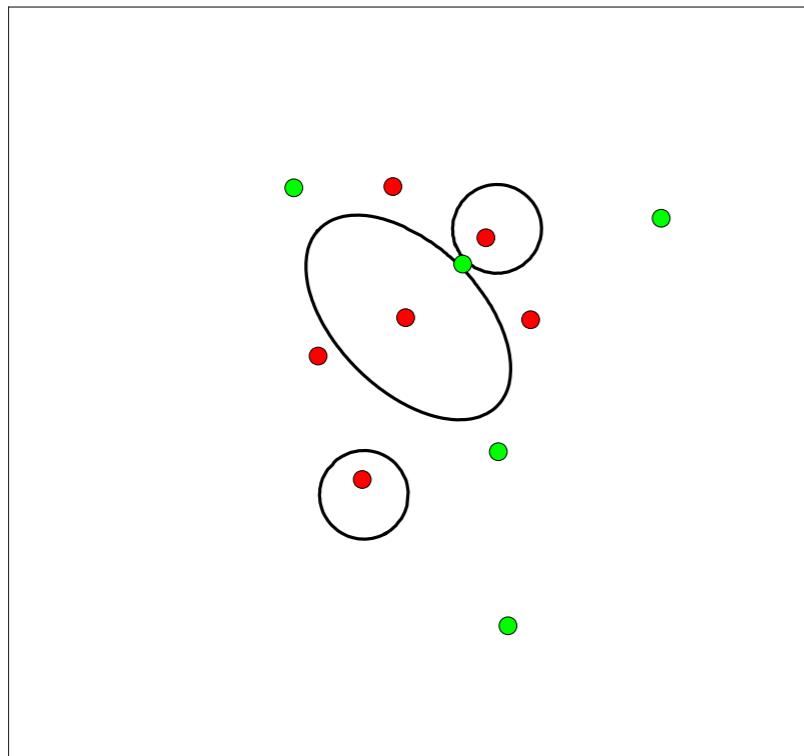
$$\det(\text{Hess } g)(\bullet) \neq 0$$

infinitely many routing points

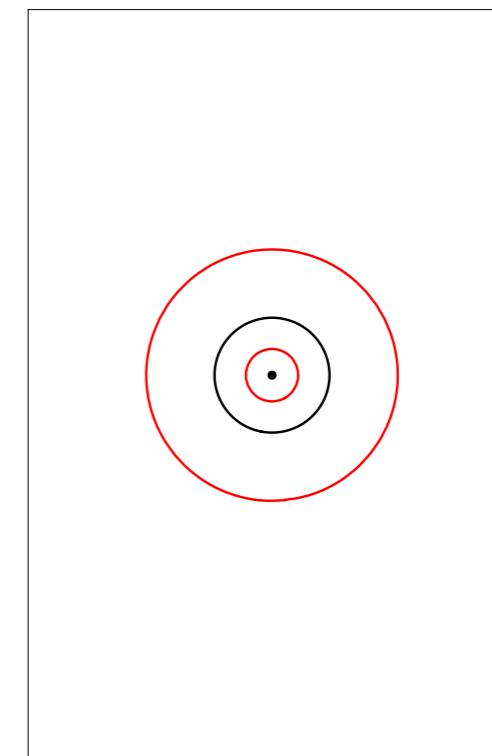


# 1. Correctness: What if $g$ is not a routing function?

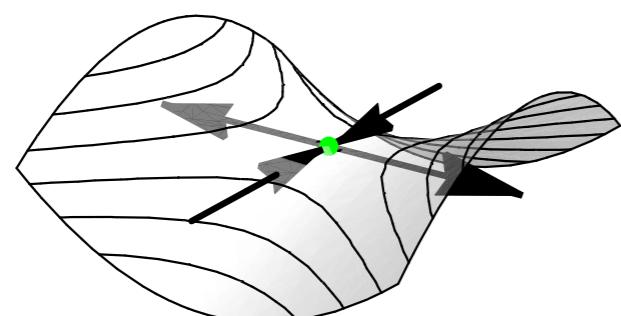
$$g_1 = \frac{f_1^2}{(x_1^2 + x_2^2 + 1)^{\deg(f_1)+1}}$$



$$g_2 = \frac{f_2^2}{(x_1^2 + x_2^2 + 1)^{\deg(f_2)+1}}$$

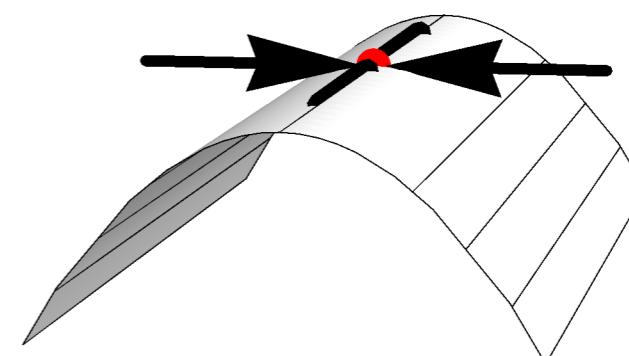


finitely many routing points



$$\det(\text{Hess } g)(\bullet) \neq 0$$

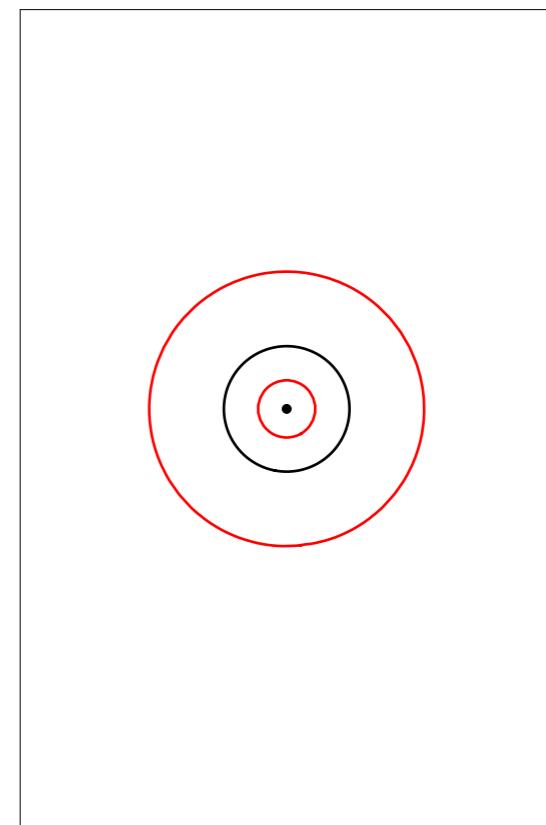
infinitely many routing points



$$\det(\text{Hess } g)(\bullet) = 0$$

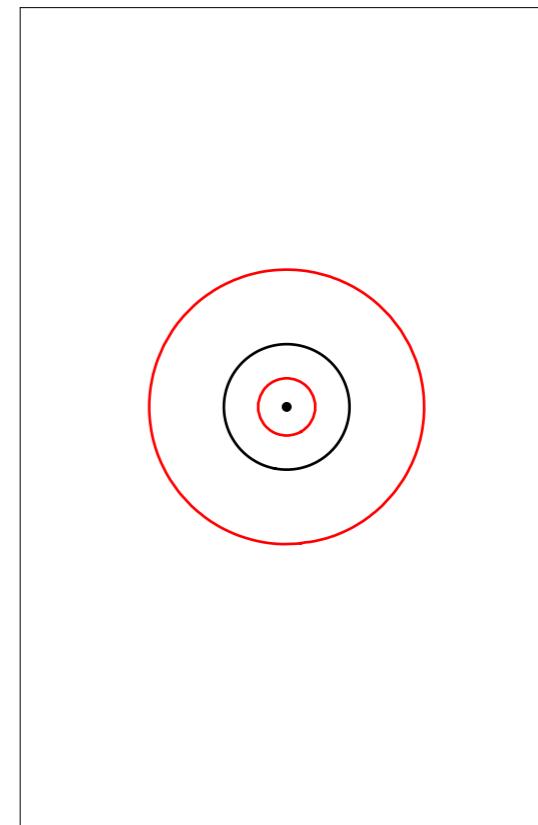
# 1. Correctness: Fixing by Perturbation

$$g_2 = \frac{f_2^2}{(x_1^2 + x_2^2 + 1)^{\deg(f_2)+1}}$$

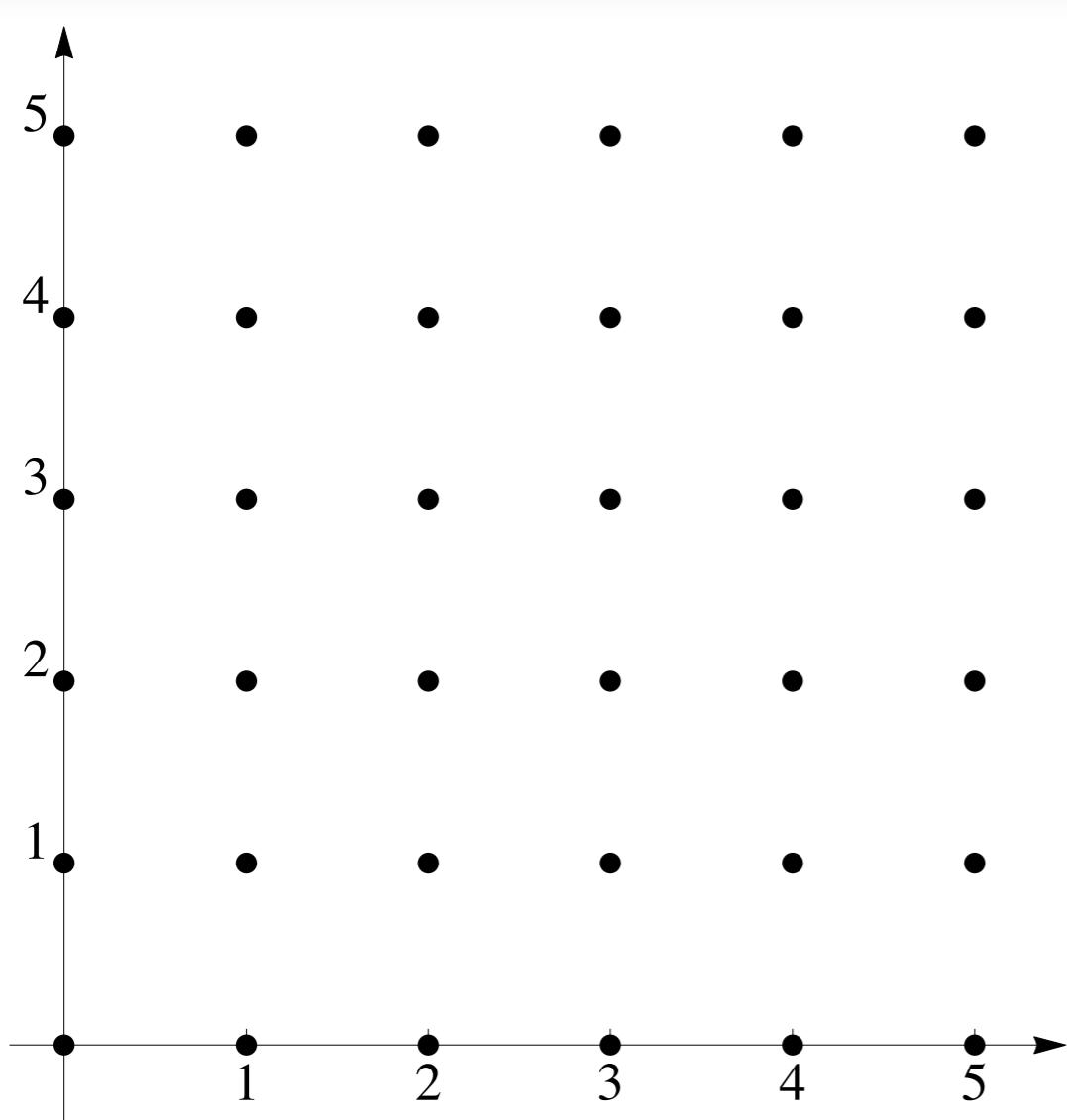


# 1. Correctness: Fixing by Perturbation

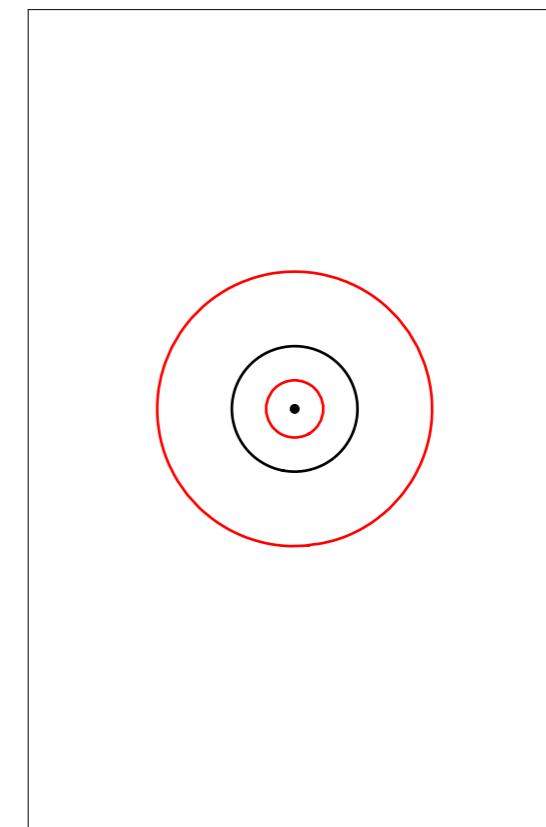
$$g_2 = \frac{f_2^2}{((x_1 - 0)^2 + (x_2 - 0)^2 + 1)^{\deg(f_2)+1}}$$



# 1. Correctness: Fixing by Perturbation

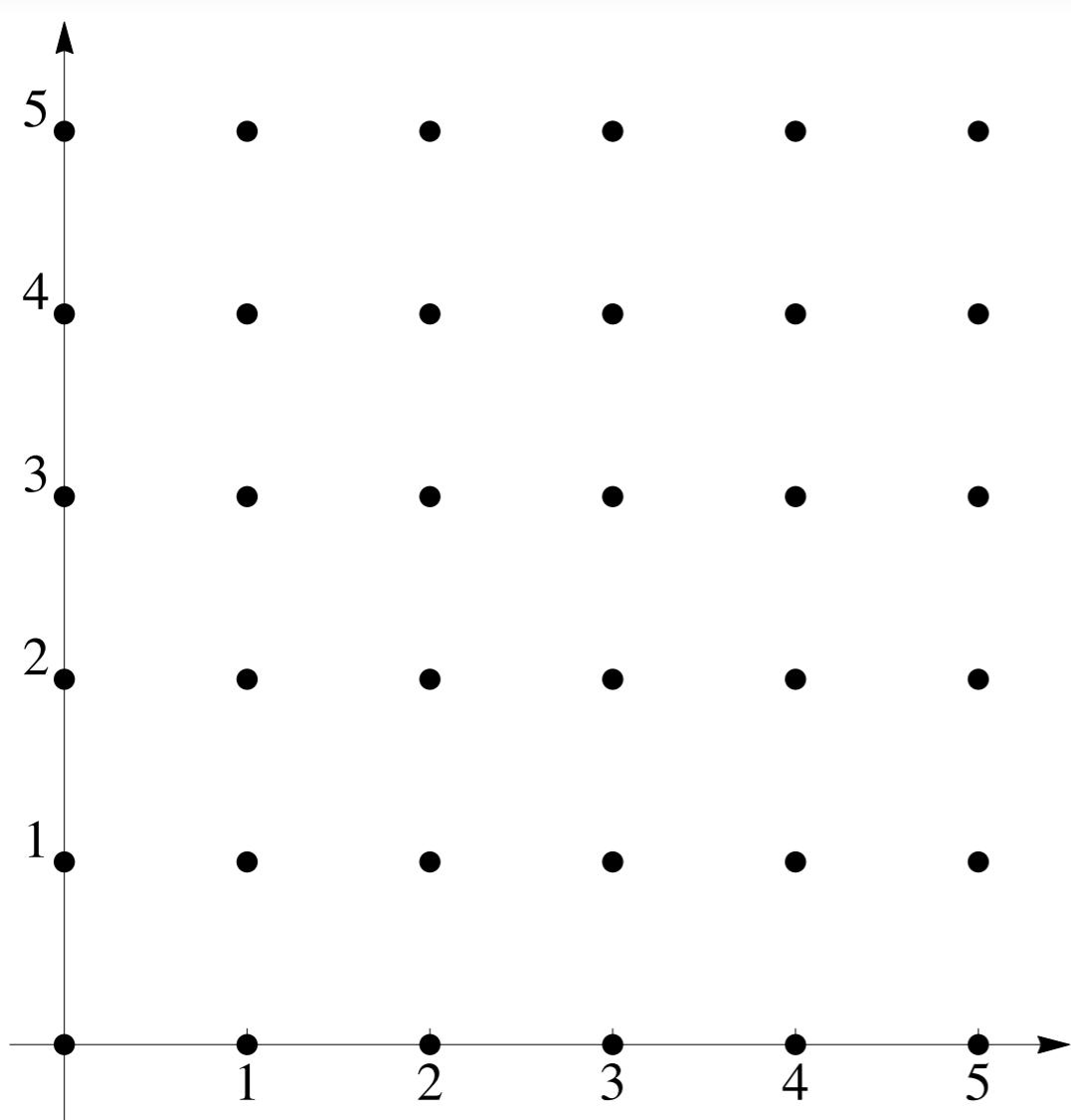


$$g_2 = \frac{f_2^2}{((x_1 - 0)^2 + (x_2 - 0)^2 + 1)^{\deg(f_2)+1}}$$

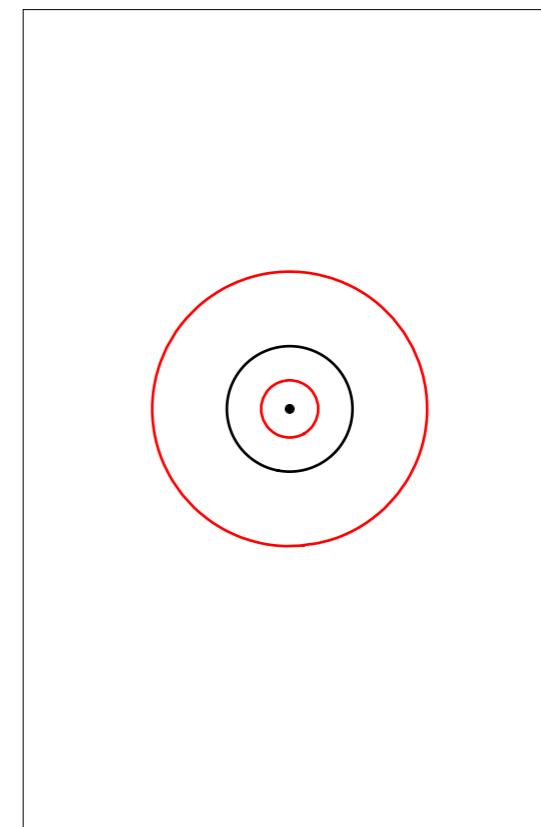


Graded lexicographic order

# 1. Correctness: Fixing by Perturbation



$$g_2 = \frac{f_2^2}{((x_1 - 0)^2 + (x_2 - 0)^2 + 1)^{\deg(f_2)+1}}$$



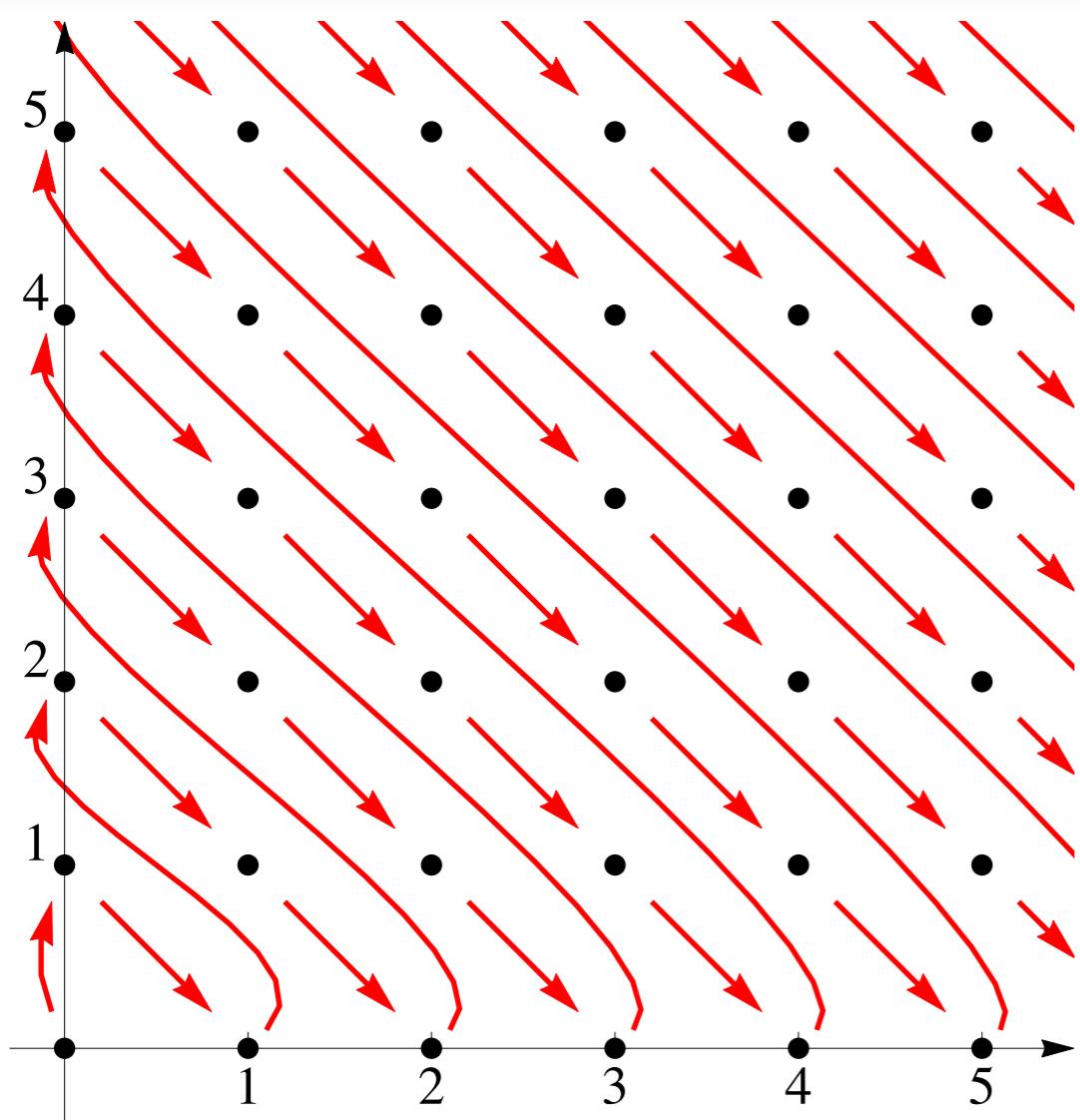
**Graded lexicographic order**

$x^\alpha >_{\text{grlex}} x^\beta$  if  
 $\deg(x^\alpha) > \deg(x^\beta)$

or

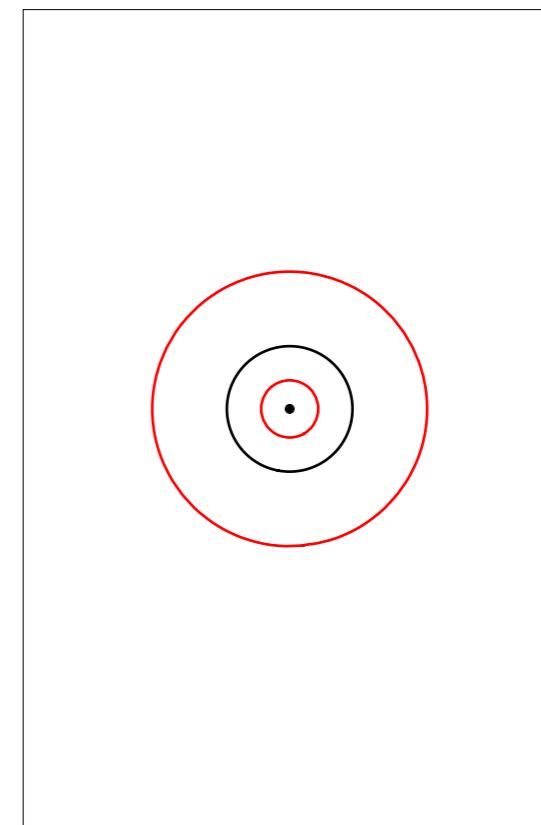
$\deg(x^\alpha) = \deg(x^\beta)$  and  $x^\alpha >_{\text{lex}} x^\beta$

# 1. Correctness: Fixing by Perturbation



Graded lexicographic order

$$g_2 = \frac{f_2^2}{((x_1 - 0)^2 + (x_2 - 0)^2 + 1)^{\deg(f_2)+1}}$$

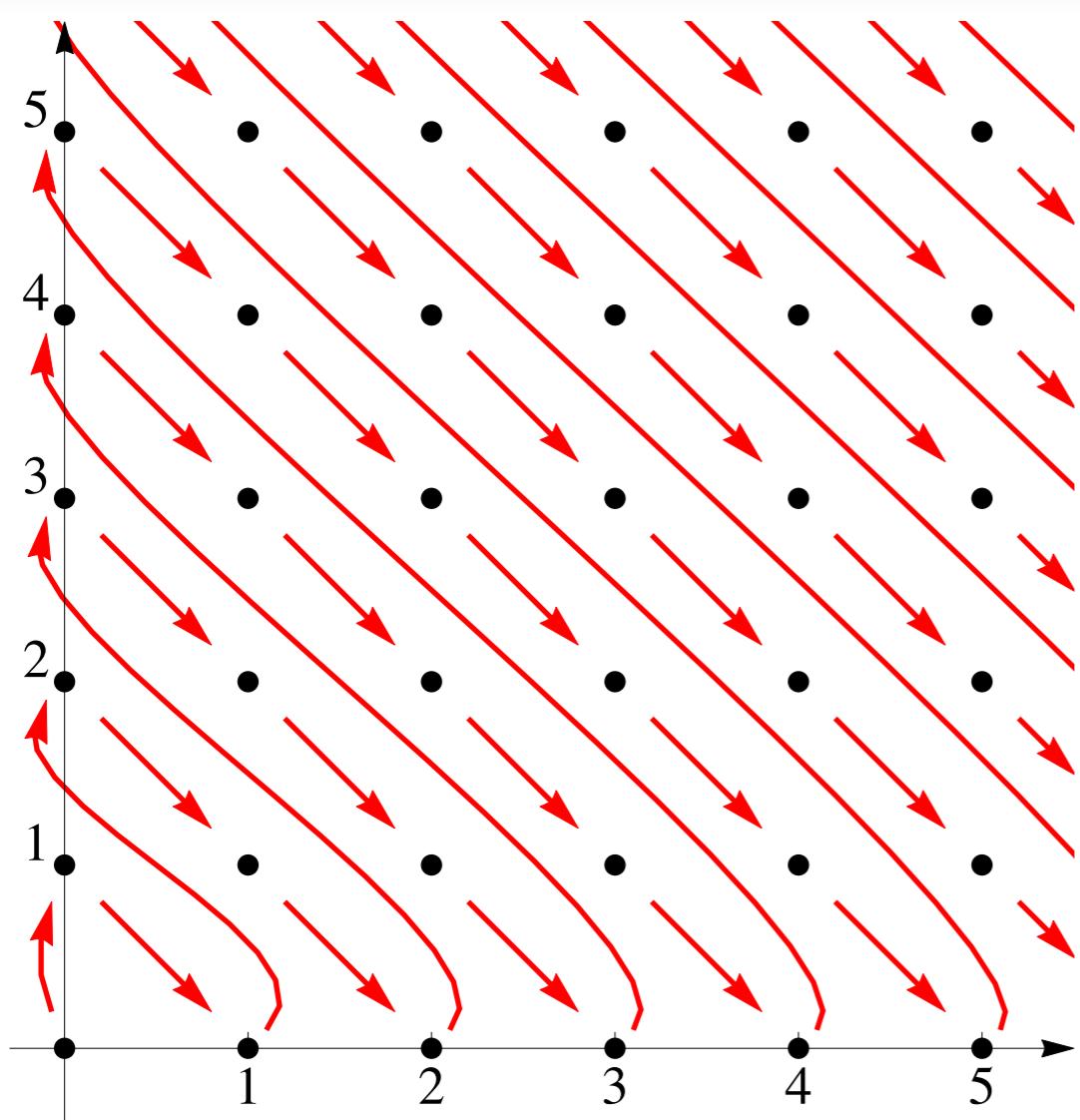


$x^\alpha >_{\text{grlex}} x^\beta$  if  
 $\deg(x^\alpha) > \deg(x^\beta)$

or

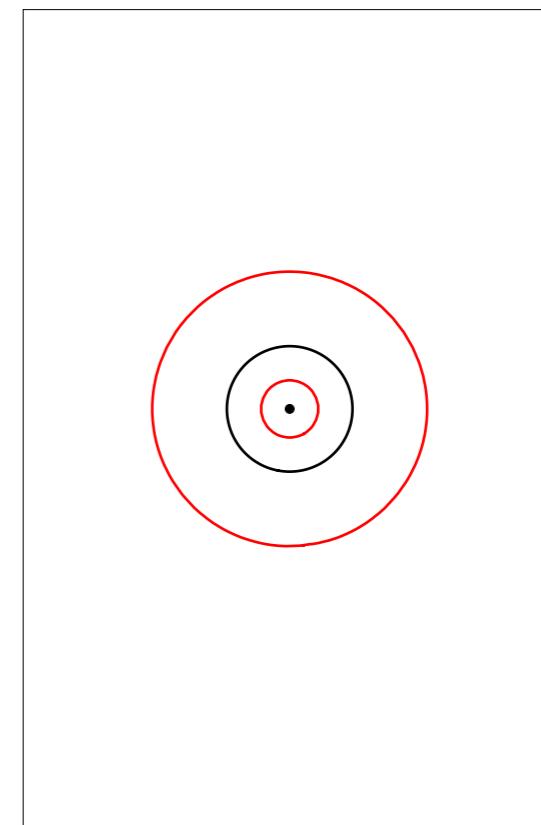
$\deg(x^\alpha) = \deg(x^\beta)$  and  $x^\alpha >_{\text{lex}} x^\beta$

# 1. Correctness: Fixing by Perturbation



Graded lexicographic order

$$g_2 = \frac{f_2^2}{((x_1 - 0)^2 + (x_2 - 1)^2 + 1)^{\deg(f_2)+1}}$$

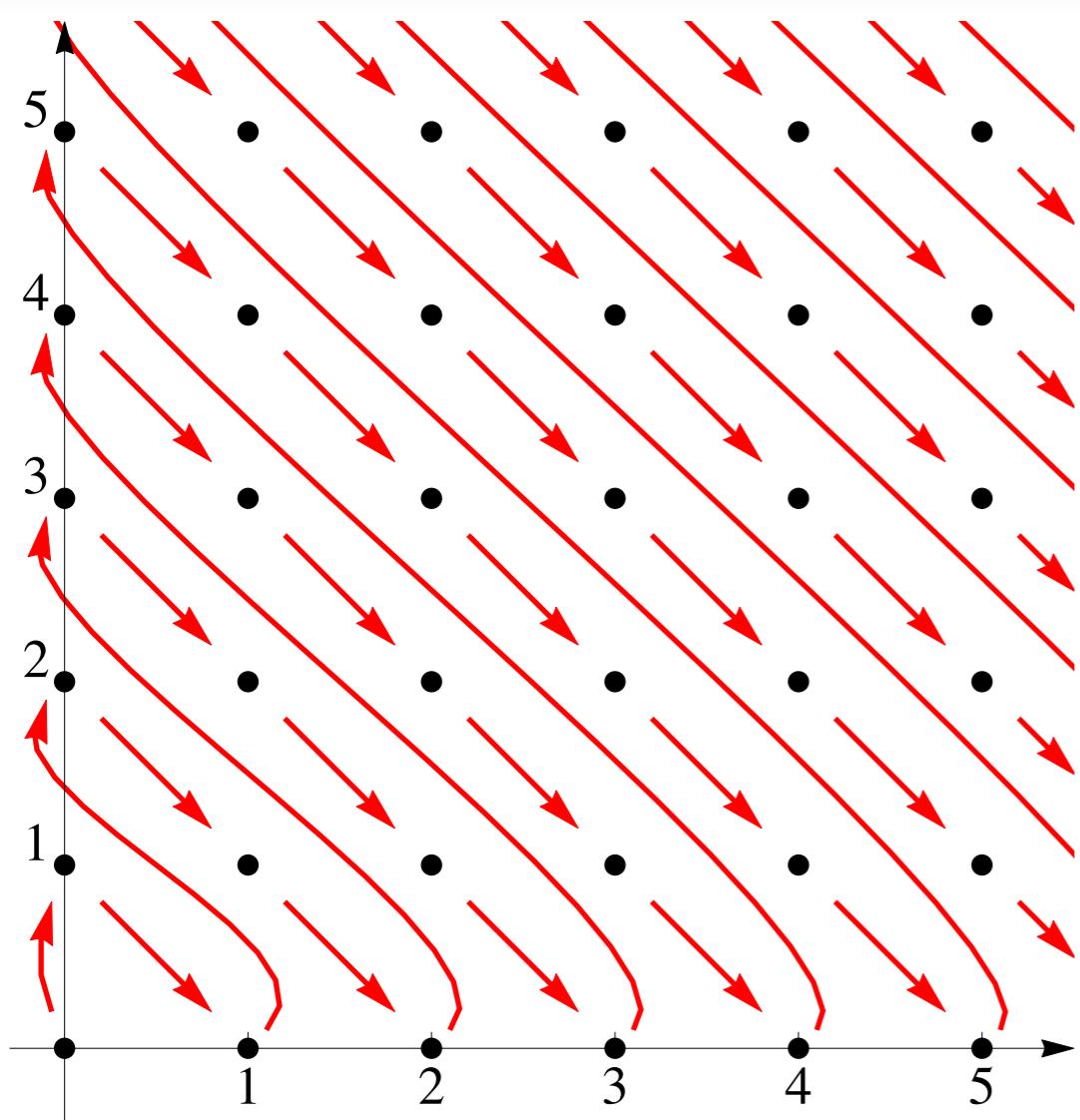


$x^\alpha >_{\text{grlex}} x^\beta$  if  
 $\deg(x^\alpha) > \deg(x^\beta)$

or

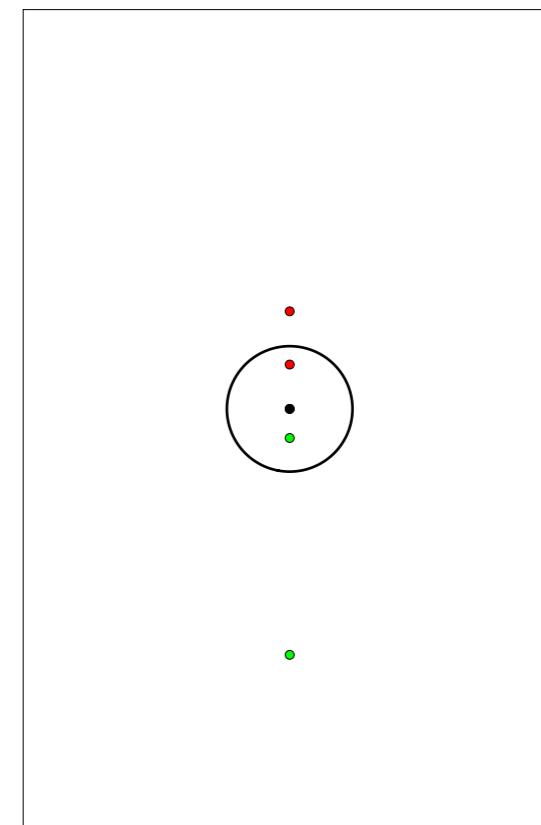
$\deg(x^\alpha) = \deg(x^\beta)$  and  $x^\alpha >_{\text{lex}} x^\beta$

# 1. Correctness: Fixing by Perturbation



Graded lexicographic order

$$g_2 = \frac{f_2^2}{((x_1 - 0)^2 + (x_2 - 1)^2 + 1)^{\deg(f_2)+1}}$$



$x^\alpha >_{\text{grlex}} x^\beta$  if  
 $\deg(x^\alpha) > \deg(x^\beta)$

or

$\deg(x^\alpha) = \deg(x^\beta)$  and  $x^\alpha >_{\text{lex}} x^\beta$

## 2. Termination: Theorem

## 2. Termination: Theorem

$$\forall f \in \mathbb{Z}[\mathbf{x}_1, \dots, \mathbf{x}_n]$$

$\exists$  semialgebraic set  $S \subset \mathbb{R}^n$

$$\text{codim } S < n$$

$$\forall (c_1, \dots, c_n) \in S$$

$$\forall \gamma \neq 0 \in \mathbb{R}$$

$$g = \frac{f^2}{((\mathbf{x}_1 - c_1)^2 + \dots + (\mathbf{x}_n - c_n)^2 + 1)^\gamma}$$

- $g$  has finitely many routing points
- all routing points of  $g$  are nondegenerate

## 2. Termination: Proof Sketch

## 2. Termination: Proof Sketch

**Scratchwork**

## 2. Termination: Proof Sketch

### Scratchwork

$$U(\mathbf{x}) = (x_1 - c_1)^2 + \cdots + (x_n - c_n)^2 + 1$$

## 2. Termination: Proof Sketch

### Scratchwork

$$U(\mathbf{x}) = (x_1 - c_1)^2 + \cdots + (x_n - c_n)^2 + 1$$

$$g = \frac{f^2}{U^\gamma}$$

## 2. Termination: Proof Sketch

### Scratchwork

$$U(\mathbf{x}) = (x_1 - c_1)^2 + \cdots + (x_n - c_n)^2 + 1$$

$$g = \frac{f^2}{U^\gamma}$$

$$\nabla g(x) = 0 \wedge g(x) \neq 0$$

## 2. Termination: Proof Sketch

### Scratchwork

$$U(\mathbf{x}) = (x_1 - c_1)^2 + \cdots + (x_n - c_n)^2 + 1$$

$$g = \frac{f^2}{U^\gamma}$$

$$\nabla g(x) = 0 \wedge g(x) \neq 0 \iff 2\nabla f(x)U(x) - \gamma f(x)\nabla U(x) = 0 \wedge f(x) \neq 0$$

## 2. Termination: Proof Sketch

### Scratchwork

$$U(\mathbf{x}) = (\mathbf{x}_1 - c_1)^2 + \cdots + (\mathbf{x}_n - c_n)^2 + 1$$

$$g = \frac{f^2}{U^\gamma}$$

$$\nabla g(x) = 0 \wedge g(x) \neq 0 \iff 2\nabla f(x)U(x) - \gamma f(x)\nabla U(x) = 0 \wedge f(x) \neq 0$$

## 2. Termination: Proof Sketch

### Scratchwork

$$U(\mathbf{x}) = (\mathbf{x}_1 - c_1)^2 + \cdots + (\mathbf{x}_n - c_n)^2 + 1$$

$$g = \frac{f^2}{U^\gamma}$$

$$\nabla g(x) = 0 \wedge g(x) \neq 0 \iff 2\nabla f(x)U(x) - \gamma f(x)\nabla U(x) = 0 \wedge f(x) \neq 0$$

$$2\nabla f(x)U(x) - \gamma f(x)\nabla U(x) = 0 \xrightarrow{\text{rewrite}} c_i = -\partial_{\mathbf{x}_i} f(x) \frac{U(x)}{\gamma f(x)} + x_i \quad i = 1, \dots, n$$

## 2. Termination: Proof Sketch

### Scratchwork

$$U(\mathbf{x}) = (x_1 - c_1)^2 + \cdots + (x_n - c_n)^2 + 1$$

$$g = \frac{f^2}{U^\gamma}$$

$$\nabla g(x) = 0 \wedge g(x) \neq 0 \iff 2\nabla f(x)U(x) - \gamma f(x)\nabla U(x) = 0 \wedge f(x) \neq 0$$

$$2\nabla f(x)U(x) - \gamma f(x)\nabla U(x) = 0 \xrightarrow{\text{rewrite}} c_i = -\partial_{x_i} f(x) \frac{U(x)}{\gamma f(x)} + x_i \quad i = 1, \dots, n$$

## 2. Termination: Proof Sketch

### Scratchwork

$$U(\mathbf{x}) = (\mathbf{x}_1 - c_1)^2 + \cdots + (\mathbf{x}_n - c_n)^2 + 1$$

$$g = \frac{f^2}{U^\gamma}$$

$$\nabla g(x) = 0 \wedge g(x) \neq 0 \iff 2\nabla f(x)U(x) - \gamma f(x)\nabla U(x) = 0 \wedge f(x) \neq 0$$

$$2\nabla f(x)U(x) - \gamma f(x)\nabla U(x) = 0 \xrightarrow{\text{rewrite}} c_i = -\partial_{\mathbf{x}_i} f(x)t + x_i \quad i = 1, \dots, n$$
$$t = \frac{U(x)}{\gamma f(x)}$$

## 2. Termination: Proof Sketch

### Scratchwork

$$U(\mathbf{x}) = (x_1 - c_1)^2 + \cdots + (x_n - c_n)^2 + 1$$

$$g = \frac{f^2}{U^\gamma}$$

$$\nabla g(x) = 0 \wedge g(x) \neq 0 \iff 2\nabla f(x)U(x) - \gamma f(x)\nabla U(x) = 0 \wedge f(x) \neq 0$$

$$2\nabla f(x)U(x) - \gamma f(x)\nabla U(x) = 0 \xrightarrow{\text{rewrite}} c_i = -\partial_{x_i} f(x)t + x_i \quad i = 1, \dots, n$$

$$t = \frac{U(x)}{\gamma f(x)}$$

## 2. Termination: Proof Sketch

### Scratchwork

$$U(\mathbf{x}) = (x_1 - c_1)^2 + \cdots + (x_n - c_n)^2 + 1$$

$$g = \frac{f^2}{U^\gamma}$$

$$\nabla g(x) = 0 \wedge g(x) \neq 0 \iff 2\nabla f(x)U(x) - \gamma f(x)\nabla U(x) = 0 \wedge f(x) \neq 0$$

$$2\nabla f(x)U(x) - \gamma f(x)\nabla U(x) = 0 \xrightarrow{\text{rewrite}} c_i = -\partial_{x_i} f(x)t + x_i \quad i = 1, \dots, n$$
$$\boxed{\gamma = \frac{U(x)}{tf(x)}}$$

## 2. Termination: Proof Sketch

### Scratchwork

$$U(\mathbf{x}) = (\mathbf{x}_1 - c_1)^2 + \cdots + (\mathbf{x}_n - c_n)^2 + 1$$

$$g = \frac{f^2}{U^\gamma}$$

$$\nabla g(x) = 0 \wedge g(x) \neq 0 \iff 2\nabla f(x)U(x) - \gamma f(x)\nabla U(x) = 0 \wedge f(x) \neq 0$$

$$2\nabla f(x)U(x) - \gamma f(x)\nabla U(x) = 0 \xrightarrow{\text{rewrite}} c_i = -\partial_{\mathbf{x}_i} f(x)t + x_i \quad i = 1, \dots, n$$
$$\gamma = \frac{U(x)}{tf(x)}$$

## 2. Termination: Proof Sketch

### Scratchwork

$$U(\mathbf{x}) = (x_1 - c_1)^2 + \cdots + (x_n - c_n)^2 + 1$$

$$g = \frac{f^2}{U^\gamma}$$

$$\nabla g(x) = 0 \wedge g(x) \neq 0 \iff \exists t \neq 0$$

$$c_i = -\partial_{x_i} f(x)t + x_i \wedge f(x) \neq 0 \quad i = 1, \dots, n$$

$$\gamma = \frac{U(x)}{tf(x)}$$

## 2. Termination: Proof Sketch

### Scratchwork

$$U(\mathbf{x}) = (x_1 - c_1)^2 + \cdots + (x_n - c_n)^2 + 1$$

$$g = \frac{f^2}{U^\gamma}$$

$$\nabla g(x) = 0 \wedge g(x) \neq 0 \iff \exists t \neq 0$$

$$c_i = -\partial_{x_i} f(x)t + x_i \wedge f(x) \neq 0 \quad i = 1, \dots, n$$

$$\gamma = \frac{U(x)}{tf(x)}$$

### Idea

## 2. Termination: Proof Sketch

### Scratchwork

$$U(\mathbf{x}) = (\mathbf{x}_1 - c_1)^2 + \cdots + (\mathbf{x}_n - c_n)^2 + 1$$

$$g = \frac{f^2}{U^\gamma}$$

$$\nabla g(x) = 0 \wedge g(x) \neq 0 \iff \exists t \neq 0$$

$$c_i = -\partial_{\mathbf{x}_i} f(x)t + x_i \wedge f(x) \neq 0 \quad i = 1, \dots, n$$

$$\gamma = \frac{U(x)}{tf(x)}$$

### Idea

Let  $p = (p_1, \dots, p_n)$

$$p_i(\mathbf{x}, \mathbf{t}) = -\partial_{\mathbf{x}_i} f(\mathbf{x})\mathbf{t} + \mathbf{x}_i$$

## 2. Termination: Proof Sketch

### Scratchwork

$$U(\mathbf{x}) = (x_1 - c_1)^2 + \cdots + (x_n - c_n)^2 + 1$$

$$g = \frac{f^2}{U^\gamma}$$

$$\nabla g(x) = 0 \wedge g(x) \neq 0 \iff \exists t \neq 0$$

$$c_i = \boxed{-\partial_{x_i} f(x)t + x_i} \wedge f(x) \neq 0 \quad i = 1, \dots, n$$

$$\gamma = \frac{U(x)}{tf(x)}$$

### Idea

Let  $p = (p_1, \dots, p_n)$

$$p_i(\mathbf{x}, \mathbf{t}) = \boxed{-\partial_{x_i} f(\mathbf{x})\mathbf{t} + x_i}$$

## 2. Termination: Proof Sketch

### Scratchwork

$$U(\mathbf{x}) = (\mathbf{x}_1 - c_1)^2 + \cdots + (\mathbf{x}_n - c_n)^2 + 1$$

$$g = \frac{f^2}{U^\gamma}$$

$$\nabla g(x) = 0 \wedge g(x) \neq 0 \iff \exists t \neq 0$$

$$c_i = -\partial_{\mathbf{x}_i} f(x)t + x_i \wedge f(x) \neq 0 \quad i = 1, \dots, n$$

$$\gamma = \frac{U(x)}{tf(x)}$$

### Idea

Let  $p = (p_1, \dots, p_n)$

$$p_i(\mathbf{x}, \mathbf{t}) = -\partial_{\mathbf{x}_i} f(\mathbf{x})\mathbf{t} + \mathbf{x}_i$$

Choose  $(c_1, \dots, c_n) \in \mathbb{R}^n \setminus \{\text{critical values of } p\}$

## 2. Termination: Proof Sketch

### Scratchwork

$$U(\mathbf{x}) = (\mathbf{x}_1 - c_1)^2 + \cdots + (\mathbf{x}_n - c_n)^2 + 1$$

$$g = \frac{f^2}{U^\gamma}$$

$$\nabla g(x) = 0 \wedge g(x) \neq 0 \iff \exists t \neq 0$$

$$c_i = -\partial_{\mathbf{x}_i} f(x)t + \mathbf{x}_i \wedge f(x) \neq 0 \quad i = 1, \dots, n$$

$$\gamma = \frac{U(x)}{tf(x)}$$

### Idea

Let  $p = (p_1, \dots, p_n)$

$$p_i(\mathbf{x}, \mathbf{t}) = -\partial_{\mathbf{x}_i} f(\mathbf{x})\mathbf{t} + \mathbf{x}_i$$

Choose  $(c_1, \dots, c_n) \in \underbrace{\mathbb{R}^n \setminus \{\text{critical values of } p\}}_S$

## 2. Termination: Proof Sketch

### Scratchwork

$$U(\mathbf{x}) = (\mathbf{x}_1 - c_1)^2 + \cdots + (\mathbf{x}_n - c_n)^2 + 1$$

$$g = \frac{f^2}{U^\gamma}$$

$$\nabla g(x) = 0 \wedge g(x) \neq 0 \iff \exists t \neq 0 \quad c_i = -\partial_{\mathbf{x}_i} f(x)t + \mathbf{x}_i \wedge f(x) \neq 0 \quad i = 1, \dots, n$$

$$\gamma = \frac{U(x)}{tf(x)}$$

### Idea

Let  $p = (p_1, \dots, p_n)$

$$p_i(\mathbf{x}, \mathbf{t}) = -\partial_{\mathbf{x}_i} f(\mathbf{x})\mathbf{t} + \mathbf{x}_i$$

Choose  $(c_1, \dots, c_n) \in \underbrace{\mathbb{R}^n \setminus \{\text{critical values of } p\}}_S$

Let  $q(\mathbf{x}, \mathbf{t}) = \frac{(\mathbf{x}_1 - c_1)^2 + \cdots + (\mathbf{x}_n - c_n)^2 + 1}{tf(\mathbf{x})}$

## 2. Termination: Proof Sketch

### Scratchwork

$$U(\mathbf{x}) = (\mathbf{x}_1 - c_1)^2 + \cdots + (\mathbf{x}_n - c_n)^2 + 1$$

$$g = \frac{f^2}{U^\gamma}$$

$$\nabla g(x) = 0 \wedge g(x) \neq 0 \iff \exists t \neq 0$$

$$c_i = -\partial_{\mathbf{x}_i} f(x)t + \mathbf{x}_i \wedge f(x) \neq 0 \quad i = 1, \dots, n$$

$$\gamma = \frac{U(x)}{tf(x)}$$

### Idea

Let  $p = (p_1, \dots, p_n)$

$$p_i(\mathbf{x}, \mathbf{t}) = -\partial_{\mathbf{x}_i} f(\mathbf{x})\mathbf{t} + \mathbf{x}_i$$

Choose  $(c_1, \dots, c_n) \in \underbrace{\mathbb{R}^n \setminus \{\text{critical values of } p\}}_S$

Let  $q(\mathbf{x}, \mathbf{t}) = \frac{(\mathbf{x}_1 - c_1)^2 + \cdots + (\mathbf{x}_n - c_n)^2 + 1}{tf(\mathbf{x})}$

## 2. Termination: Proof Sketch

### Scratchwork

$$U(\mathbf{x}) = (\mathbf{x}_1 - c_1)^2 + \cdots + (\mathbf{x}_n - c_n)^2 + 1$$

$$g = \frac{f^2}{U^\gamma}$$

$$\nabla g(x) = 0 \wedge g(x) \neq 0 \iff \exists t \neq 0 \quad c_i = -\partial_{\mathbf{x}_i} f(x)t + \mathbf{x}_i \wedge f(x) \neq 0 \quad i = 1, \dots, n$$

$$\gamma = \frac{U(x)}{tf(x)}$$

### Idea

Let  $p = (p_1, \dots, p_n)$

$$p_i(\mathbf{x}, \mathbf{t}) = -\partial_{\mathbf{x}_i} f(\mathbf{x})\mathbf{t} + \mathbf{x}_i$$

Choose  $(c_1, \dots, c_n) \in \underbrace{\mathbb{R}^n \setminus \{\text{critical values of } p\}}_S$

Let  $q(\mathbf{x}, \mathbf{t}) = \frac{(\mathbf{x}_1 - c_1)^2 + \cdots + (\mathbf{x}_n - c_n)^2 + 1}{tf(\mathbf{x})}$

Choose  $\gamma \in \mathbb{R} \setminus \{\text{critical values of } q\}$

## 2. Termination: Proof Sketch

### Scratchwork

$$U(\mathbf{x}) = (\mathbf{x}_1 - c_1)^2 + \cdots + (\mathbf{x}_n - c_n)^2 + 1$$

$$g = \frac{f^2}{U^\gamma}$$

$$\nabla g(x) = 0 \wedge g(x) \neq 0 \iff \exists t \neq 0 \quad c_i = -\partial_{\mathbf{x}_i} f(x)t + \mathbf{x}_i \wedge f(x) \neq 0 \quad i = 1, \dots, n$$

$$\gamma = \frac{U(x)}{tf(x)}$$

### Idea

Let  $p = (p_1, \dots, p_n)$

$$p_i(\mathbf{x}, \mathbf{t}) = -\partial_{\mathbf{x}_i} f(\mathbf{x})\mathbf{t} + \mathbf{x}_i$$

Choose  $(c_1, \dots, c_n) \in \underbrace{\mathbb{R}^n \setminus \{\text{critical values of } p\}}_S$

$$\text{Let } q(\mathbf{x}, \mathbf{t}) = \frac{(\mathbf{x}_1 - c_1)^2 + \cdots + (\mathbf{x}_n - c_n)^2 + 1}{\mathbf{t}f(\mathbf{x})}$$

Choose  $\gamma \in \mathbb{R} \setminus \{\text{critical values of } q\} = \mathbb{R}$

## 2. Termination: Proof Sketch

### Scratchwork

$$U(x) = (x_1 - c_1)^2 + \cdots + (x_n - c_n)^2 + 1$$

$$g = \frac{f^2}{U^\gamma}$$

$$\nabla g(x) = 0 \wedge g(x) \neq 0 \iff \exists t \neq 0$$

$$c_i = -\partial_{x_i} f(x)t + x_i \wedge f(x) \neq 0 \quad i = 1, \dots, n$$

$$\gamma = \frac{U(x)}{tf(x)}$$

### Idea

$$\text{Let } p = (p_1, \dots, p_n)$$

Sard's Thm + Constant Rank Thm

$$p_i(x, t) = -\partial_{x_i} f(x)t + x_i$$

$\implies$

Choose  $(c_1, \dots, c_n) \in \underbrace{\mathbb{R}^n \setminus \{\text{critical values of } p\}}_S$

$$\text{Let } q(x, t) = \frac{(x_1 - c_1)^2 + \cdots + (x_n - c_n)^2 + 1}{tf(x)}$$

Choose  $\gamma \in \mathbb{R} \setminus \{\text{critical values of } q\} = \mathbb{R}$

dimension of

$$\left\{ (x, t) \in \mathbb{R}^n \times \mathbb{R} \mid \begin{array}{l} p(x, t) - c = 0 \wedge \\ q(x, t) - \gamma = 0 \wedge \\ f(x) \neq 0 \wedge \\ t \neq 0 \wedge \\ \gamma \neq 0 \end{array} \right\}$$

is zero

## 2. Termination: Proof Sketch

### Scratchwork

$$U(x) = (x_1 - c_1)^2 + \cdots + (x_n - c_n)^2 + 1$$

$$g = \frac{f^2}{U^\gamma}$$

$$\nabla g(x) = 0 \wedge g(x) \neq 0 \iff \exists t \neq 0 \quad c_i = -\partial_{x_i} f(x)t + x_i \wedge f(x) \neq 0 \quad i = 1, \dots, n$$

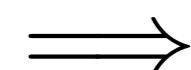
$$\gamma = \frac{U(x)}{tf(x)}$$

### Idea

Let  $p = (p_1, \dots, p_n)$

Sard's Thm + Constant Rank Thm

$$p_i(x, t) = -\partial_{x_i} f(x)t + x_i$$



Choose  $(c_1, \dots, c_n) \in \underbrace{\mathbb{R}^n \setminus \{\text{critical values of } p\}}_S$

$$\text{Let } q(x, t) = \frac{(x_1 - c_1)^2 + \cdots + (x_n - c_n)^2 + 1}{tf(x)}$$

Choose  $\gamma \in \mathbb{R} \setminus \{\text{critical values of } q\} = \mathbb{R}$

dimension of

$$\left\{ (x, t) \in \mathbb{R}^n \times \mathbb{R} \mid \begin{array}{l} p(x, t) - c = 0 \wedge \\ q(x, t) - \gamma = 0 \wedge \\ f(x) \neq 0 \wedge \\ t \neq 0 \wedge \\ \gamma \neq 0 \end{array} \right\}$$

is zero

## 2. Termination: Proof Sketch

### Scratchwork

$$U(x) = (x_1 - c_1)^2 + \cdots + (x_n - c_n)^2 + 1$$

$$g = \frac{f^2}{U^\gamma}$$

$$\nabla g(x) = 0 \wedge g(x) \neq 0 \iff \exists t \neq 0 \quad p(x, t) - c = 0 \quad \wedge f(x) \neq 0$$

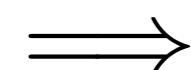
$$\gamma = \frac{U(x)}{tf(x)}$$

### Idea

Let  $p = (p_1, \dots, p_n)$

Sard's Thm + Constant Rank Thm

$$p_i(x, t) = -\partial_{x_i} f(x)t + x_i$$



Choose  $(c_1, \dots, c_n) \in \underbrace{\mathbb{R}^n \setminus \{\text{critical values of } p\}}_S$

Let  $q(x, t) = \frac{(x_1 - c_1)^2 + \cdots + (x_n - c_n)^2 + 1}{tf(x)}$

Choose  $\gamma \in \mathbb{R} \setminus \{\text{critical values of } q\} = \mathbb{R}$

dimension of

$$\left\{ (x, t) \in \mathbb{R}^n \times \mathbb{R} \mid \begin{array}{l} p(x, t) - c = 0 \wedge \\ q(x, t) - \gamma = 0 \wedge \\ f(x) \neq 0 \wedge \\ t \neq 0 \wedge \\ \gamma \neq 0 \end{array} \right\}$$

is zero

## 2. Termination: Proof Sketch

### Scratchwork

$$U(x) = (x_1 - c_1)^2 + \cdots + (x_n - c_n)^2 + 1$$

$$g = \frac{f^2}{U^\gamma}$$

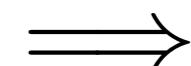
$$\nabla g(x) = 0 \wedge g(x) \neq 0 \iff \boxed{\begin{array}{l} \exists t \neq 0 \quad p(x, t) - c = 0 \\ \quad \quad \quad \wedge f(x) \neq 0 \\ q(x, t) - \gamma = 0 \end{array}}$$

### Idea

Let  $p = (p_1, \dots, p_n)$

$$p_i(x, t) = -\partial_{x_i} f(x)t + x_i$$

Sard's Thm + Constant Rank Thm



Choose  $(c_1, \dots, c_n) \in \underbrace{\mathbb{R}^n \setminus \{\text{critical values of } p\}}_S$

Let  $q(x, t) = \frac{(x_1 - c_1)^2 + \cdots + (x_n - c_n)^2 + 1}{t f(x)}$

Choose  $\gamma \in \mathbb{R} \setminus \{\text{critical values of } q\} = \mathbb{R}$

dimension of  

$$\left\{ \begin{array}{l} p(x, t) - c = 0 \\ q(x, t) - \gamma = 0 \\ f(x) \neq 0 \\ t \neq 0 \\ \gamma \neq 0 \end{array} \right\}$$
  
 is zero

## 2. Termination: Proof Sketch

### Scratchwork

$$U(\mathbf{x}) = (\mathbf{x}_1 - c_1)^2 + \cdots + (\mathbf{x}_n - c_n)^2 + 1$$

$$g = \frac{f^2}{U^\gamma}$$

$$\nabla g(x) = 0 \wedge g(x) \neq 0 \iff \exists t \neq 0 \quad p(x, t) - c = 0 \quad \wedge f(x) \neq 0$$

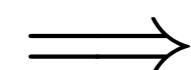
$$q(x, t) - \gamma = 0$$

### Idea

Let  $p = (p_1, \dots, p_n)$

$$p_i(\mathbf{x}, \mathbf{t}) = -\partial_{\mathbf{x}_i} f(\mathbf{x}) \mathbf{t} + \mathbf{x}_i$$

Sard's Thm + Constant Rank Thm



Choose  $(c_1, \dots, c_n) \in \underbrace{\mathbb{R}^n \setminus \{\text{critical values of } p\}}_S$

$$\text{Let } q(\mathbf{x}, \mathbf{t}) = \frac{(\mathbf{x}_1 - c_1)^2 + \cdots + (\mathbf{x}_n - c_n)^2 + 1}{\mathbf{t} f(\mathbf{x})}$$

Choose  $\gamma \in \mathbb{R} \setminus \{\text{critical values of } q\} = \mathbb{R}$

$$\left\{ \begin{array}{l} (x, t) \in \mathbb{R}^n \times \mathbb{R} \\ \left| \begin{array}{l} p(x, t) - c = 0 \quad \wedge \\ q(x, t) - \gamma = 0 \quad \wedge \\ f(x) \neq 0 \quad \wedge \\ t \neq 0 \quad \wedge \\ \gamma \neq 0 \end{array} \right. \end{array} \right.$$

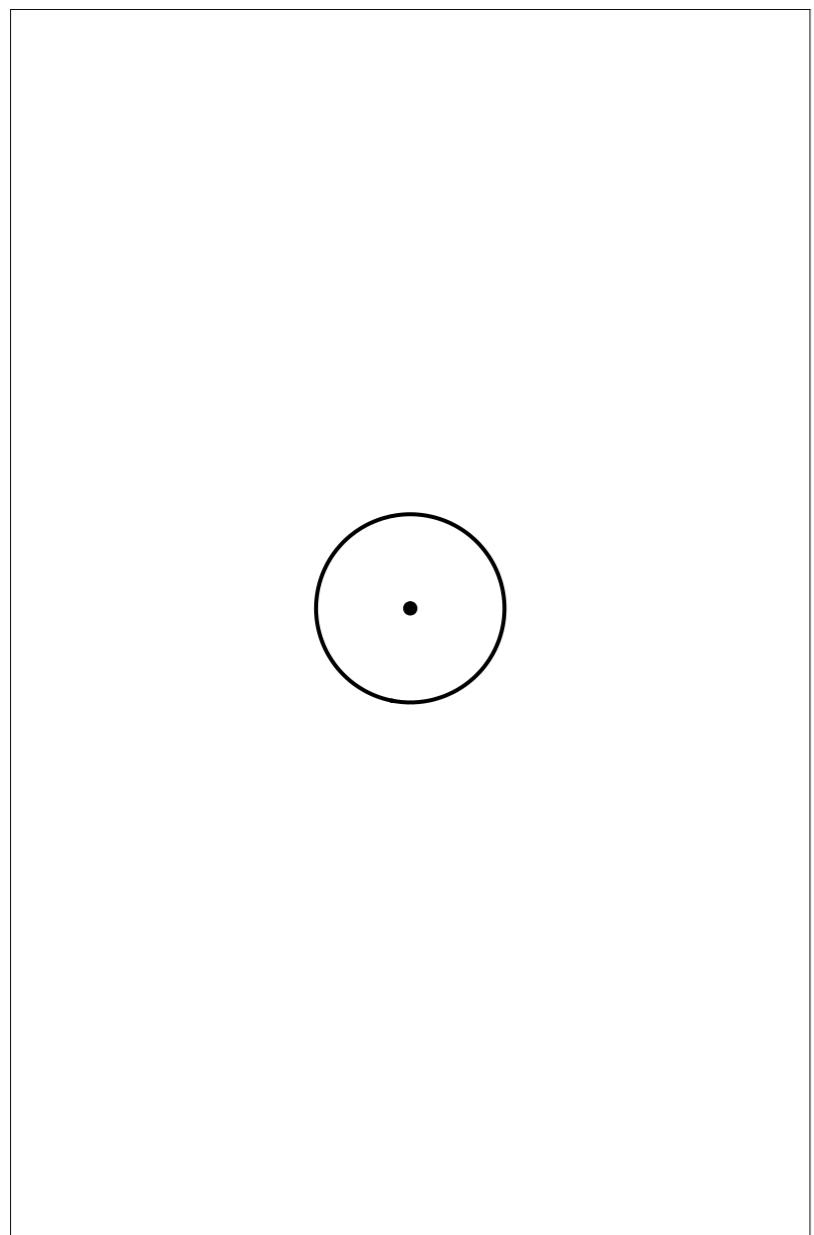
is zero

## 2. Termination: Example

$$f = \frac{1}{4} (x_1^2 + x_2^2 - 2) (x_1^2 + x_2^2)$$

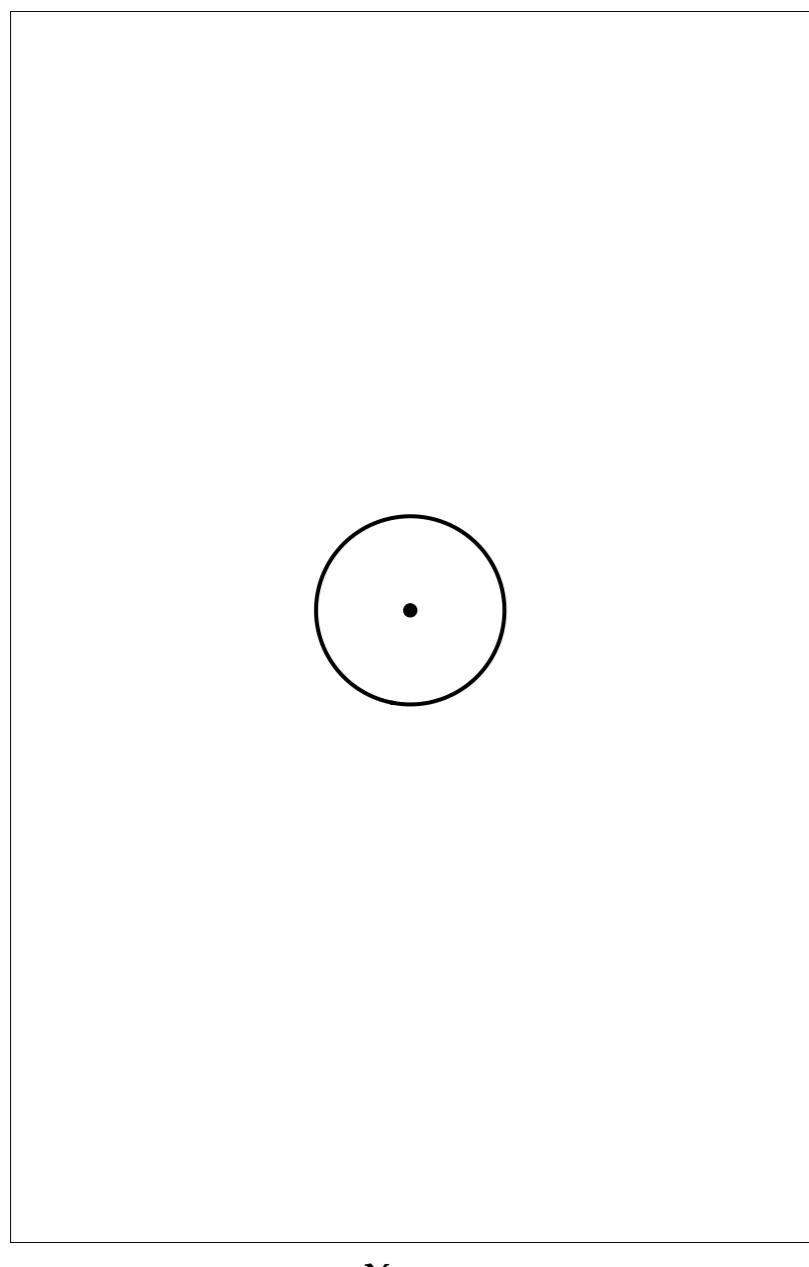
## 2. Termination: Example

$$f = \frac{1}{4} (x_1^2 + x_2^2 - 2) (x_1^2 + x_2^2)$$



## 2. Termination: Example

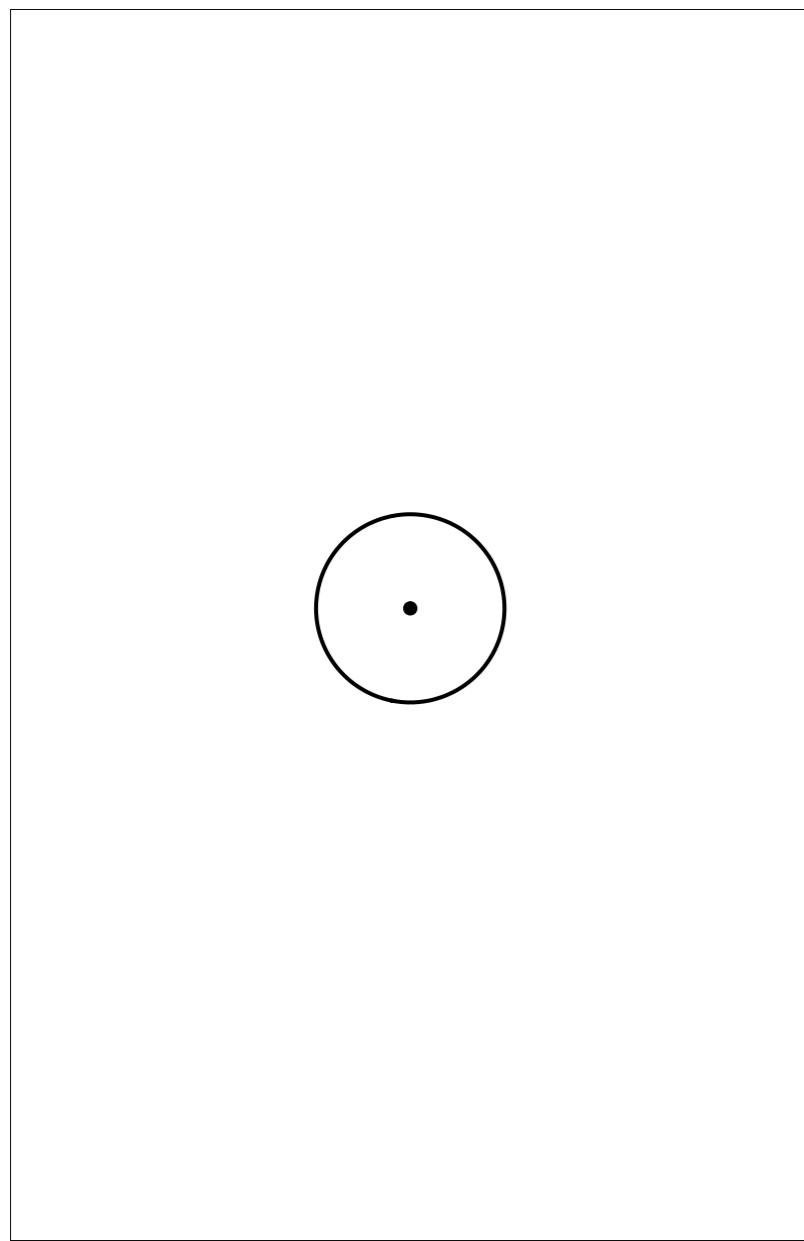
$$f = \frac{1}{4} (x_1^2 + x_2^2 - 2) (x_1^2 + x_2^2)$$



black:  $\{f=0\}$       white:  $\{f \neq 0\}$

## 2. Termination: Example

$$f = \frac{1}{4} (x_1^2 + x_2^2 - 2) (x_1^2 + x_2^2)$$

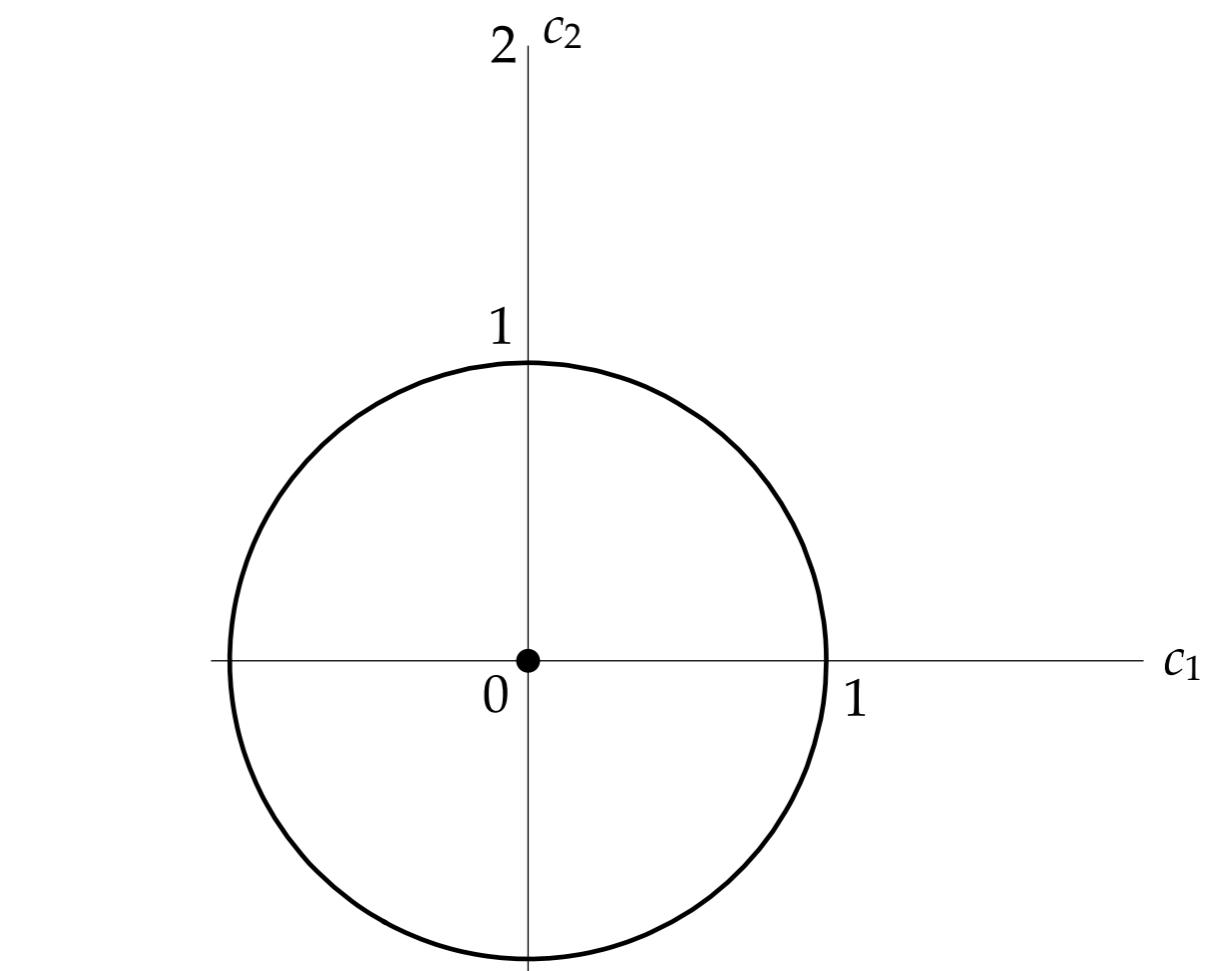


black:  $\{f=0\}$

white:  $\{f \neq 0\}$

$x_2$

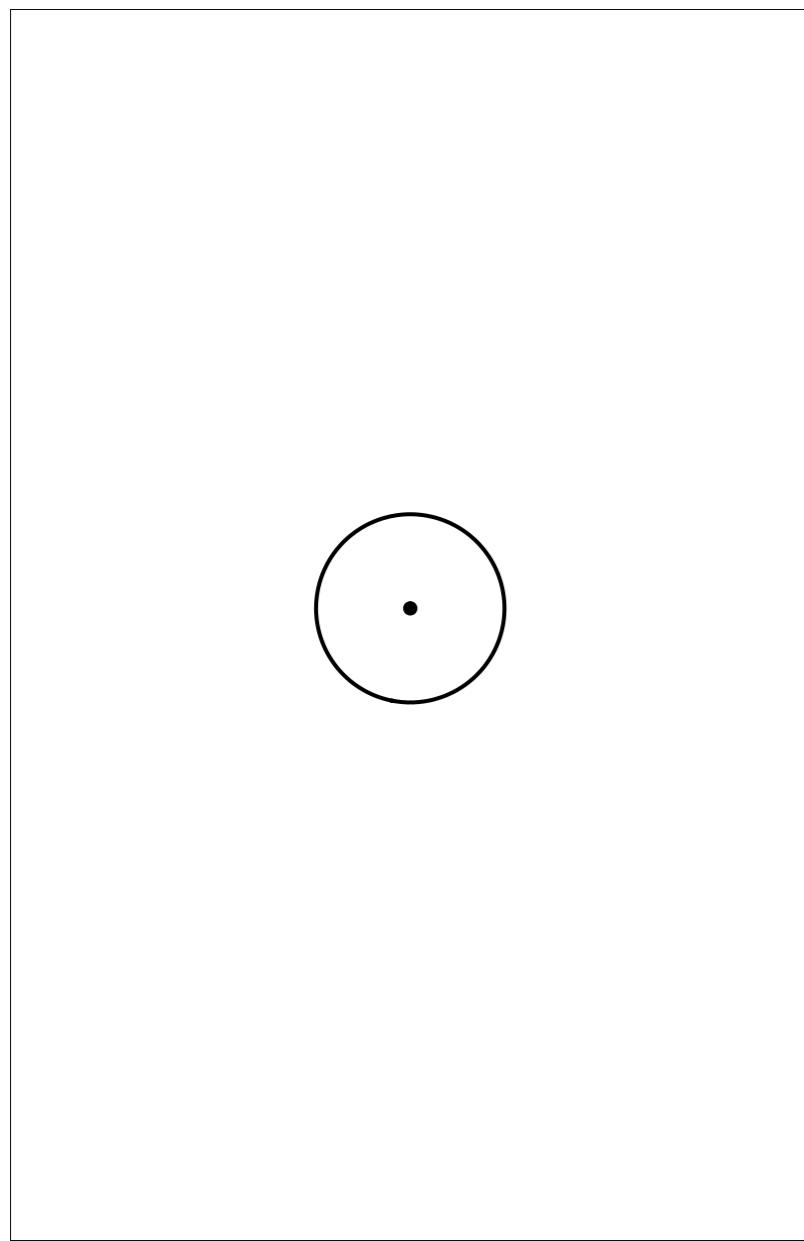
$x_1$



black: critical values of  $p$   
white:  $S = \mathbb{R}^n \setminus \{\text{critical values of } p\}$

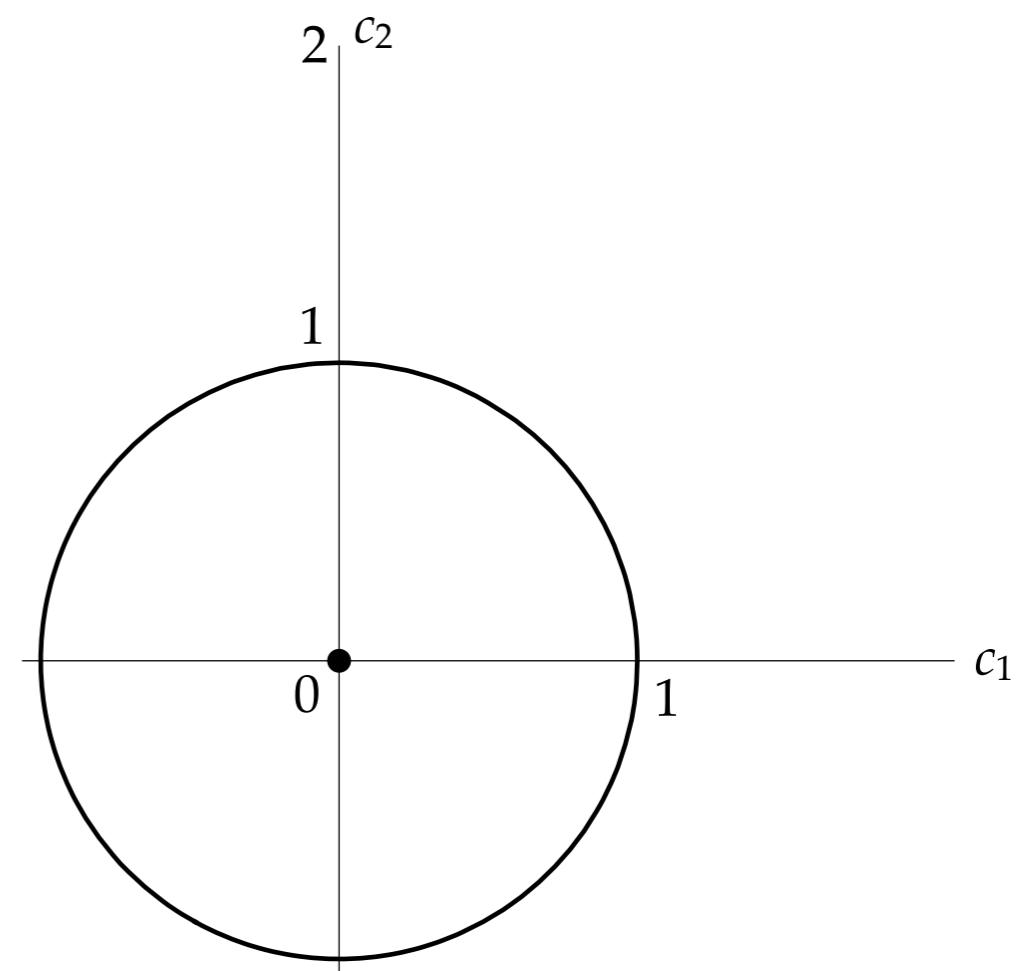
## 2. Termination: Example

$$f = \frac{1}{4} (x_1^2 + x_2^2 - 2) (x_1^2 + x_2^2)$$



black:  $\{f=0\}$

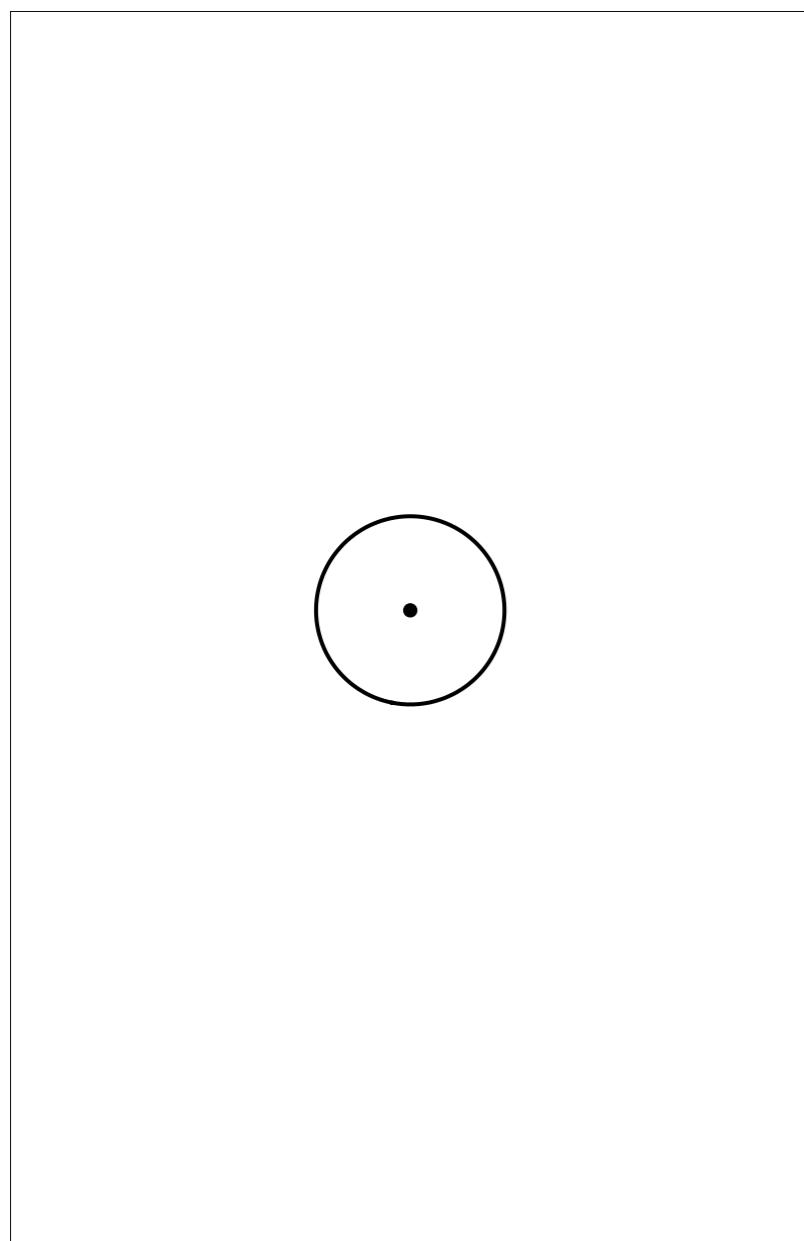
white:  $\{f \neq 0\}$



black: critical values of  $p$   
white:  $S = \mathbb{R}^n \setminus \{\text{critical values of } p\}$   
codim  $S < n = 2$

## 2. Termination: Example

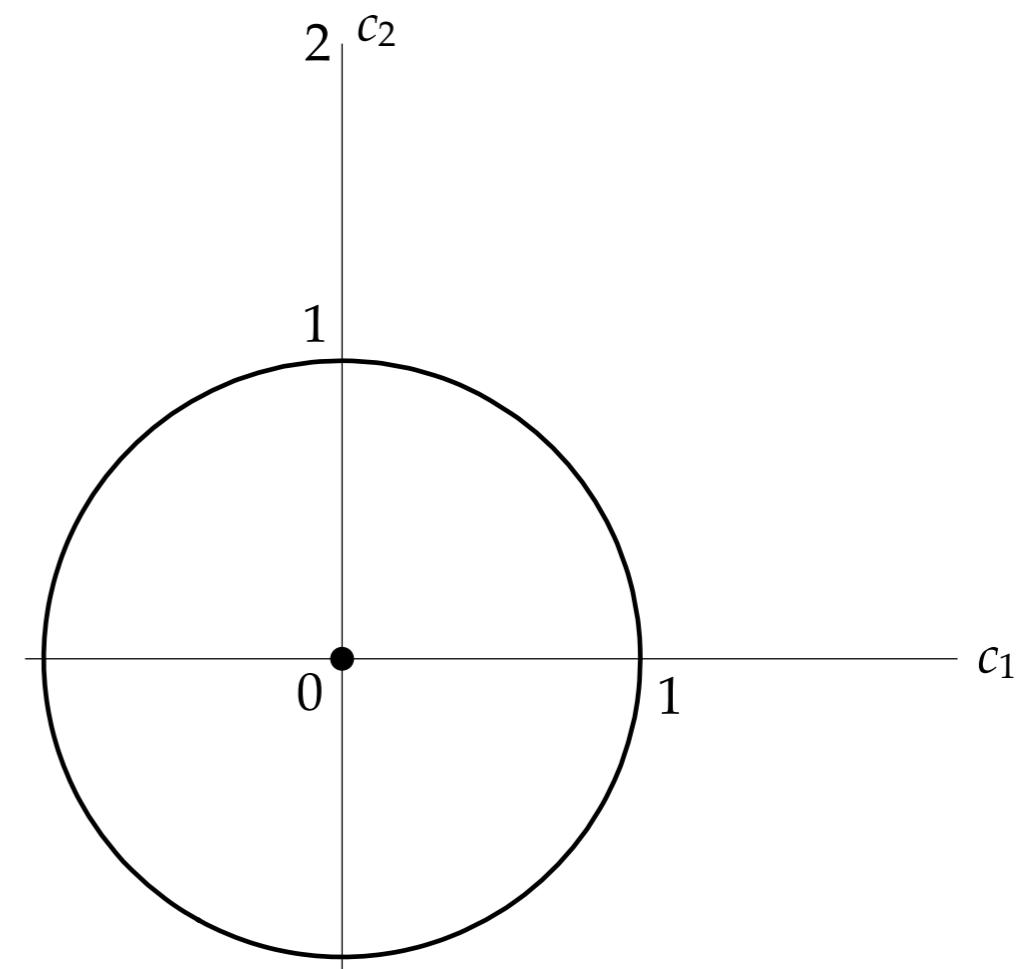
$$f = \frac{1}{4} (x_1^2 + x_2^2 - 2) (x_1^2 + x_2^2)$$



black:  $\{f=0\}$

white:  $\{f \neq 0\}$

$$g = \frac{f^2}{((x_1 - c_1)^2 + (x_2 - c_2)^2 + 1)^5}$$



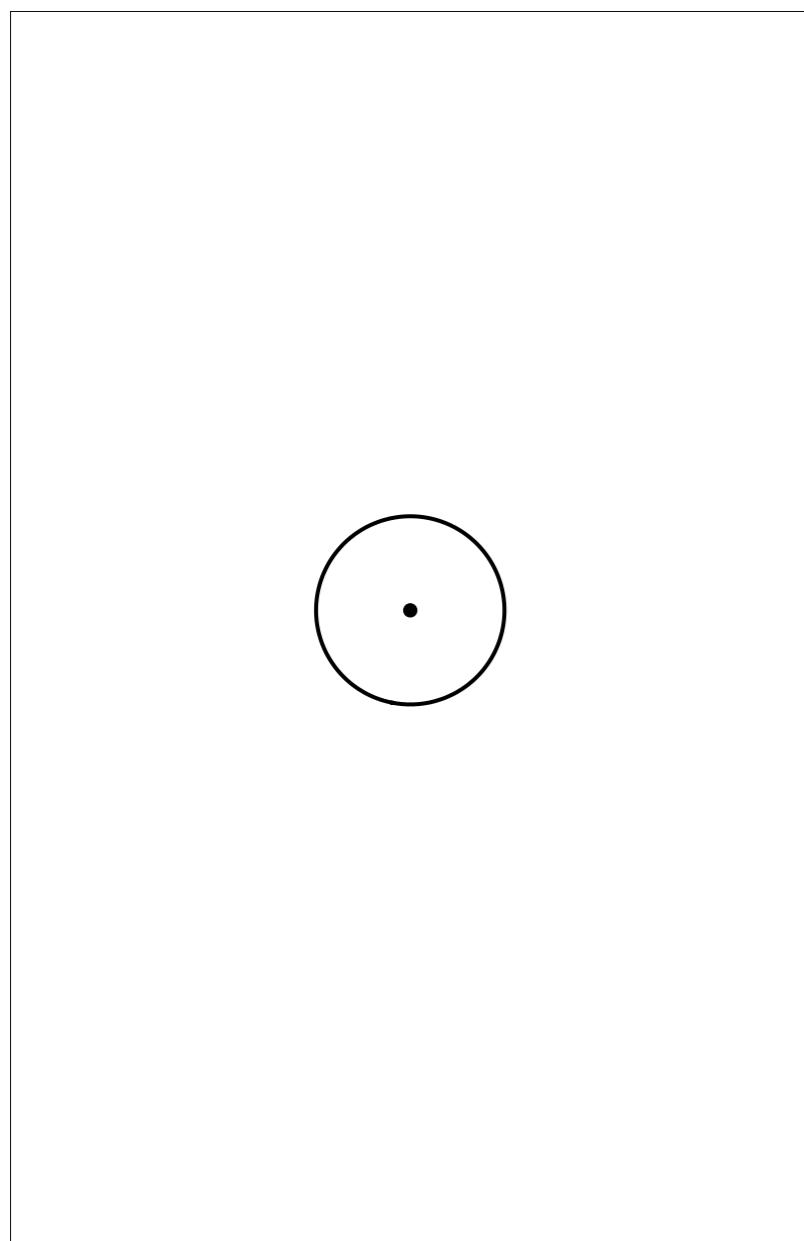
black: critical values of  $p$

white:  $S = \mathbb{R}^n \setminus \{\text{critical values of } p\}$

codim  $S < n = 2$

## 2. Termination: Example

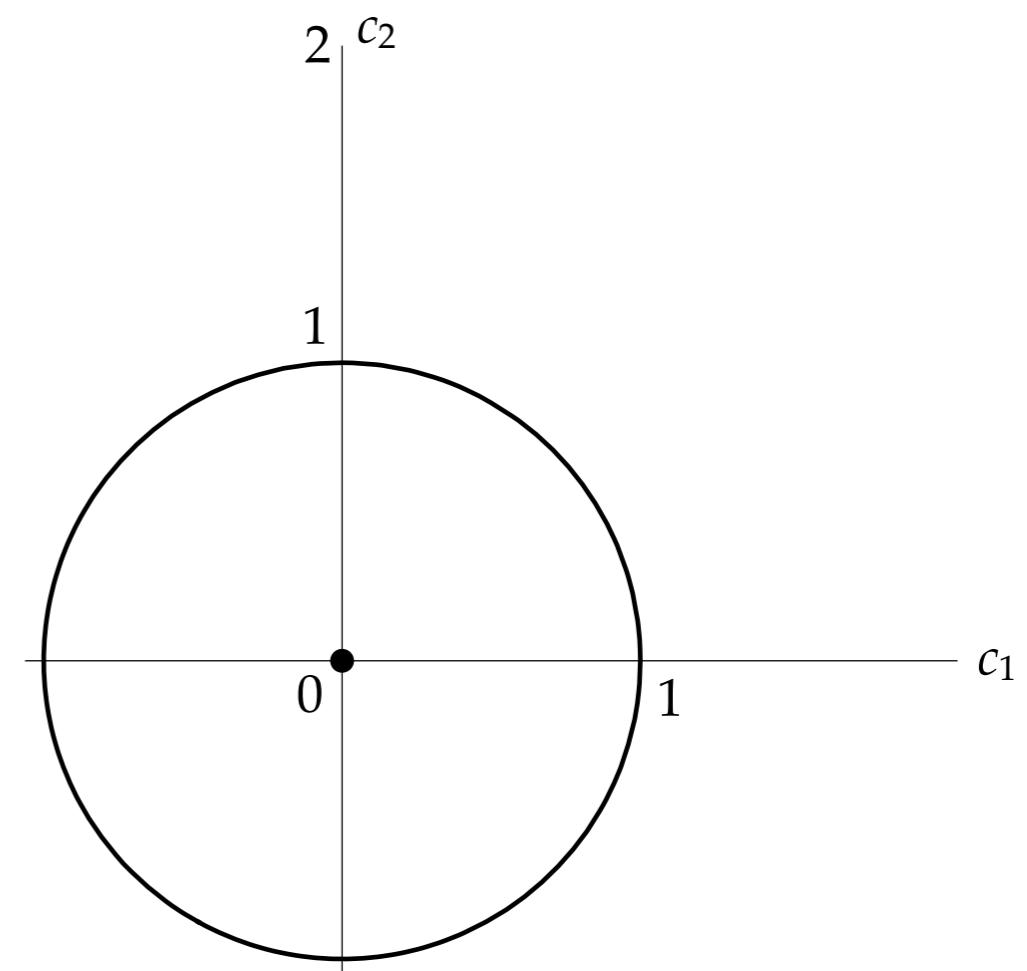
$$f = \frac{1}{4} (x_1^2 + x_2^2 - 2) (x_1^2 + x_2^2)$$



black:  $\{f=0\}$

white:  $\{f \neq 0\}$

$$g = \frac{f^2}{((x_1 - 0)^2 + (x_2 - 0)^2 + 1)^5}$$



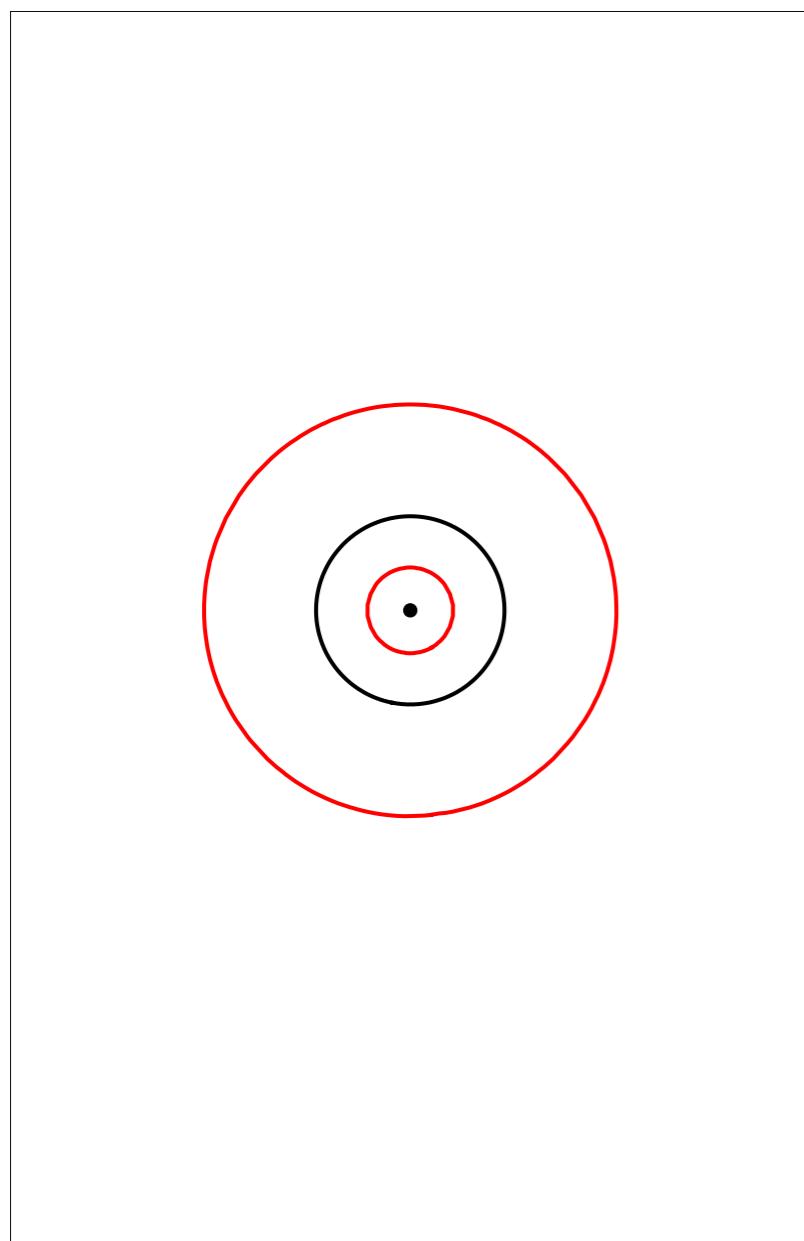
black: critical values of  $p$

white:  $S = \mathbb{R}^n \setminus \{\text{critical values of } p\}$

codim  $S < n = 2$

## 2. Termination: Example

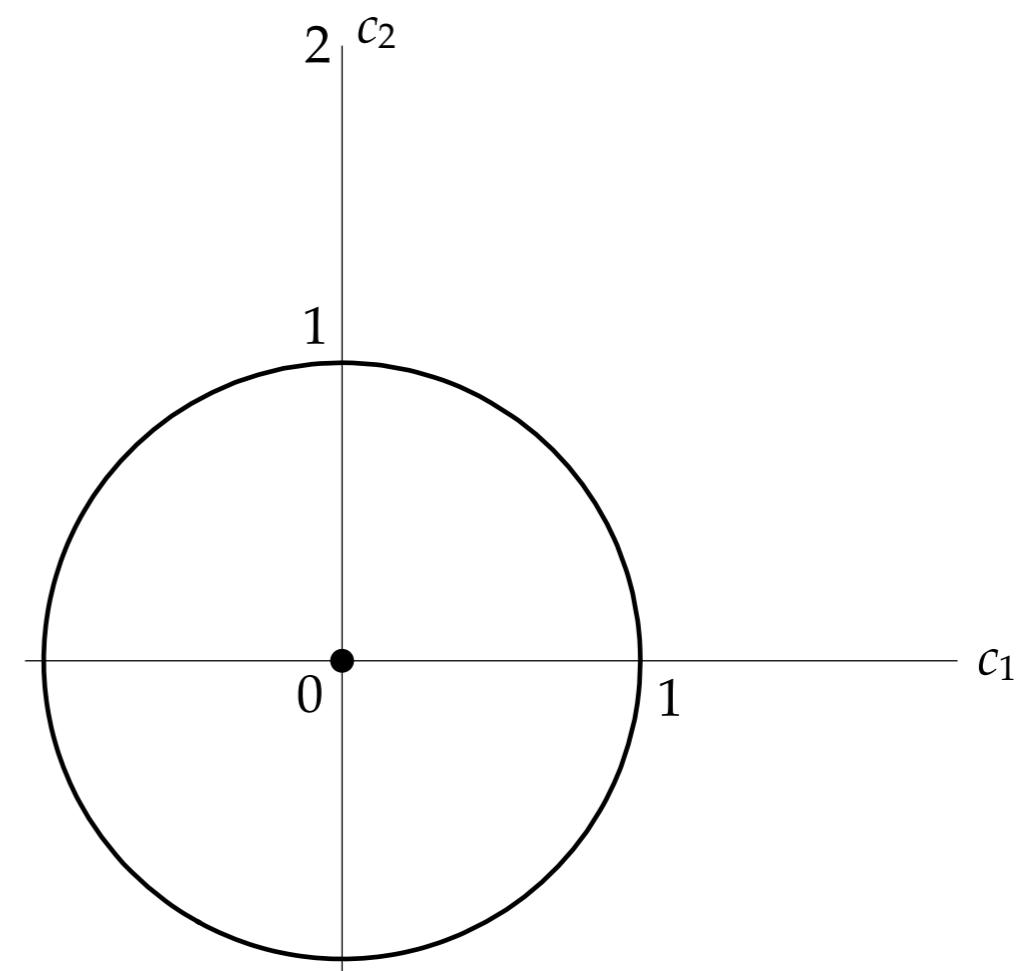
$$f = \frac{1}{4} (x_1^2 + x_2^2 - 2) (x_1^2 + x_2^2)$$



black:  $\{f=0\}$

white:  $\{f \neq 0\}$

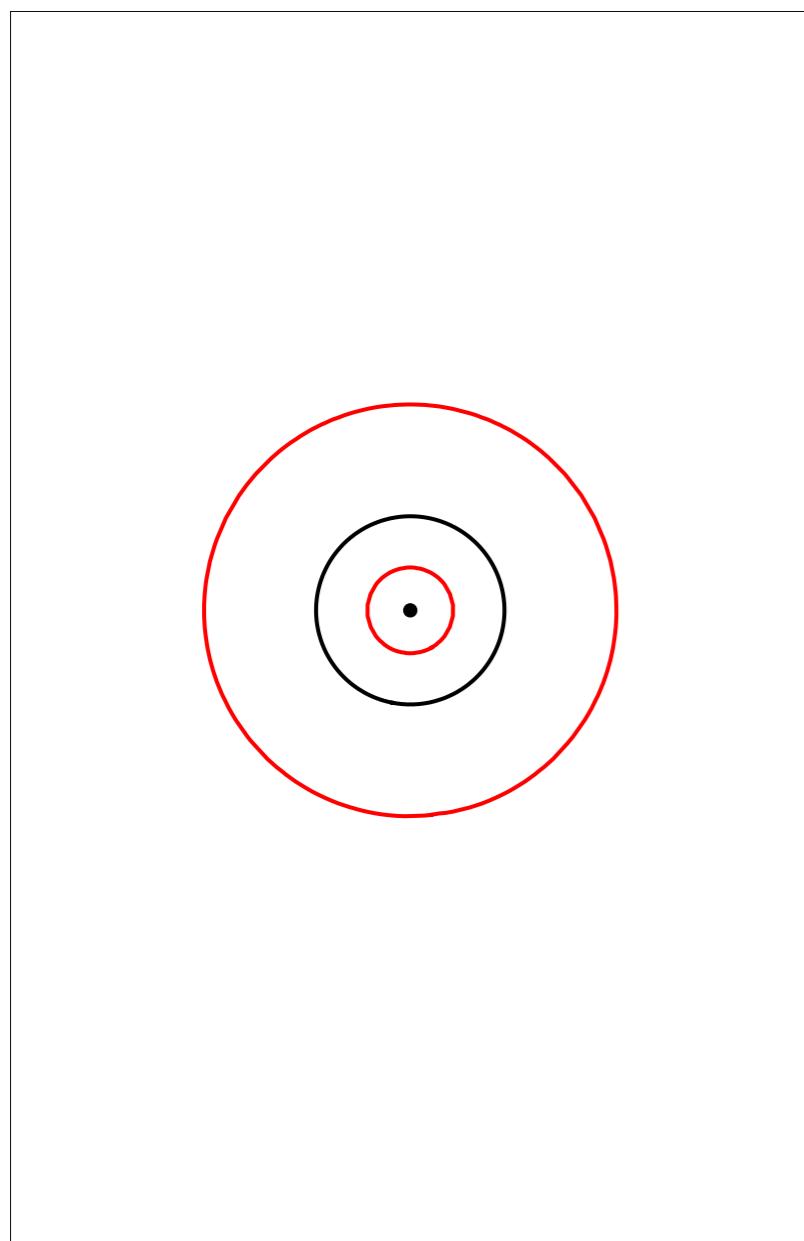
$$g = \frac{f^2}{((x_1 - 0)^2 + (x_2 - 0)^2 + 1)^5}$$



black: critical values of  $p$   
white:  $S = \mathbb{R}^n \setminus \{\text{critical values of } p\}$   
codim  $S < n = 2$

## 2. Termination: Example

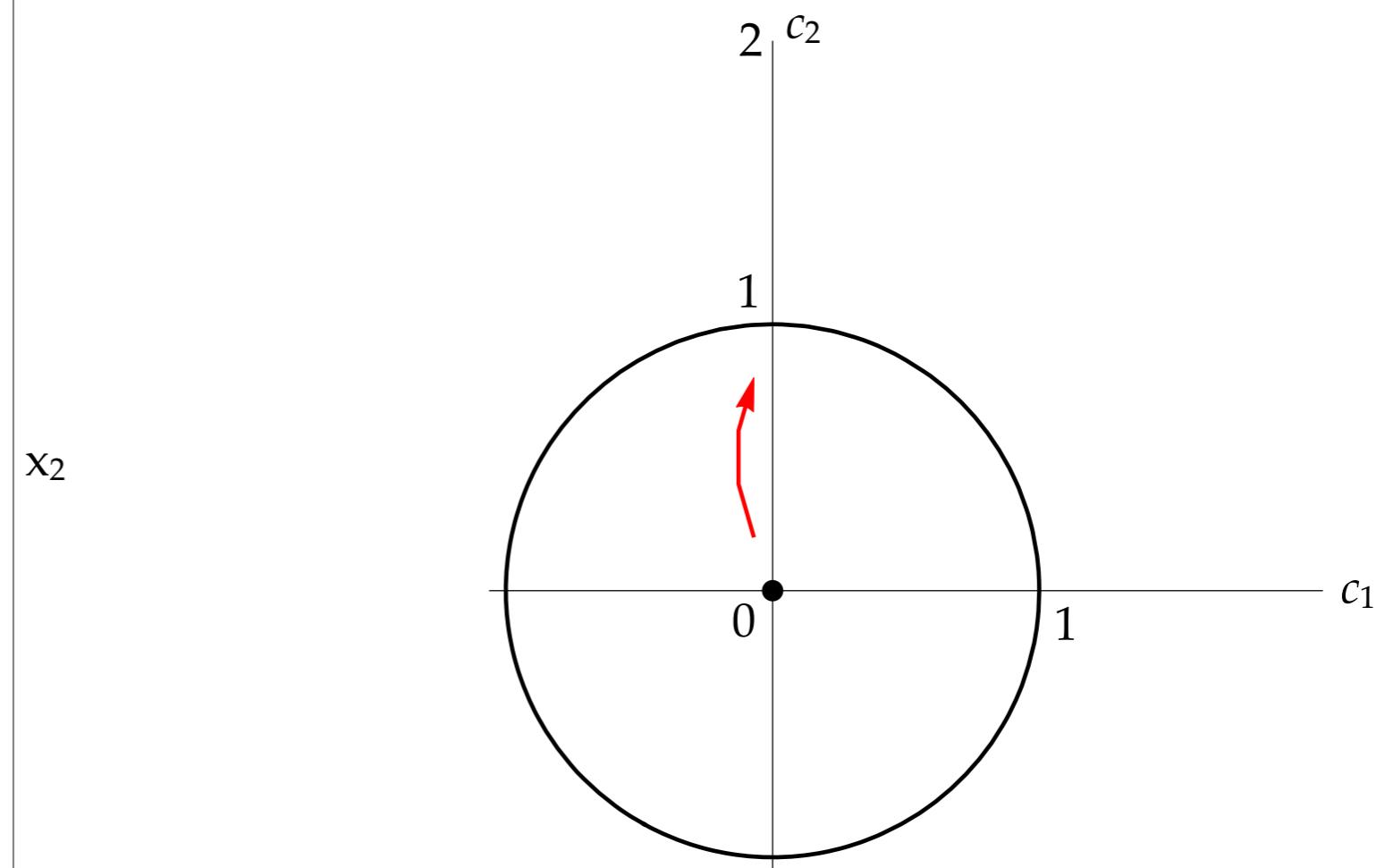
$$f = \frac{1}{4} (x_1^2 + x_2^2 - 2) (x_1^2 + x_2^2)$$



black:  $\{f=0\}$

white:  $\{f \neq 0\}$

$$g = \frac{f^2}{((x_1 - 0)^2 + (x_2 - 0)^2 + 1)^5}$$



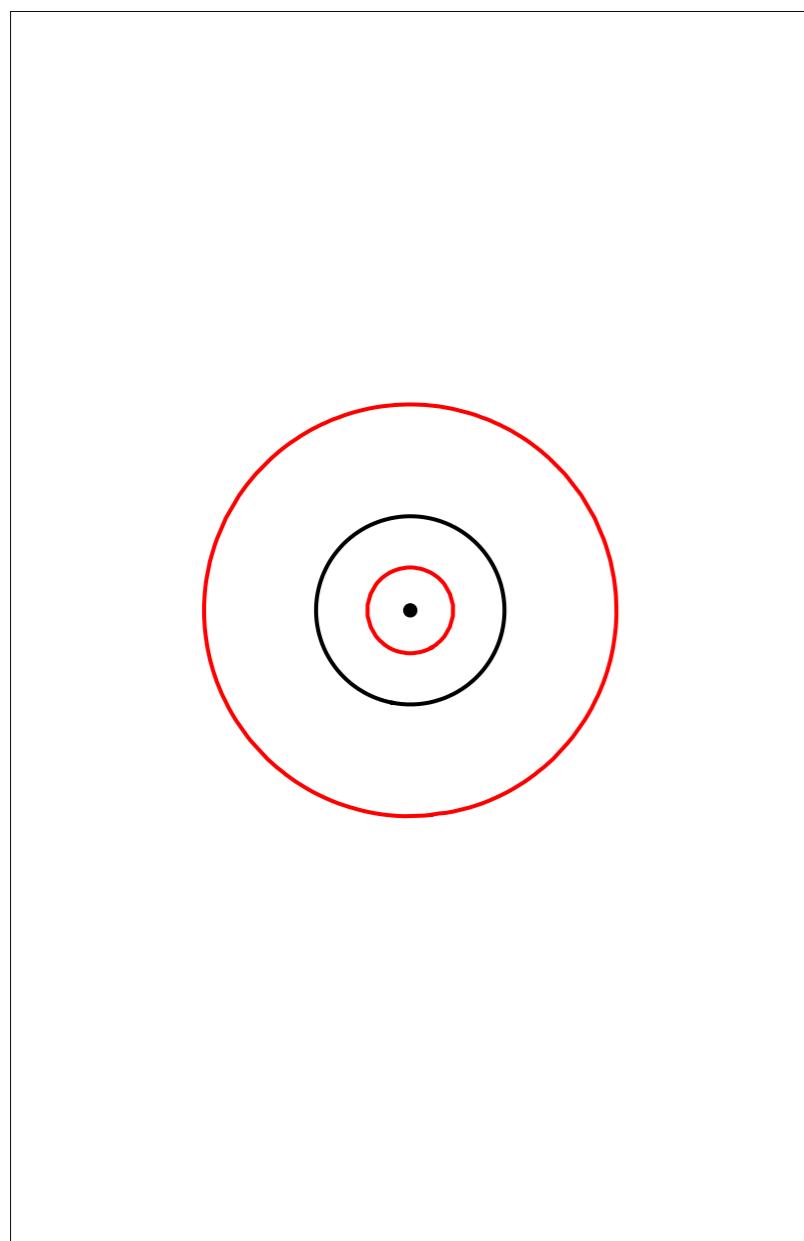
black: critical values of  $p$

white:  $S = \mathbb{R}^n \setminus \{\text{critical values of } p\}$

codim  $S < n = 2$

## 2. Termination: Example

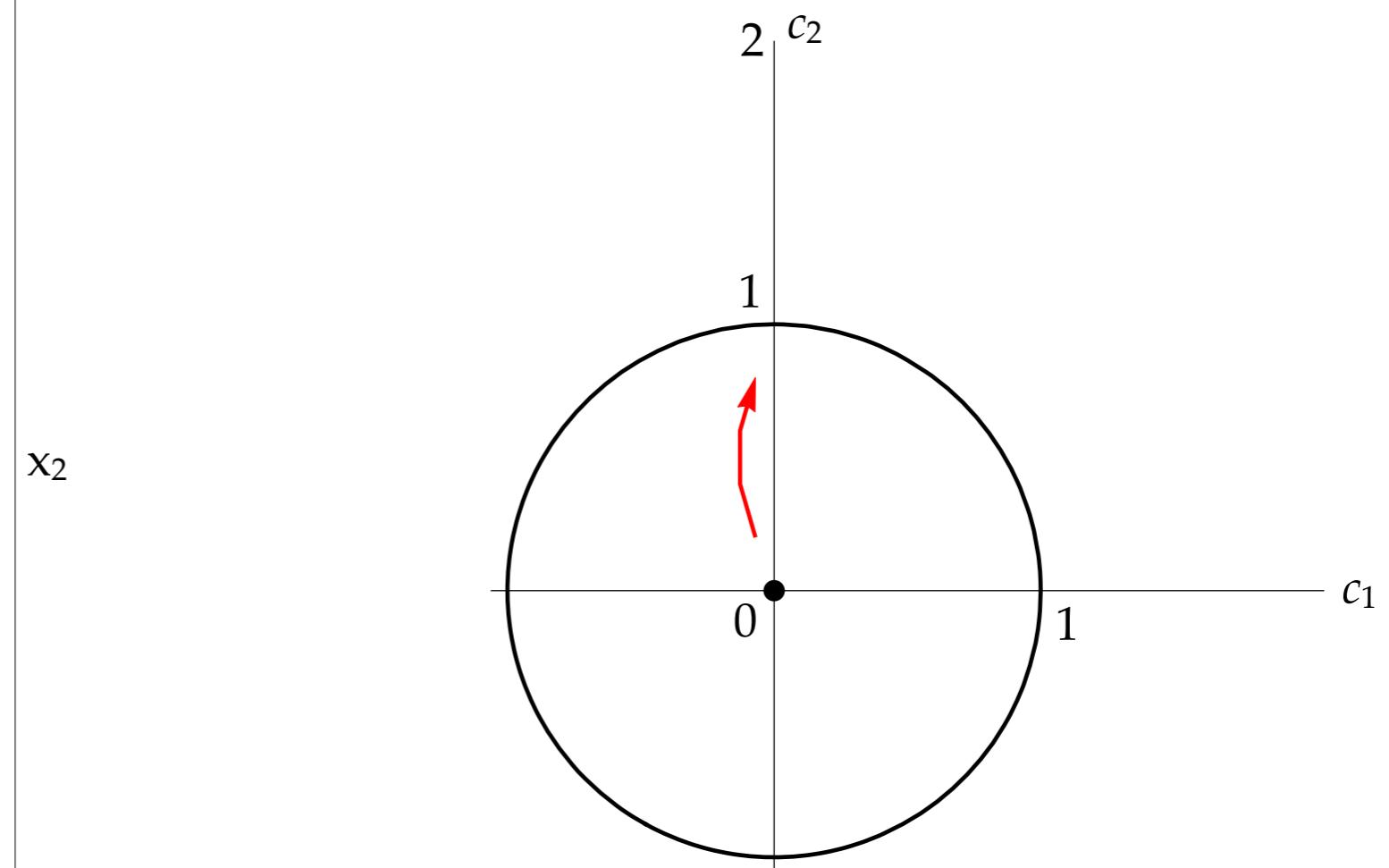
$$f = \frac{1}{4} (x_1^2 + x_2^2 - 2) (x_1^2 + x_2^2)$$



black:  $\{f=0\}$

white:  $\{f \neq 0\}$

$$g = \frac{f^2}{((x_1 - 0)^2 + (x_2 - 1)^2 + 1)^5}$$



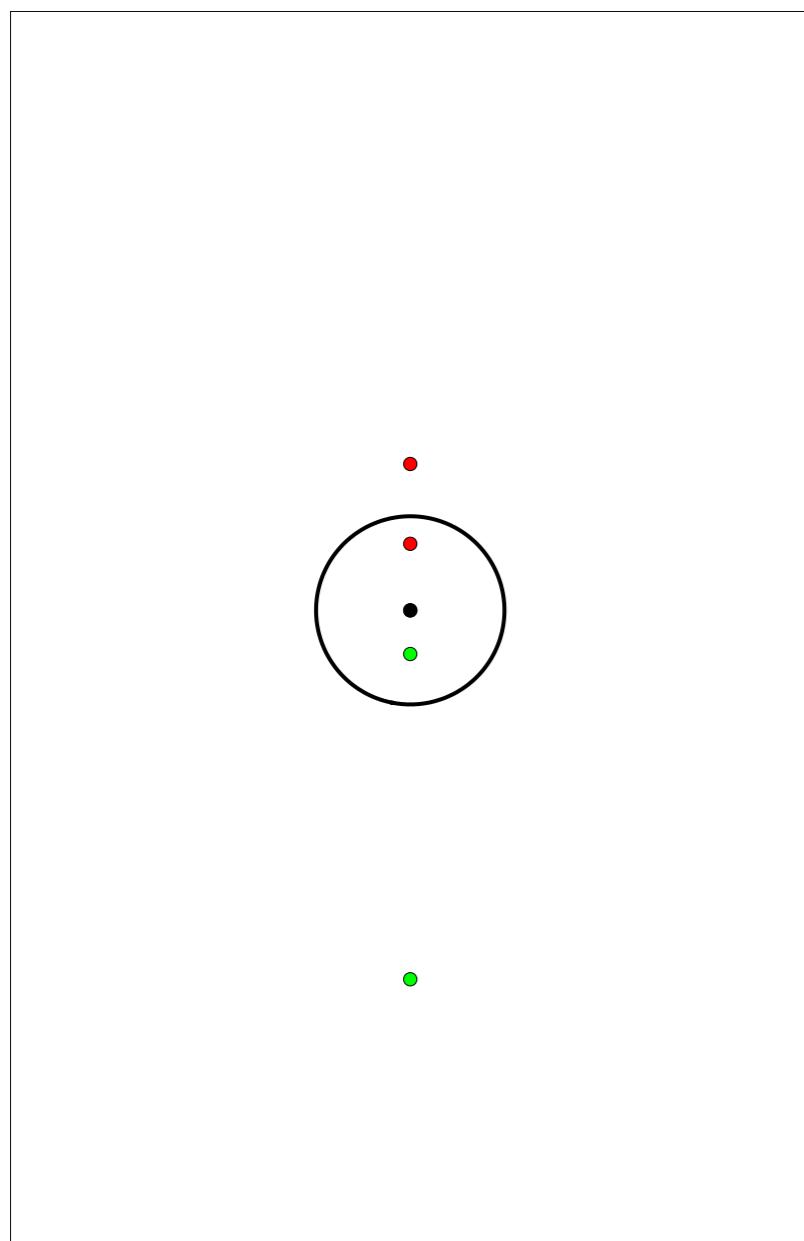
black: critical values of  $p$

white:  $S = \mathbb{R}^n \setminus \{\text{critical values of } p\}$

codim  $S < n = 2$

## 2. Termination: Example

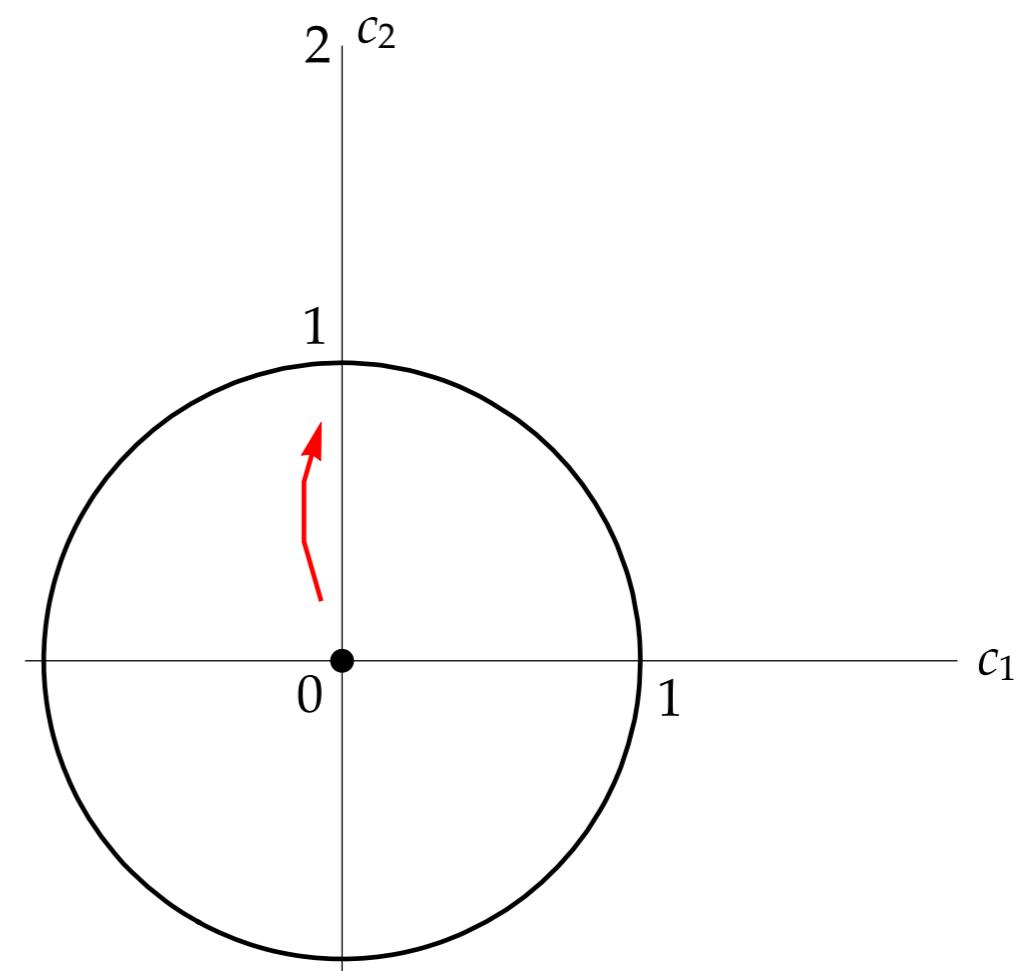
$$f = \frac{1}{4} (x_1^2 + x_2^2 - 2) (x_1^2 + x_2^2)$$



black:  $\{f=0\}$

white:  $\{f \neq 0\}$

$$g = \frac{f^2}{((x_1 - 0)^2 + (x_2 - 1)^2 + 1)^5}$$



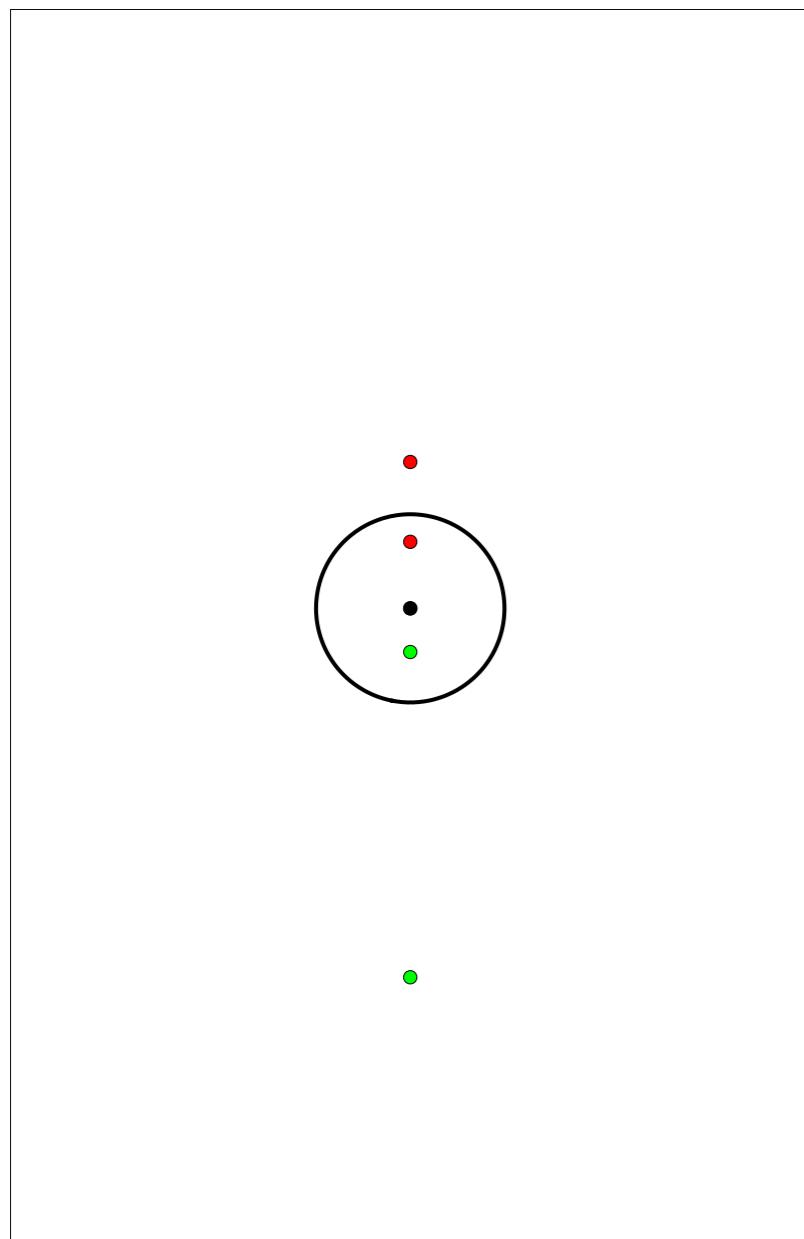
black: critical values of  $p$

white:  $S = \mathbb{R}^n \setminus \{\text{critical values of } p\}$

codim  $S < n = 2$

## 2. Termination: Example

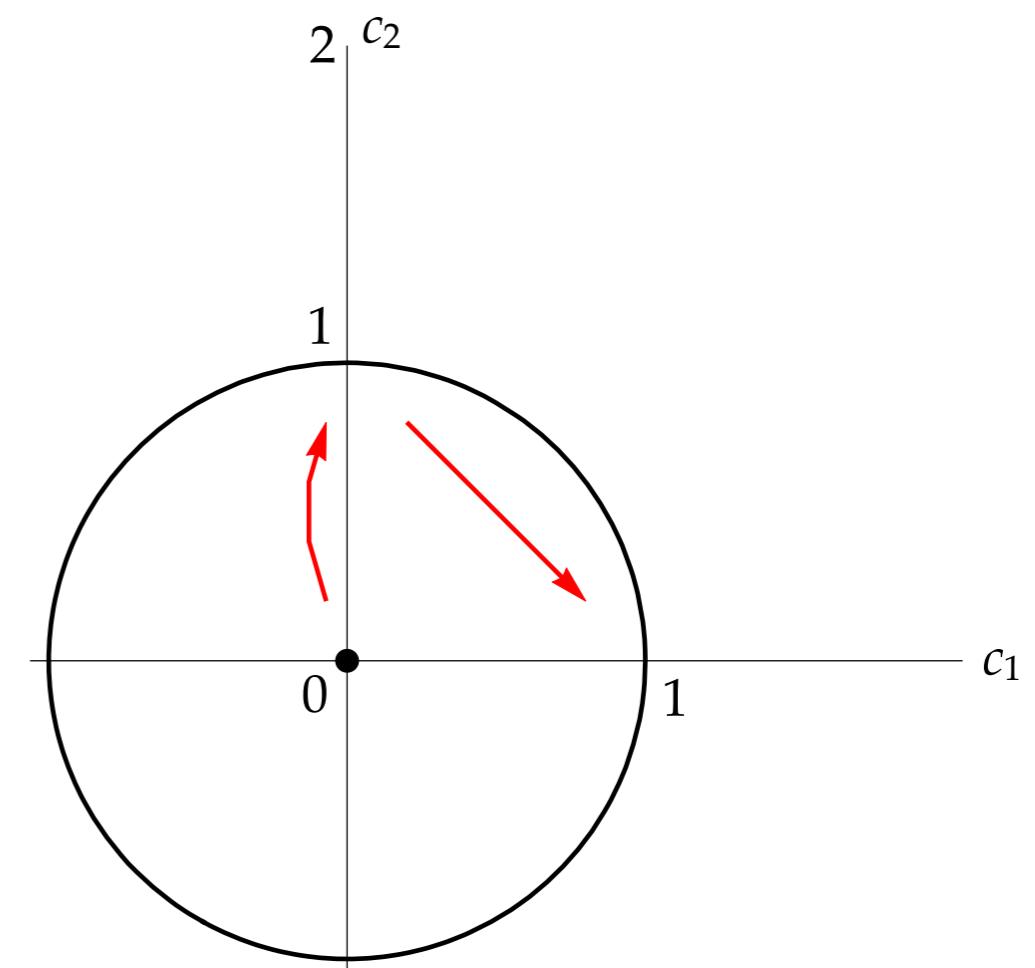
$$f = \frac{1}{4} (x_1^2 + x_2^2 - 2) (x_1^2 + x_2^2)$$



black:  $\{f=0\}$

white:  $\{f \neq 0\}$

$$g = \frac{f^2}{((x_1 - 0)^2 + (x_2 - 1)^2 + 1)^5}$$



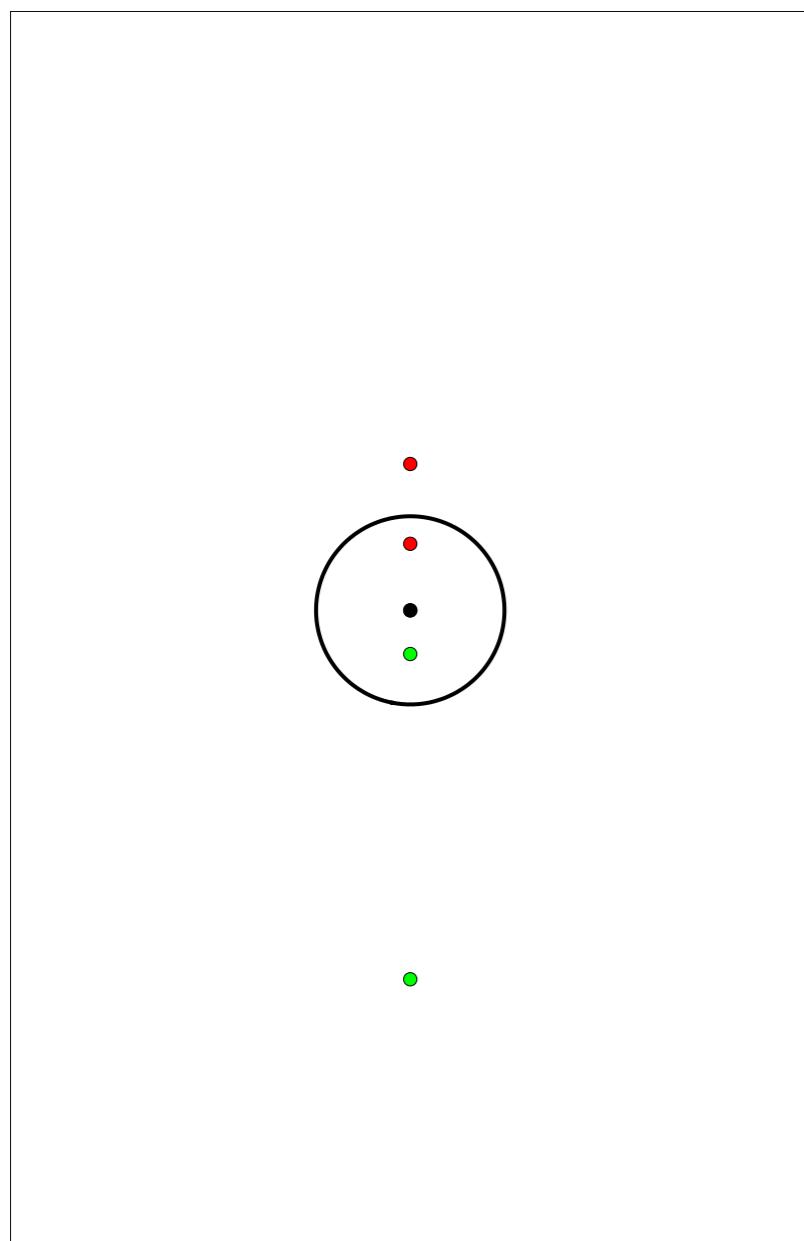
black: critical values of  $p$

white:  $S = \mathbb{R}^n \setminus \{\text{critical values of } p\}$

codim  $S < n = 2$

## 2. Termination: Example

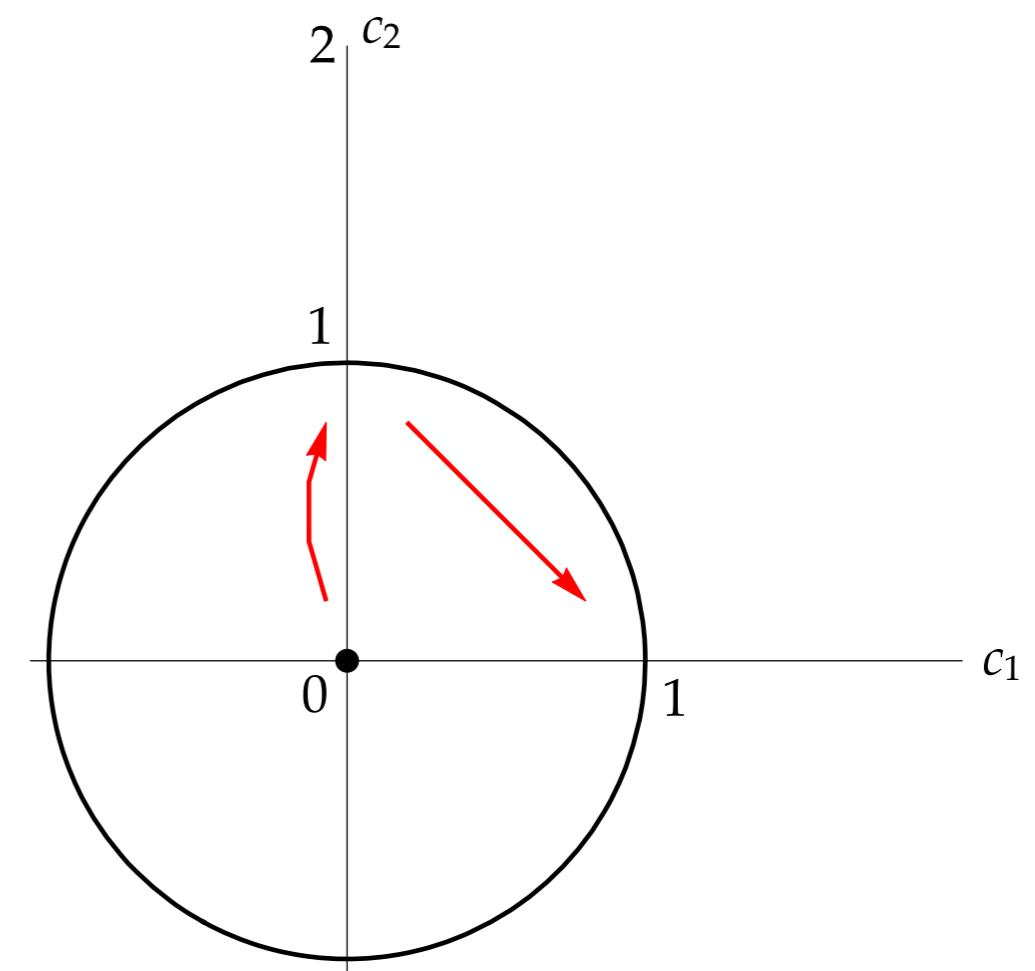
$$f = \frac{1}{4} \left( x_1^2 + x_2^2 - 2 \right) \left( x_1^2 + x_2^2 \right)$$



black:  $\{f=0\}$

white:  $\{f \neq 0\}$

$$g = \frac{f^2}{((x_1 - 1)^2 + (x_2 - 0)^2 + 1)^5}$$



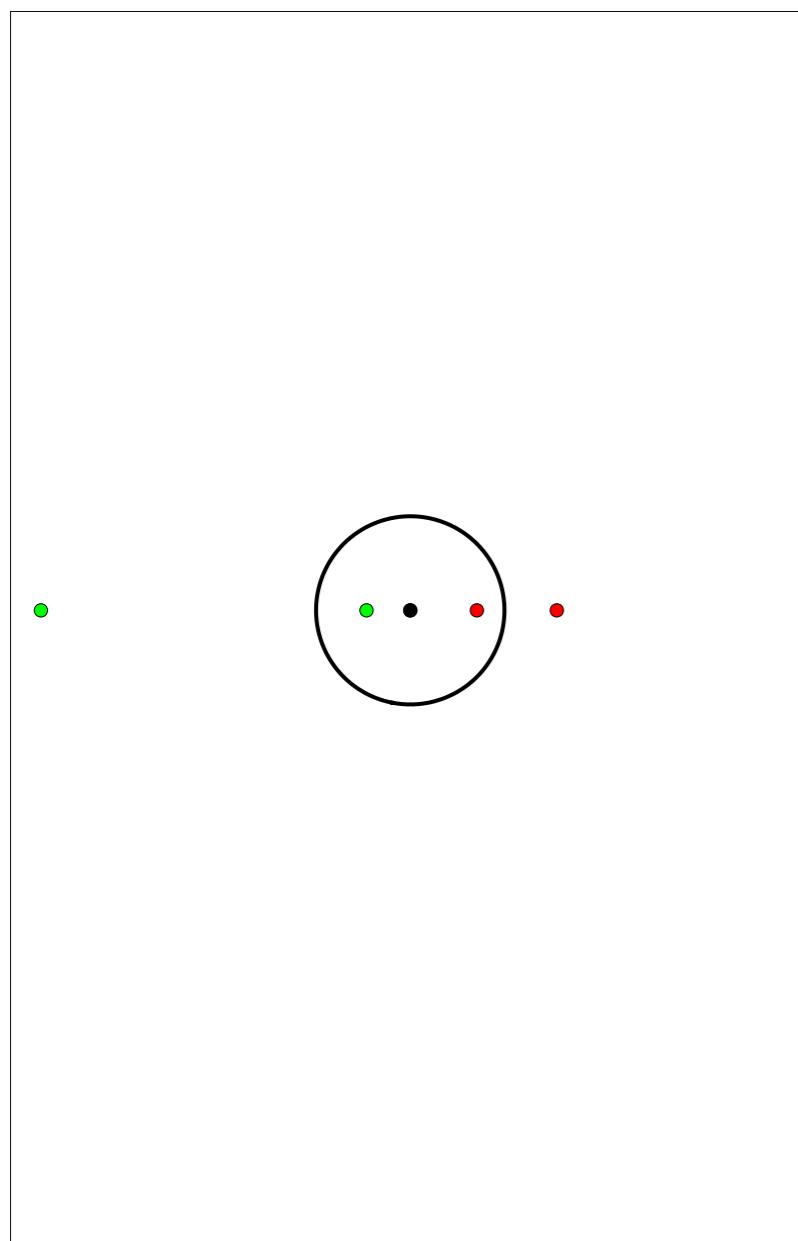
black: critical values of  $p$

white:  $S = \mathbb{R}^n \setminus \{\text{critical values of } p\}$

codim  $S < n = 2$

## 2. Termination: Example

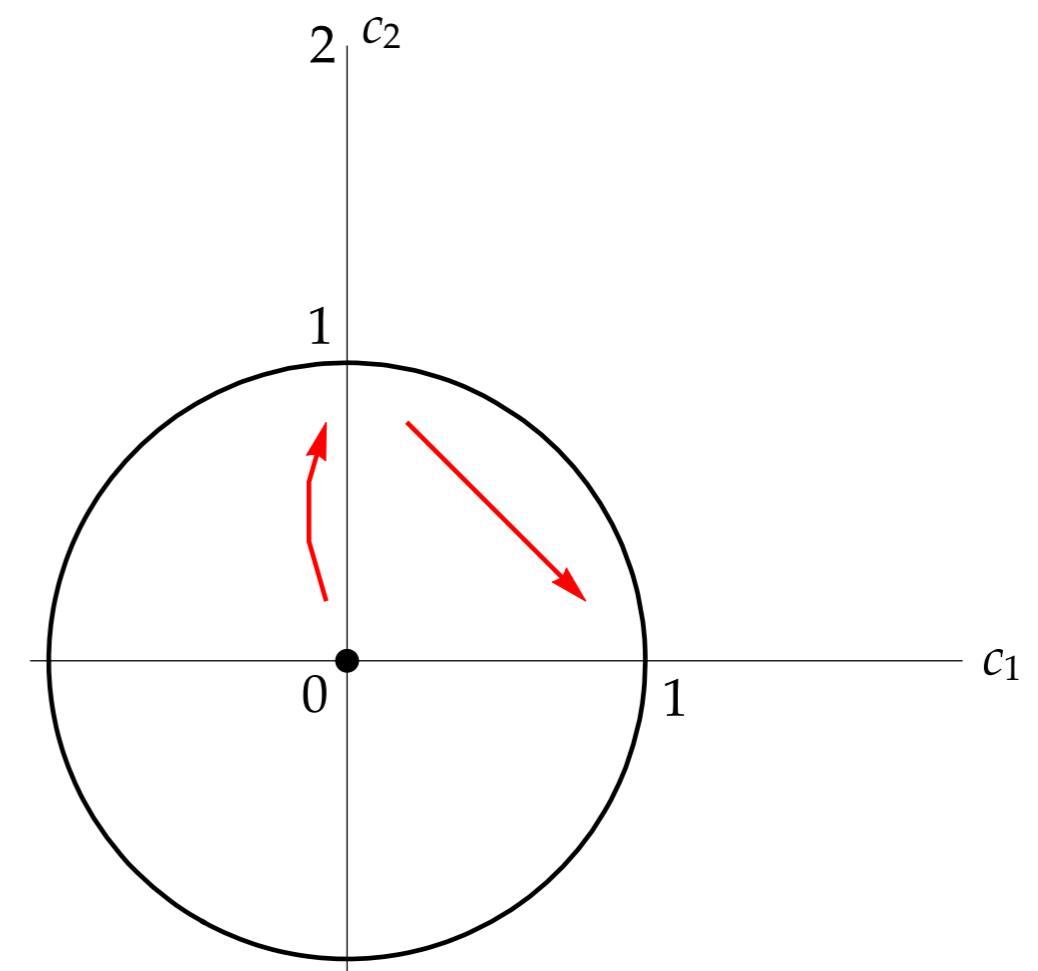
$$f = \frac{1}{4} \left( x_1^2 + x_2^2 - 2 \right) \left( x_1^2 + x_2^2 \right)$$



black:  $\{f=0\}$

white:  $\{f \neq 0\}$

$$g = \frac{f^2}{((x_1 - 1)^2 + (x_2 - 0)^2 + 1)^5}$$



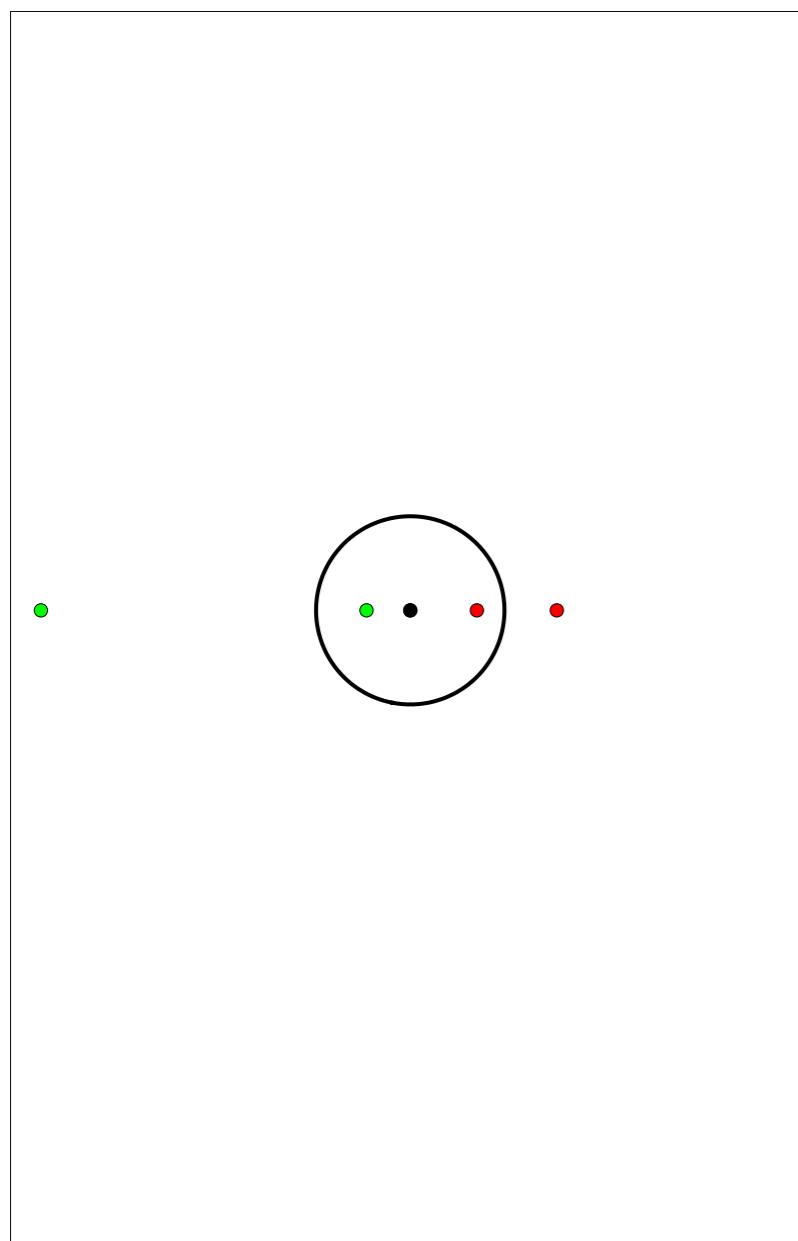
black: critical values of  $p$

white:  $S = \mathbb{R}^n \setminus \{\text{critical values of } p\}$

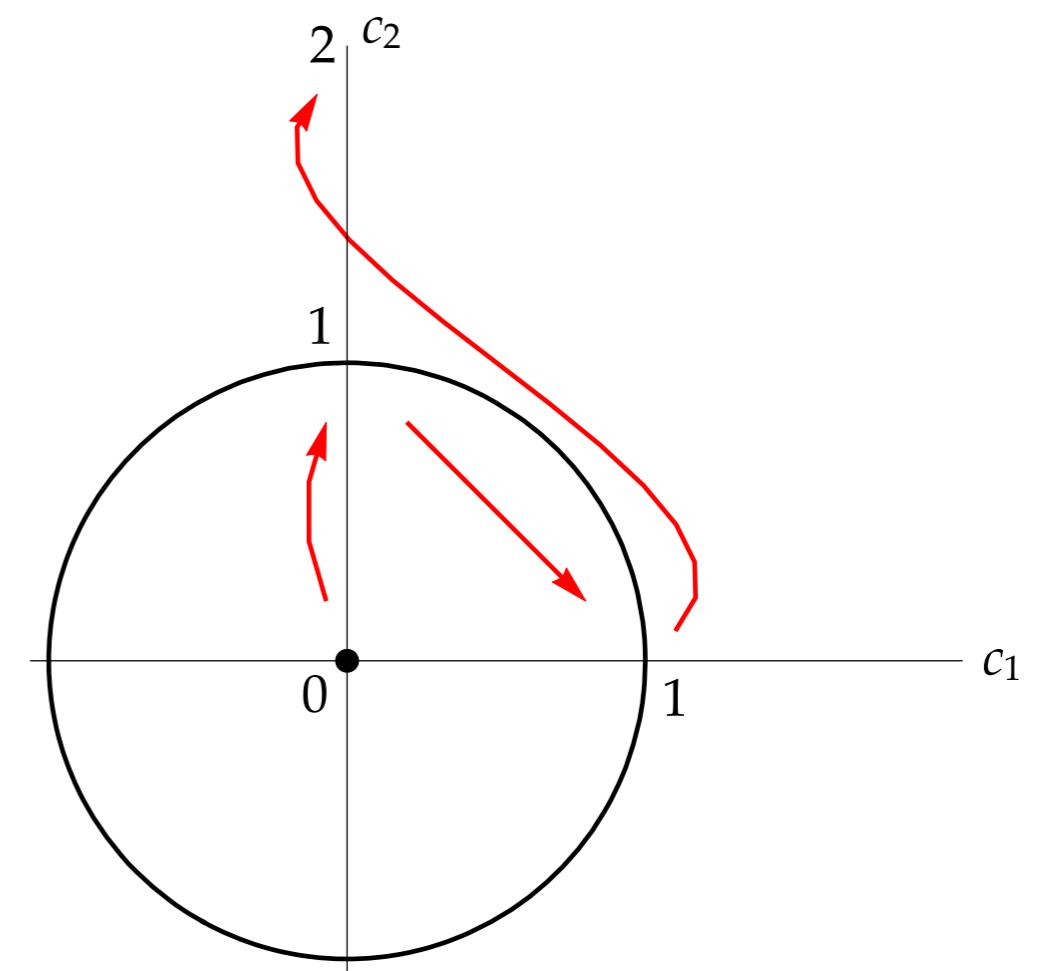
codim  $S < n = 2$

## 2. Termination: Example

$$f = \frac{1}{4} (x_1^2 + x_2^2 - 2) (x_1^2 + x_2^2)$$



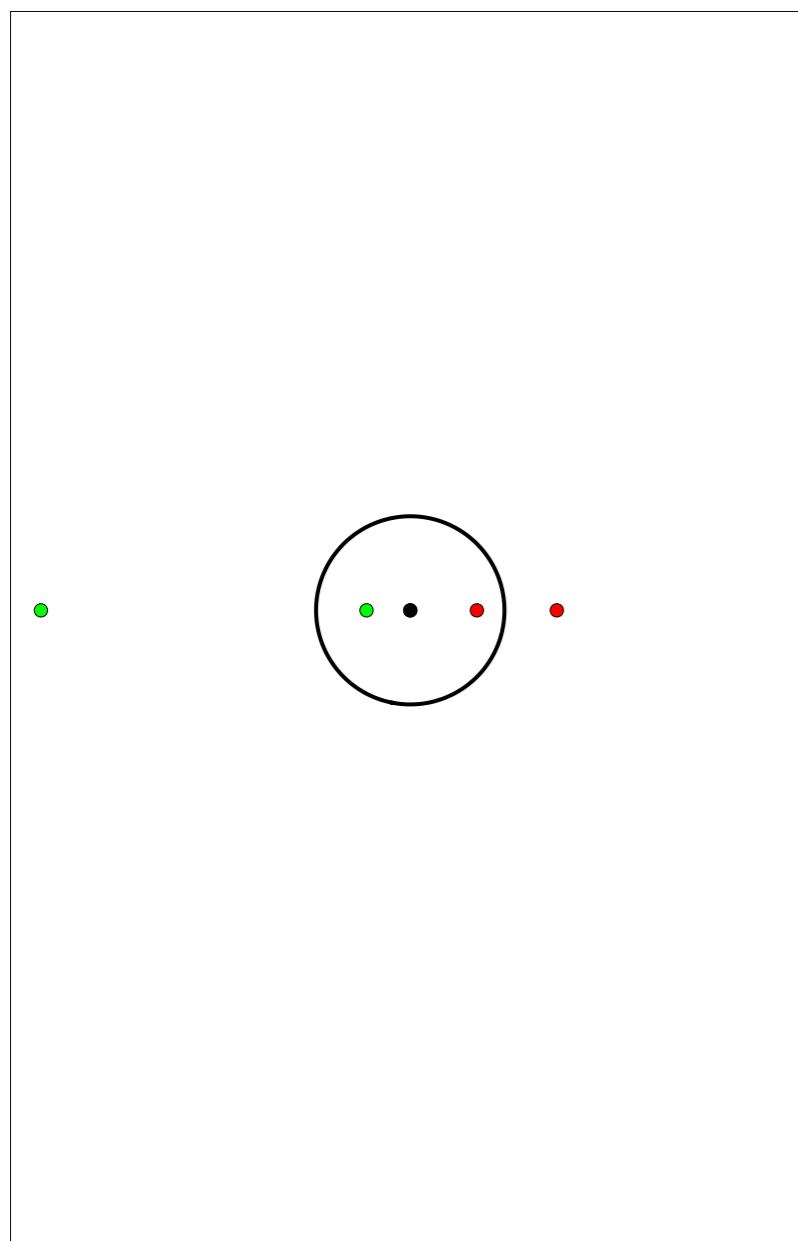
$$g = \frac{f^2}{((x_1 - 1)^2 + (x_2 - 0)^2 + 1)^5}$$



black: critical values of  $p$   
 white:  $S = \mathbb{R}^n \setminus \{\text{critical values of } p\}$   
 codim  $S < n = 2$

## 2. Termination: Example

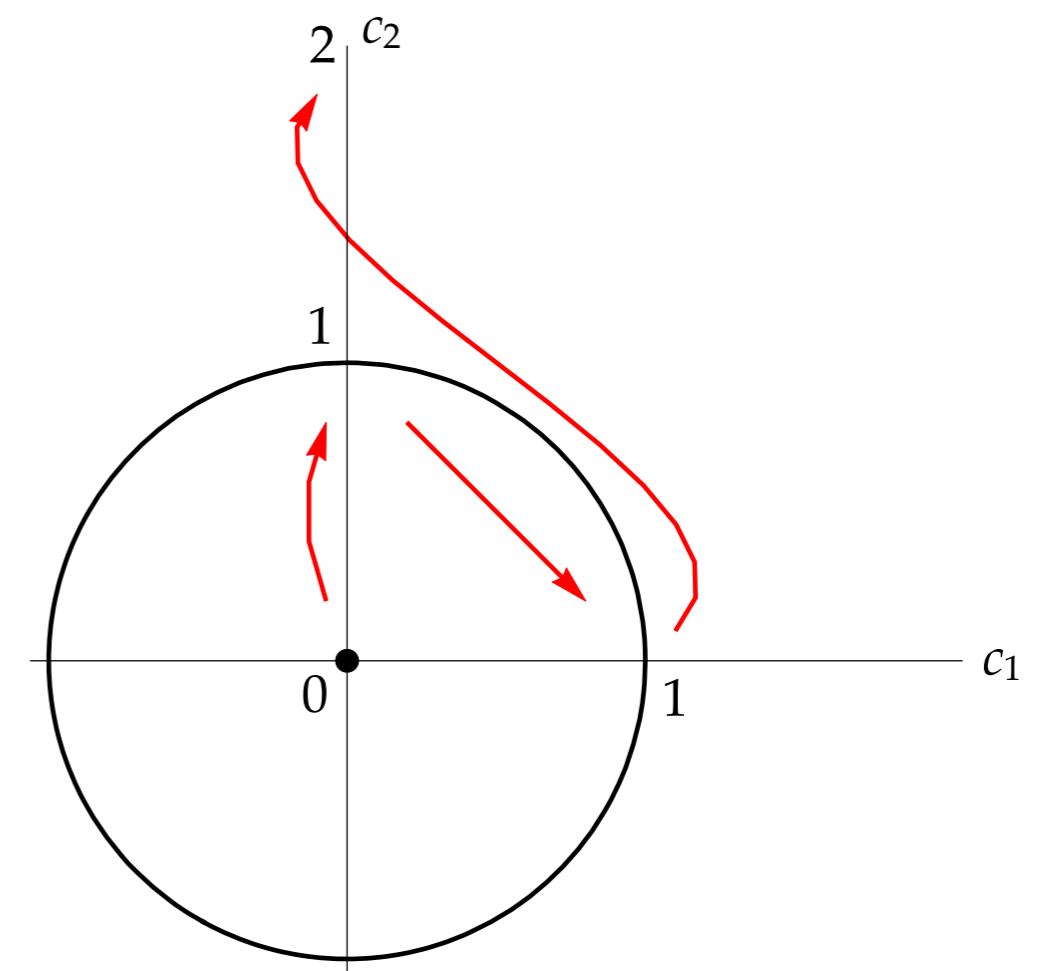
$$f = \frac{1}{4} (x_1^2 + x_2^2 - 2) (x_1^2 + x_2^2)$$



black:  $\{f=0\}$

white:  $\{f \neq 0\}$

$$g = \frac{f^2}{((x_1 - 0)^2 + (x_2 - 2)^2 + 1)^5}$$



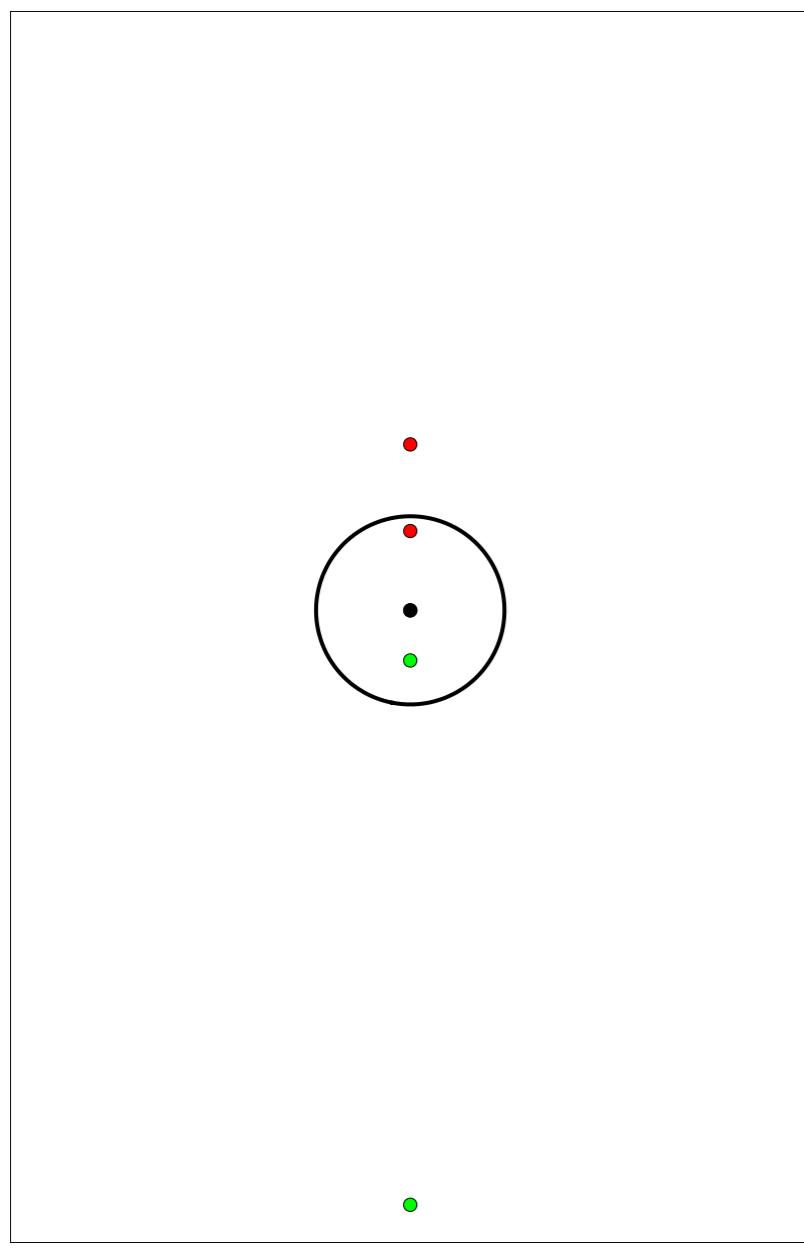
black: critical values of  $p$

white:  $S = \mathbb{R}^n \setminus \{\text{critical values of } p\}$

codim  $S < n = 2$

## 2. Termination: Example

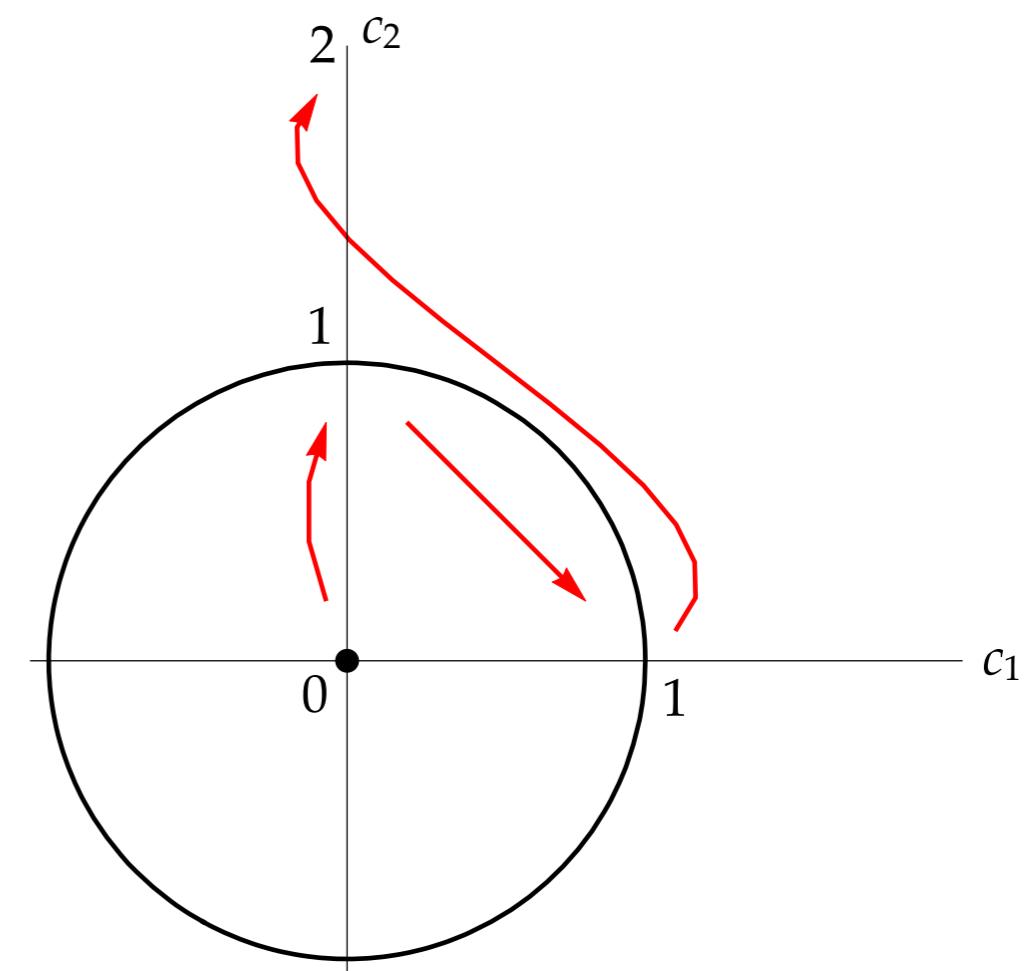
$$f = \frac{1}{4} (x_1^2 + x_2^2 - 2) (x_1^2 + x_2^2)$$



black:  $\{f=0\}$

white:  $\{f \neq 0\}$

$$g = \frac{f^2}{((x_1 - 0)^2 + (x_2 - 2)^2 + 1)^5}$$



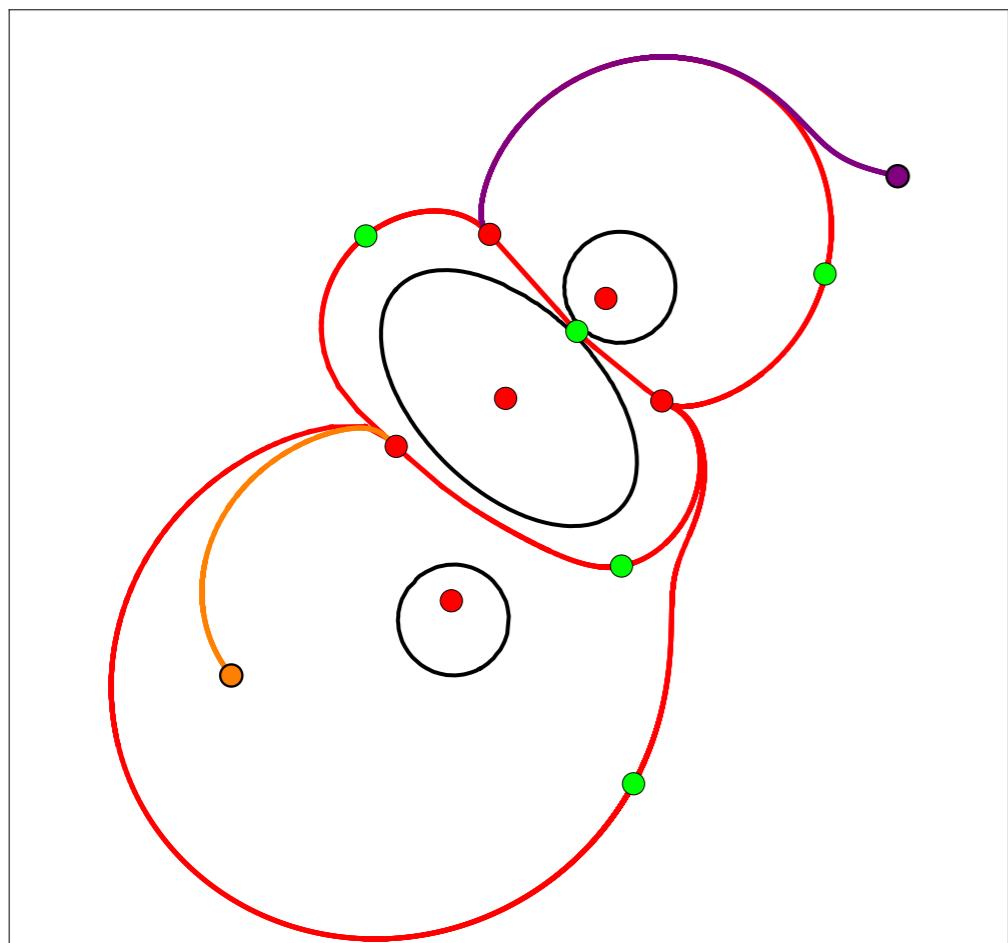
black: critical values of  $p$

white:  $S = \mathbb{R}^n \setminus \{\text{critical values of } p\}$

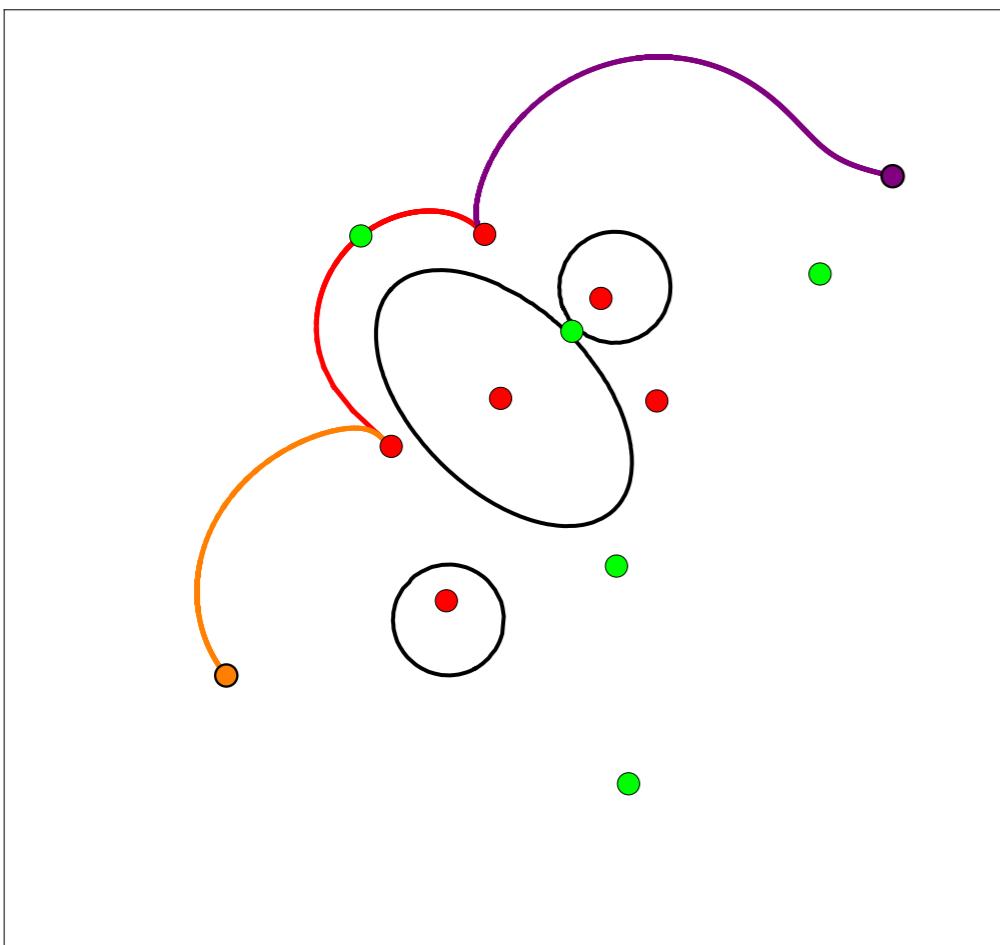
codim  $S < n = 2$

### 3. Complexity: Problem

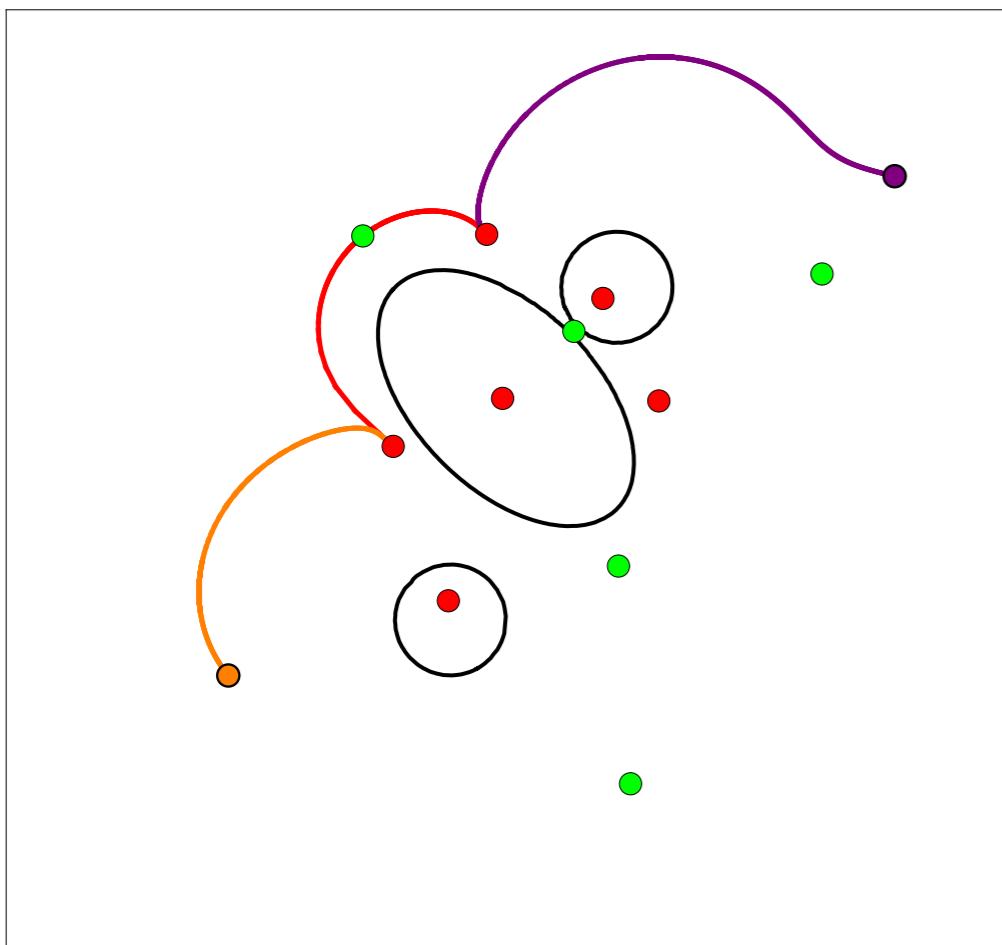
### 3. Complexity: Problem



### 3. Complexity: Problem

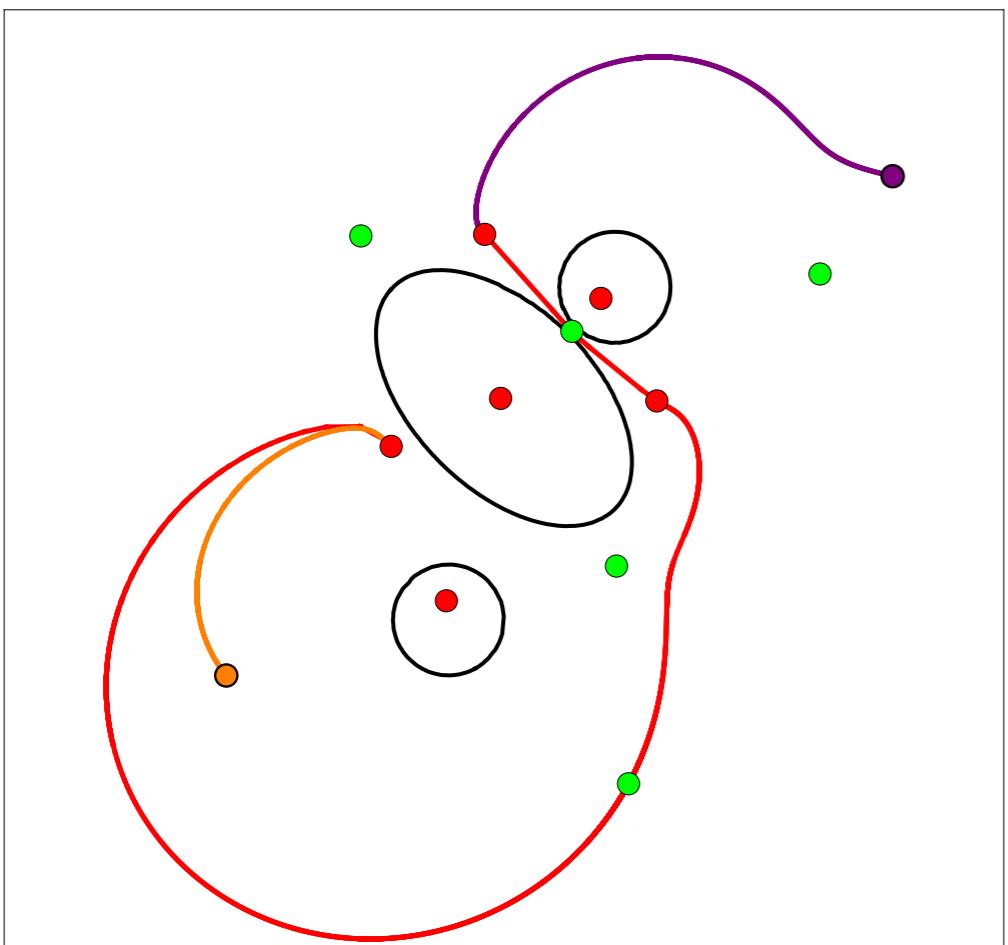


### 3. Complexity: Problem

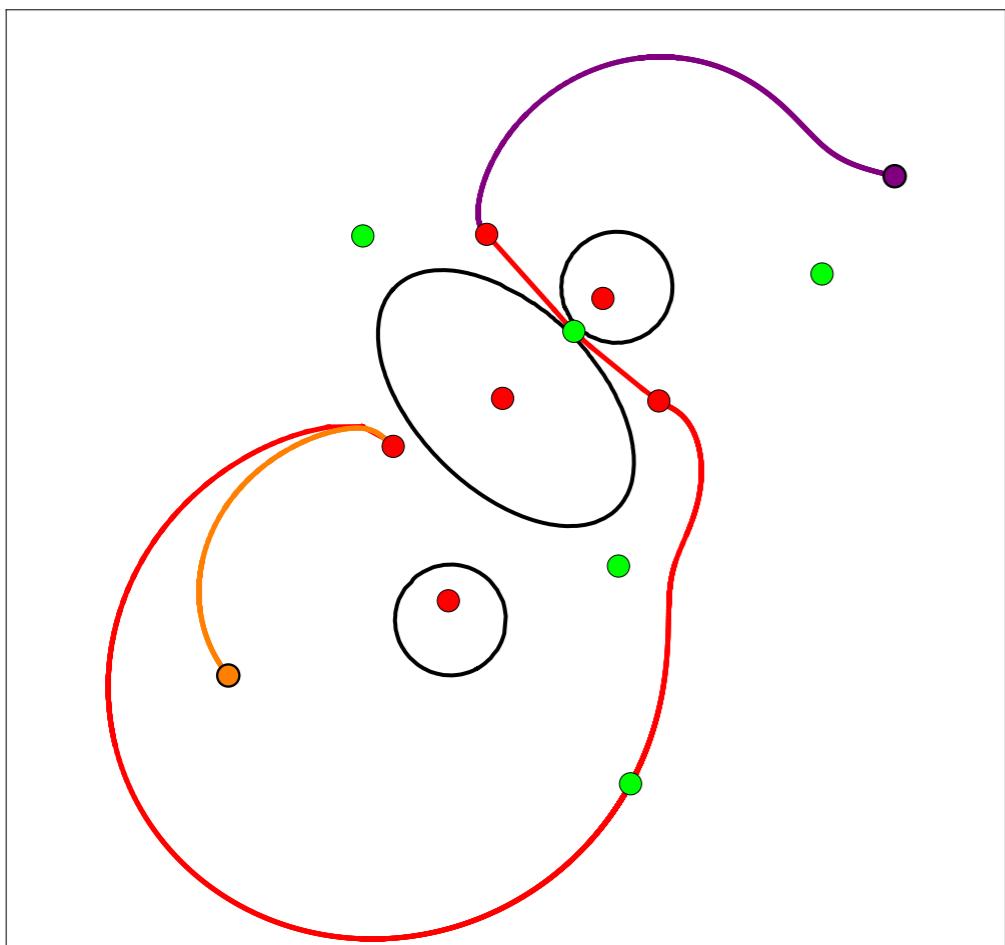


Length  $\approx 23.66$

### 3. Complexity: Problem

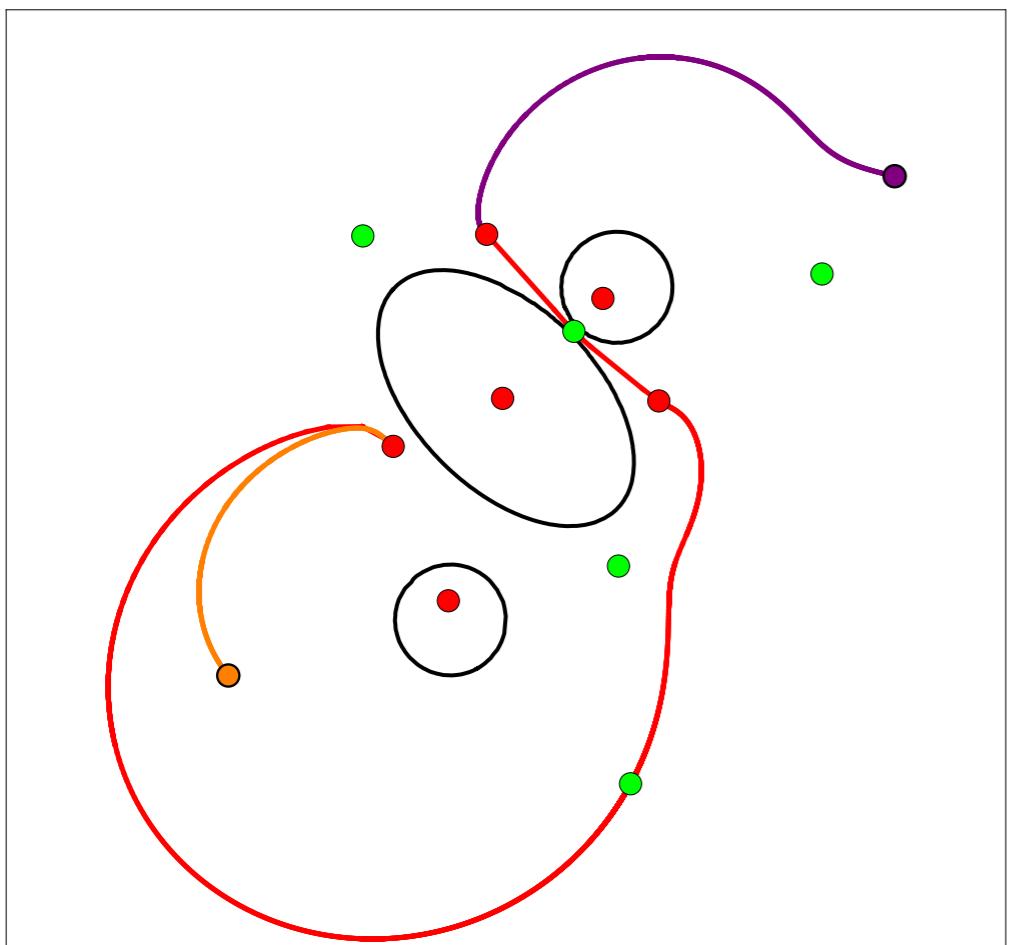


### 3. Complexity: Problem



Length  $\approx 49.18$

### 3. Complexity: Problem



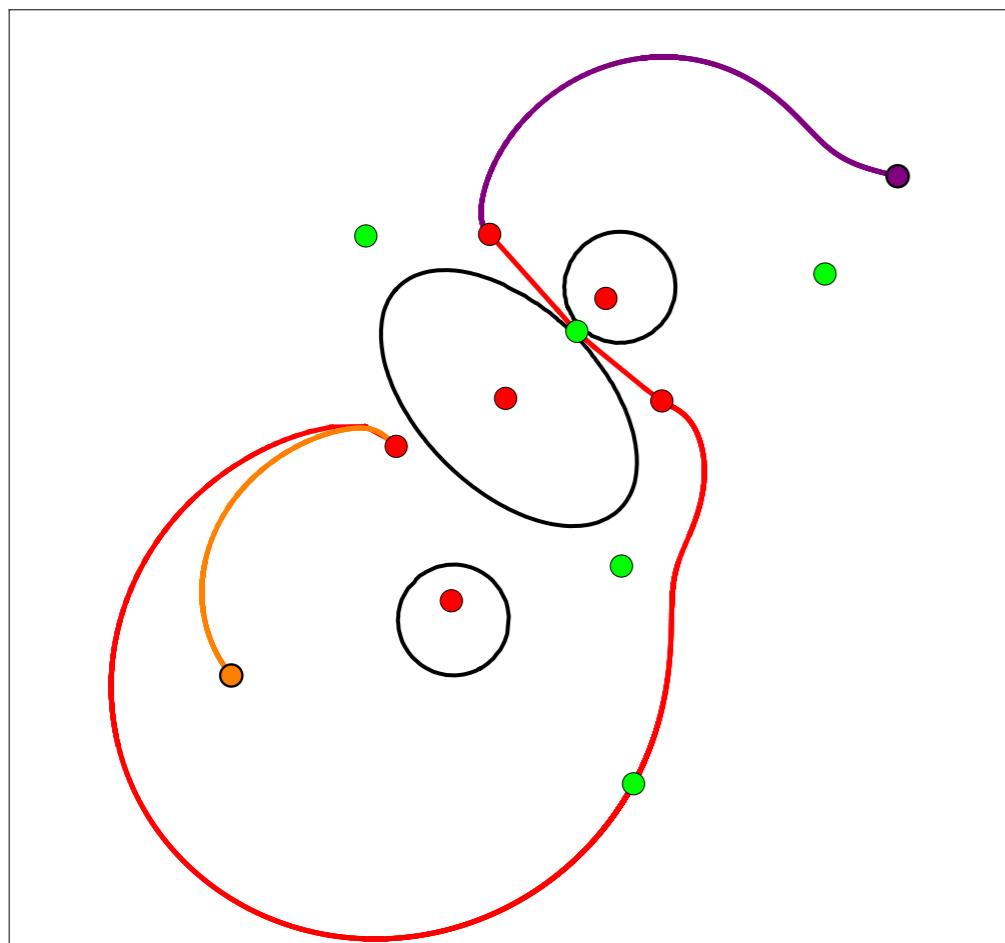
Length  $\approx 49.18$

**Given**

$$f \in \mathbb{Z}[x_1, \dots, x_n]$$

$$n \geq 2 \quad d = \deg(f) \geq 1$$

### 3. Complexity: Problem



Length  $\approx 49.18$

**Given**

$$f \in \mathbb{Z}[x_1, \dots, x_n]$$

$$n \geq 2 \quad d = \deg(f) \geq 1$$

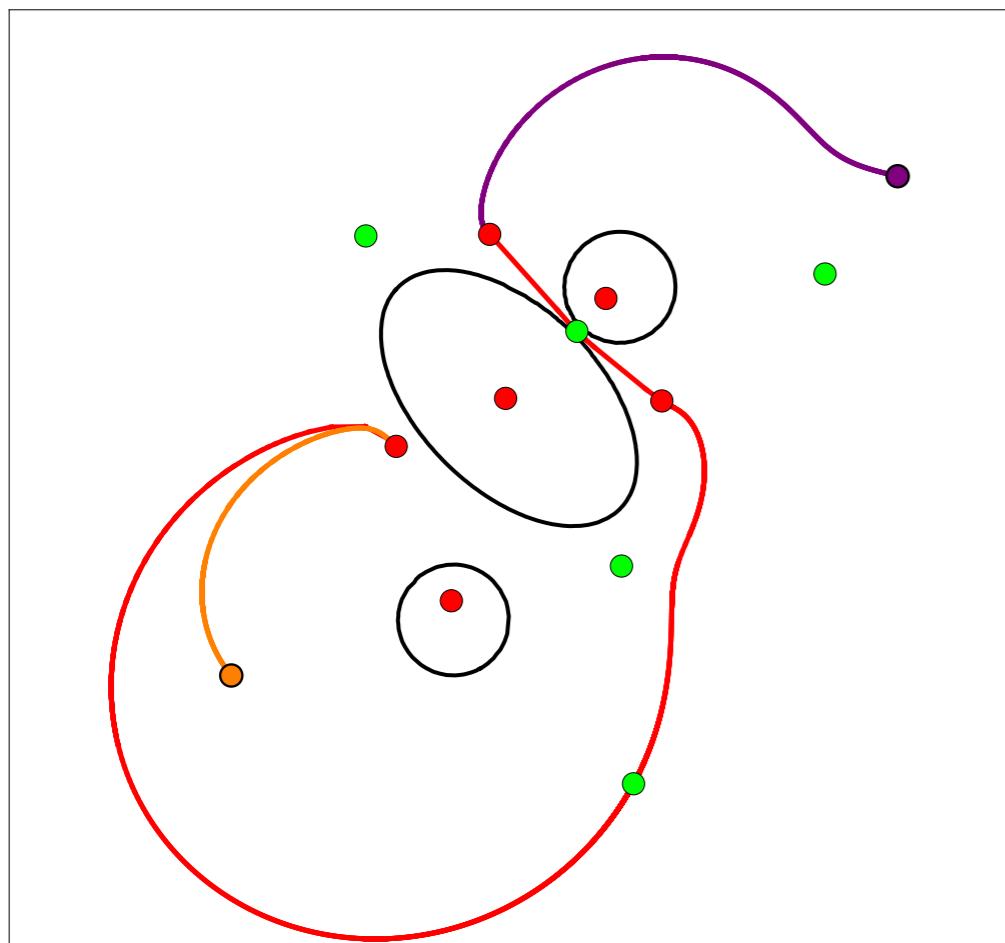
$$(c_1, \dots, c_n) \in \mathbb{Z}_{\geq 0}^n$$

such that

$$g = \frac{f(x_1, x_2)^2}{((x_1 - c_1)^2 + (x_2 - c_2)^2 + 1)^{d+1}}$$

is a routing function

### 3. Complexity: Problem



Length  $\approx 49.18$

**Given**

$$f \in \mathbb{Z}[x_1, \dots, x_n]$$

$$n \geq 2 \quad d = \deg(f) \geq 1$$

$$(c_1, \dots, c_n) \in \mathbb{Z}_{\geq 0}^n$$

such that

$$g = \frac{f(x_1, x_2)^2}{((x_1 - c_1)^2 + (x_2 - c_2)^2 + 1)^{d+1}}$$

is a routing function

**Find**  $A$  such that

$$\text{Length} \leq A(n, d, B, c_1, \dots, c_n)$$

$$B = \max |\text{coefficients of } f|$$

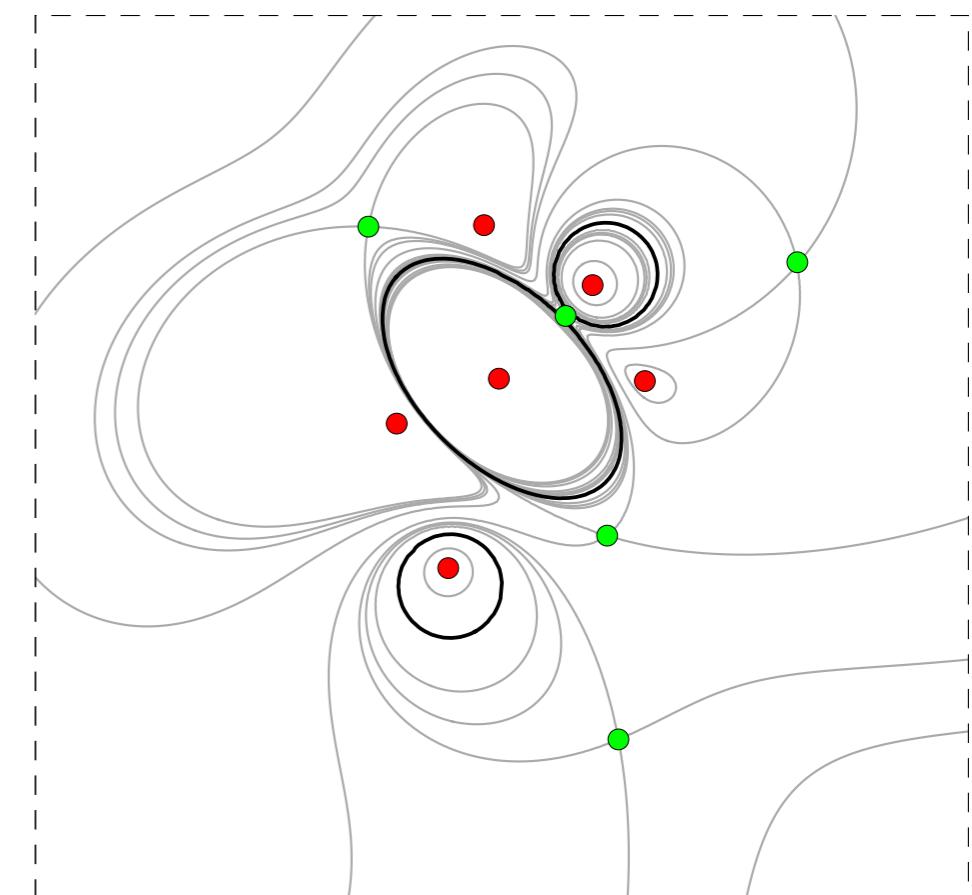
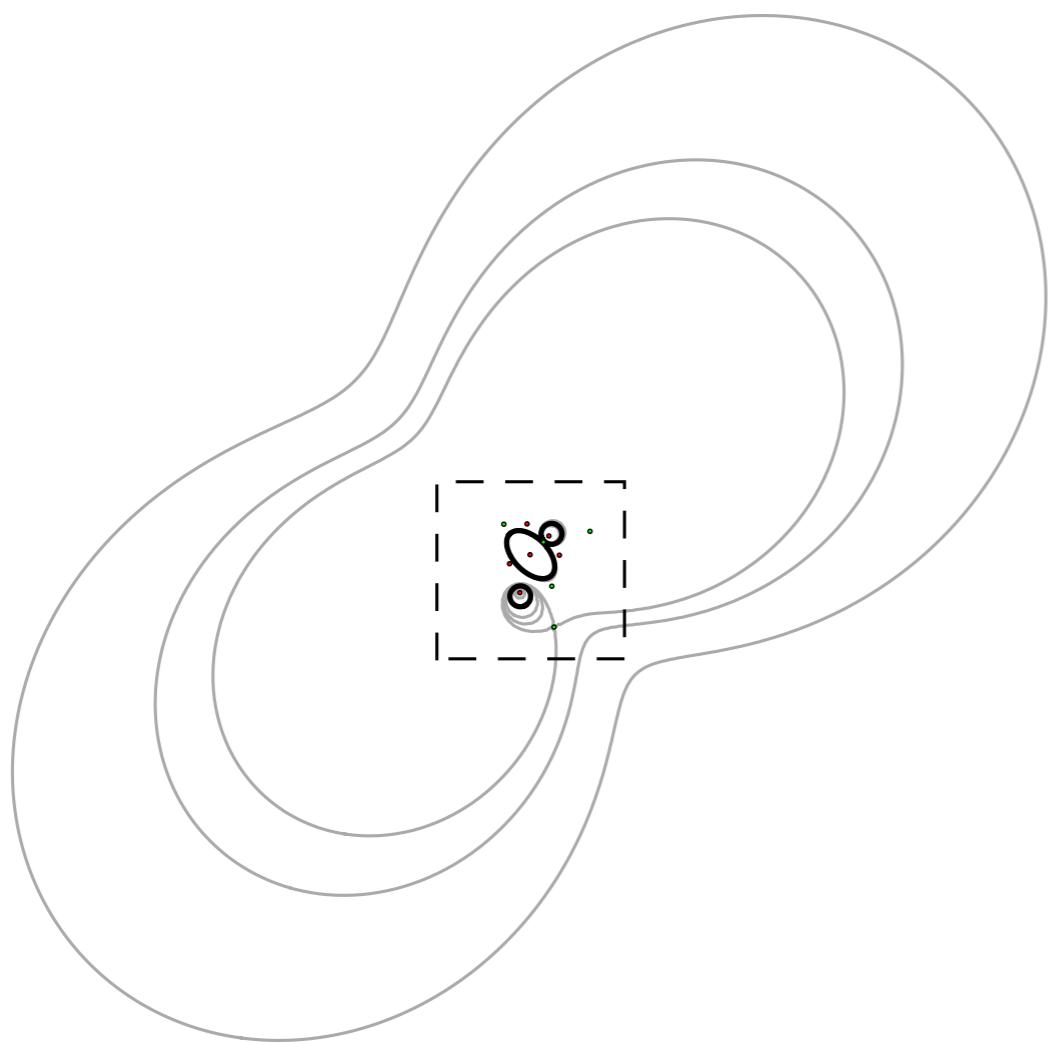
### 3. Complexity: Trajectory Bound

### 3. Complexity: Trajectory Bound

The length of any trajectory of  $\nabla g$  in a ball of radius  $r$  has length bounded by  $rC(n, d)$ .

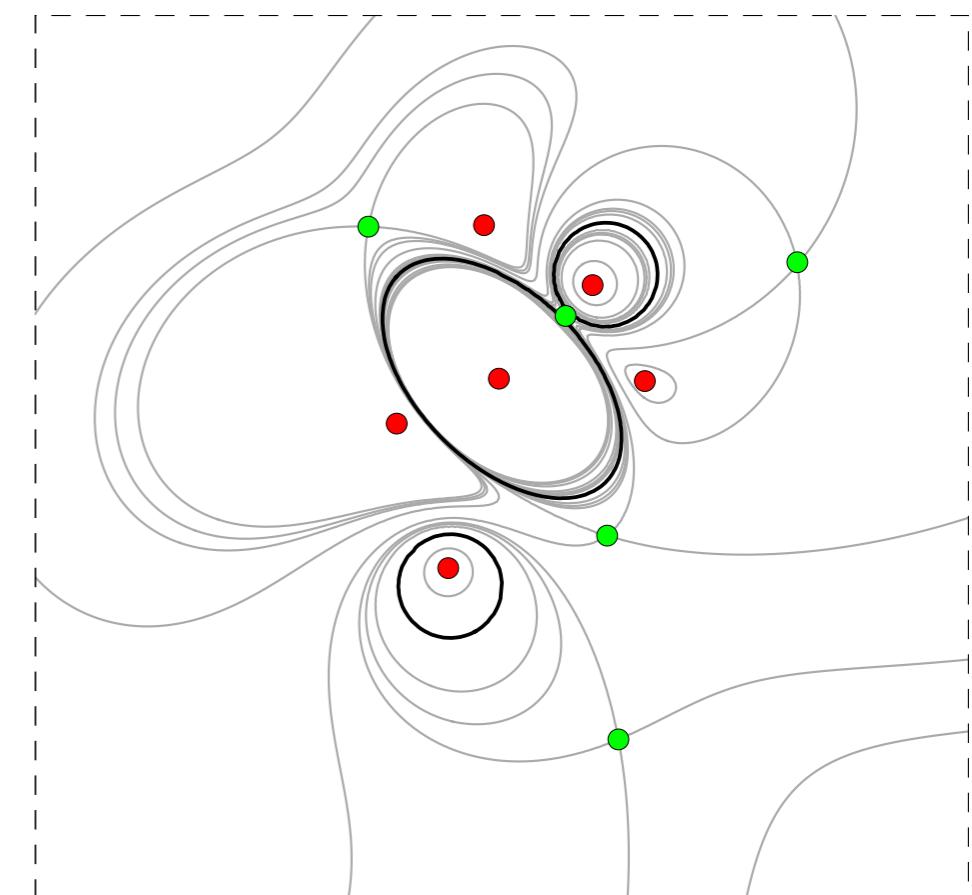
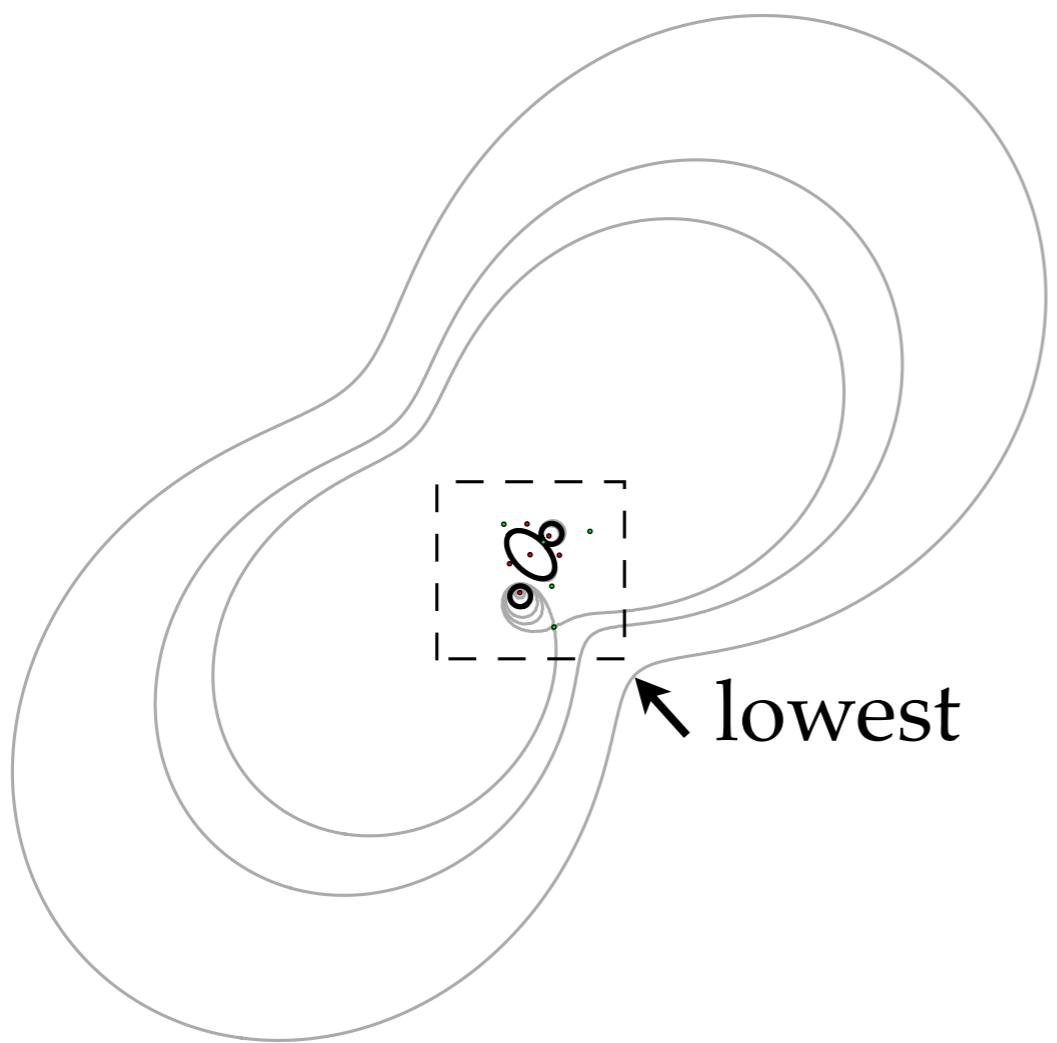
### 3. Complexity: Trajectory Bound

The length of any trajectory of  $\nabla g$  in a ball of radius  $r$  has length bounded by  $rC(n, d)$ .



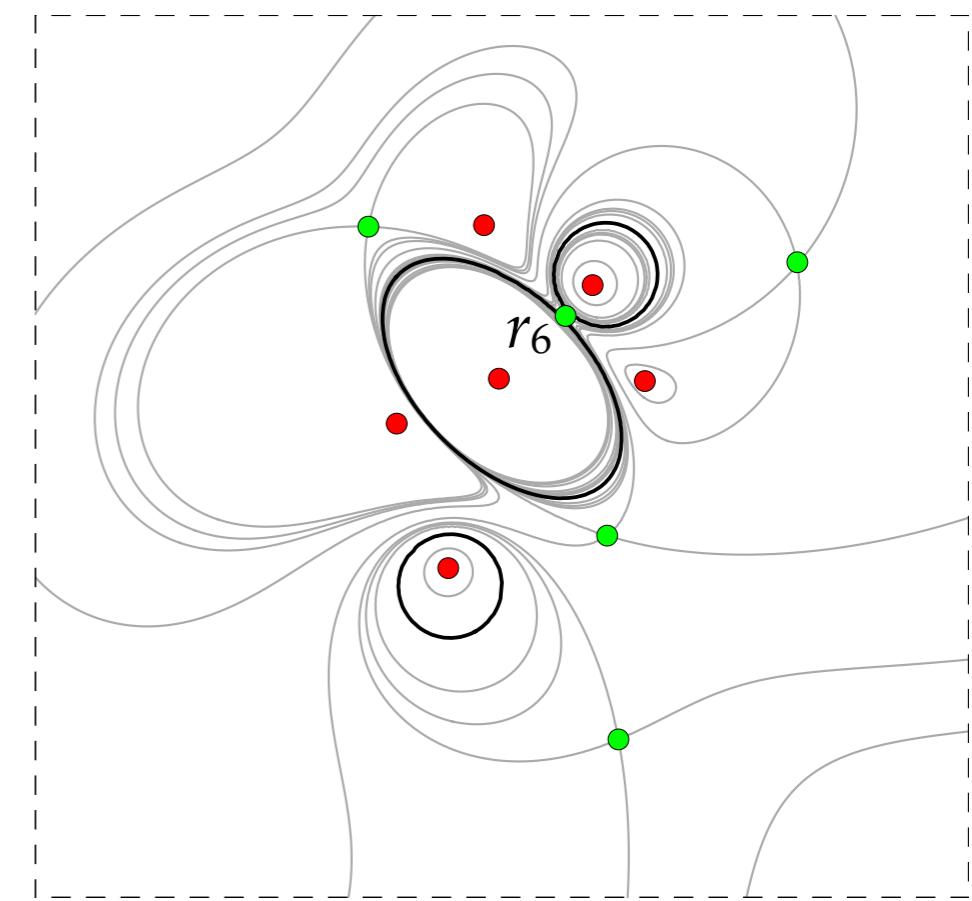
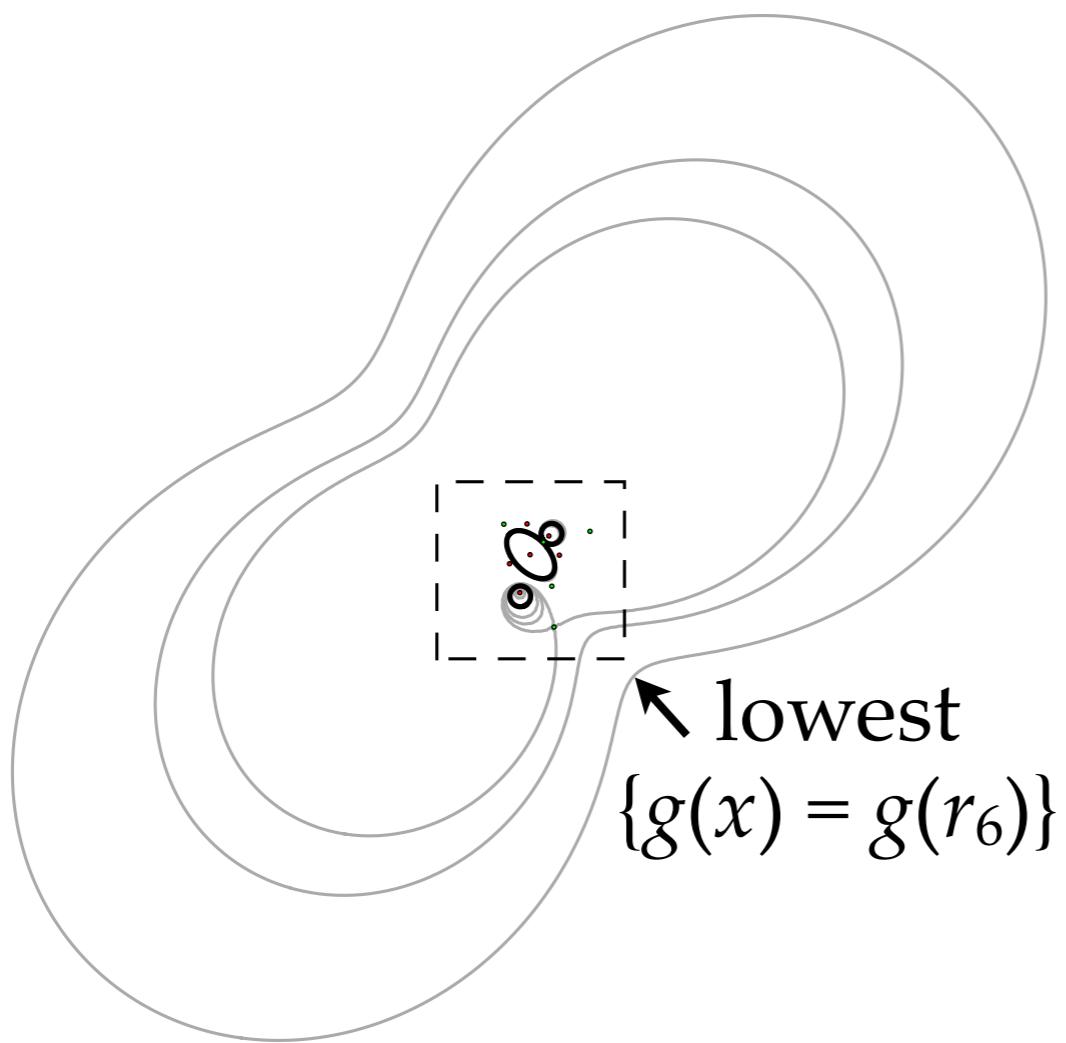
### 3. Complexity: Trajectory Bound

The length of any trajectory of  $\nabla g$  in a ball of radius  $r$  has length bounded by  $rC(n, d)$ .



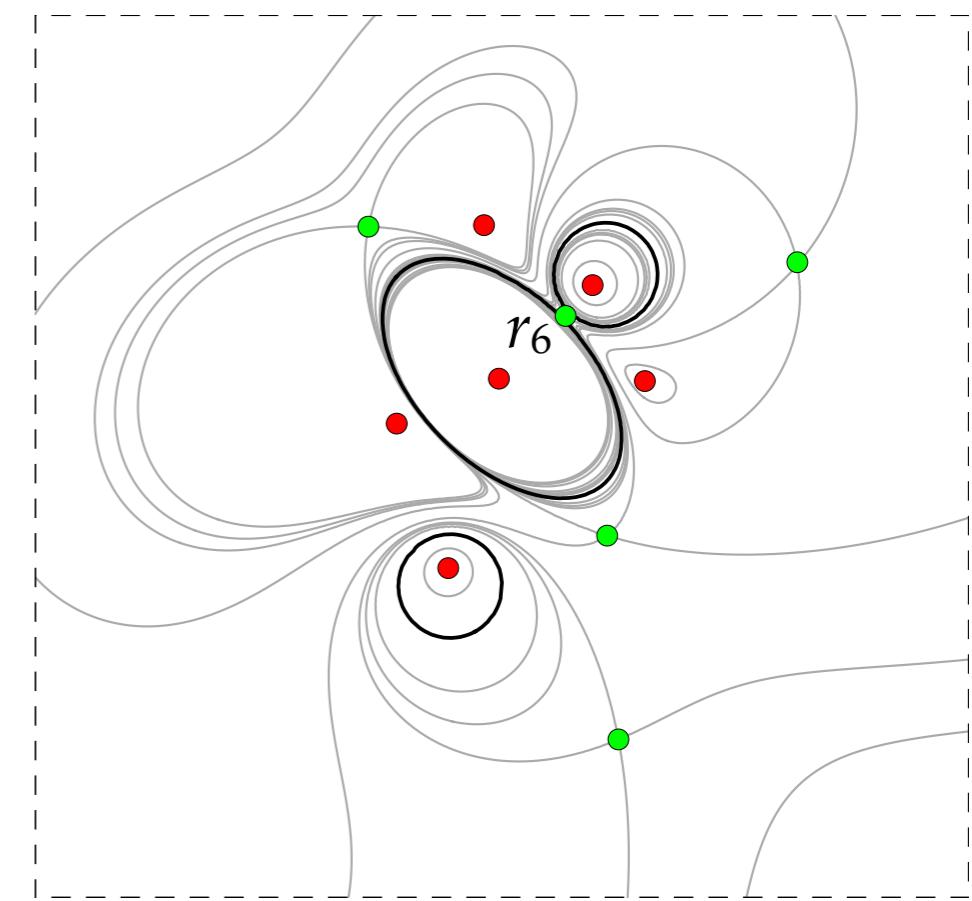
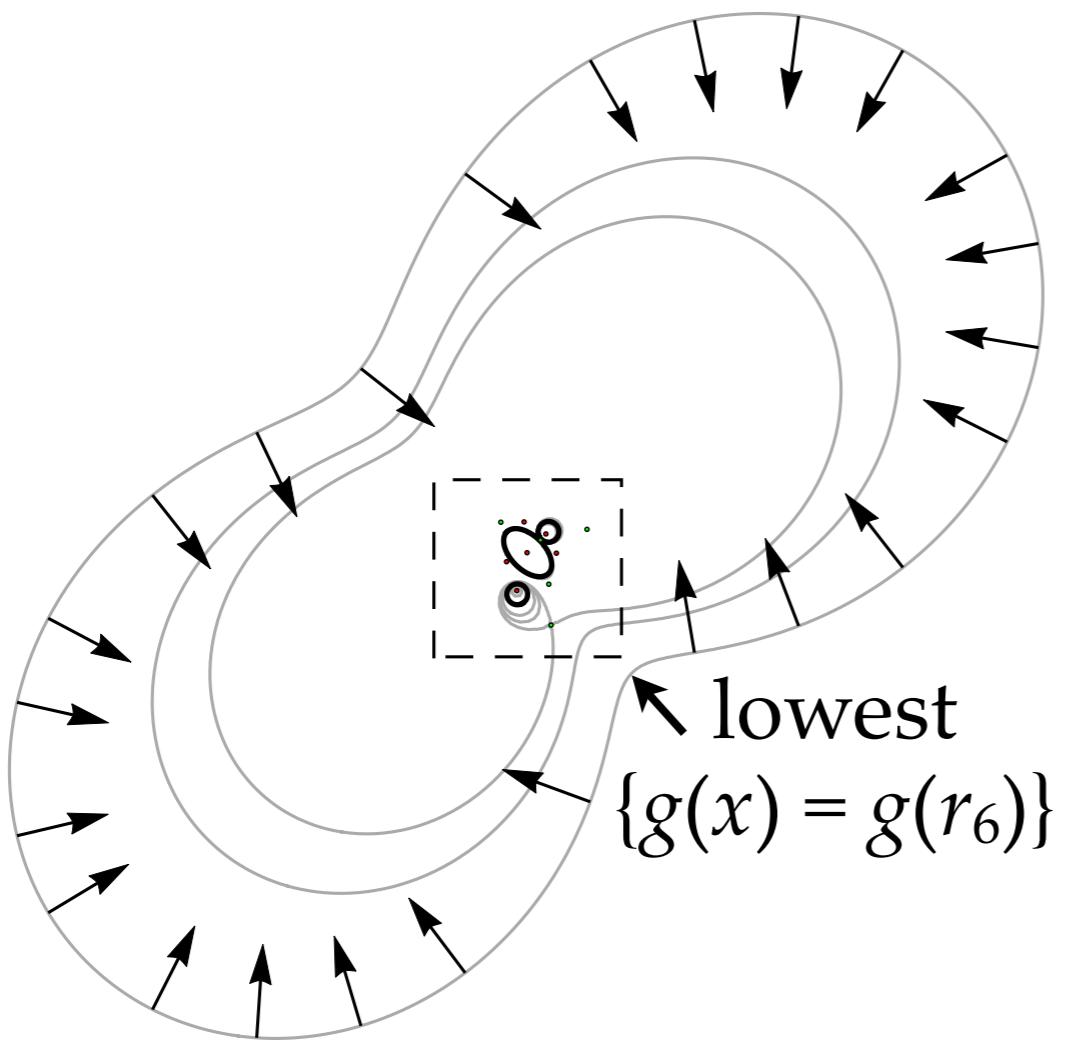
### 3. Complexity: Trajectory Bound

The length of any trajectory of  $\nabla g$  in a ball of radius  $r$  has length bounded by  $rC(n, d)$ .



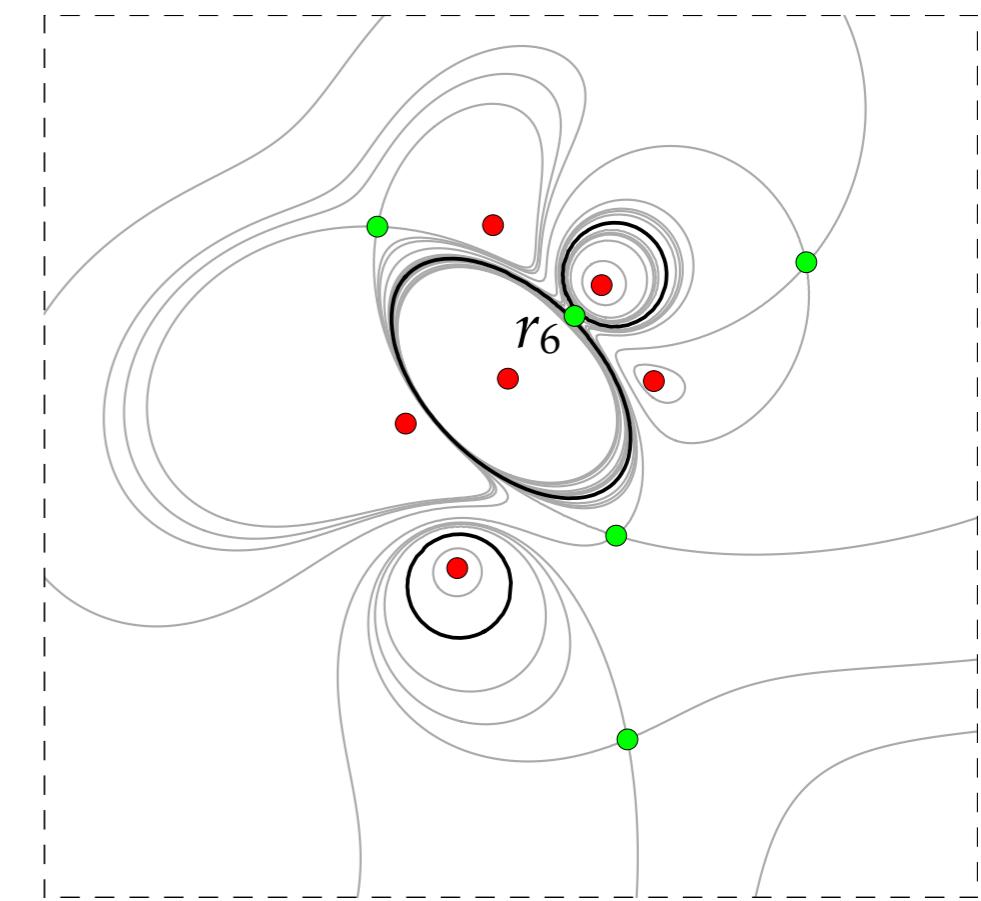
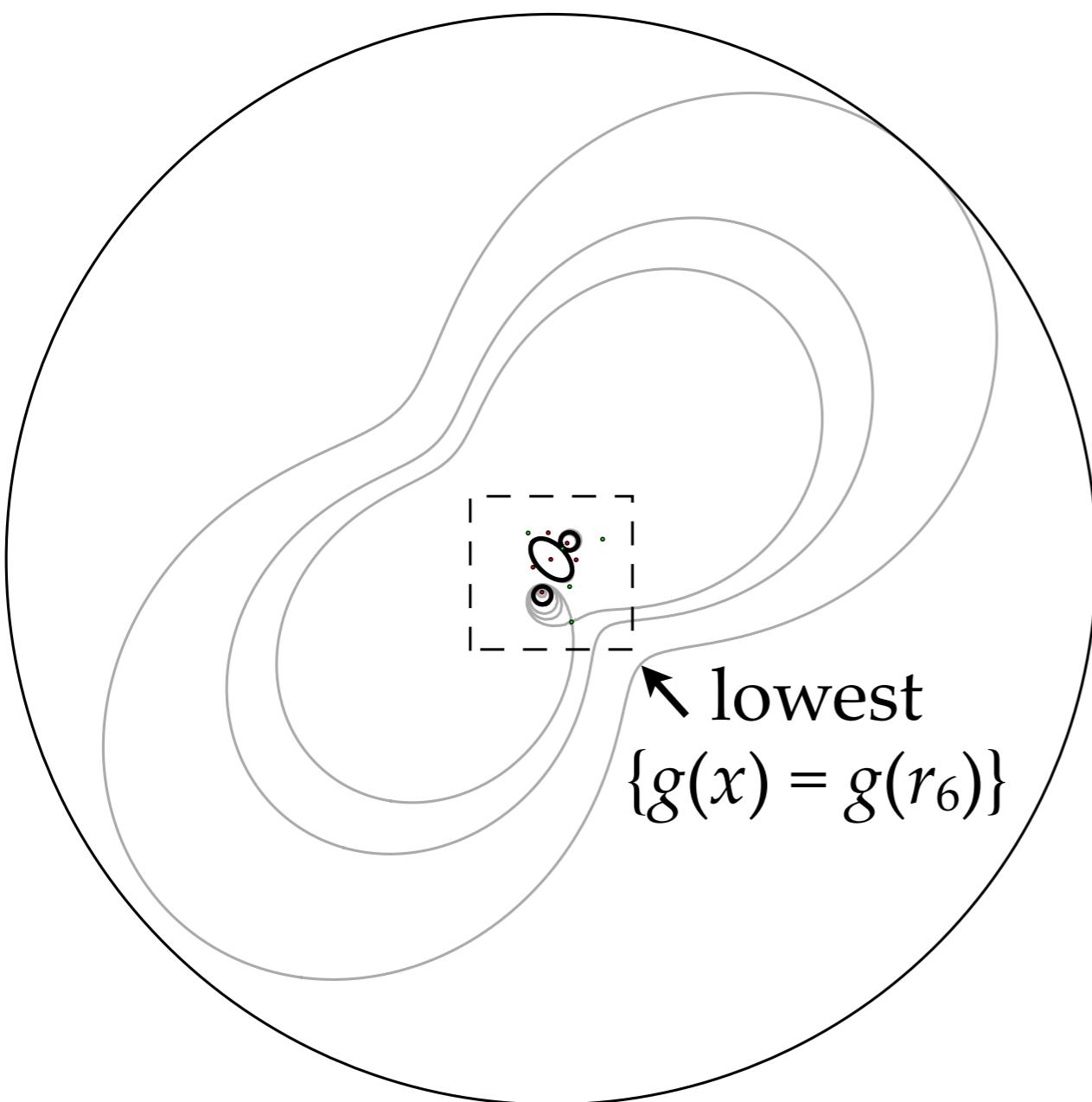
### 3. Complexity: Trajectory Bound

The length of any trajectory of  $\nabla g$  in a ball of radius  $r$  has length bounded by  $rC(n, d)$ .



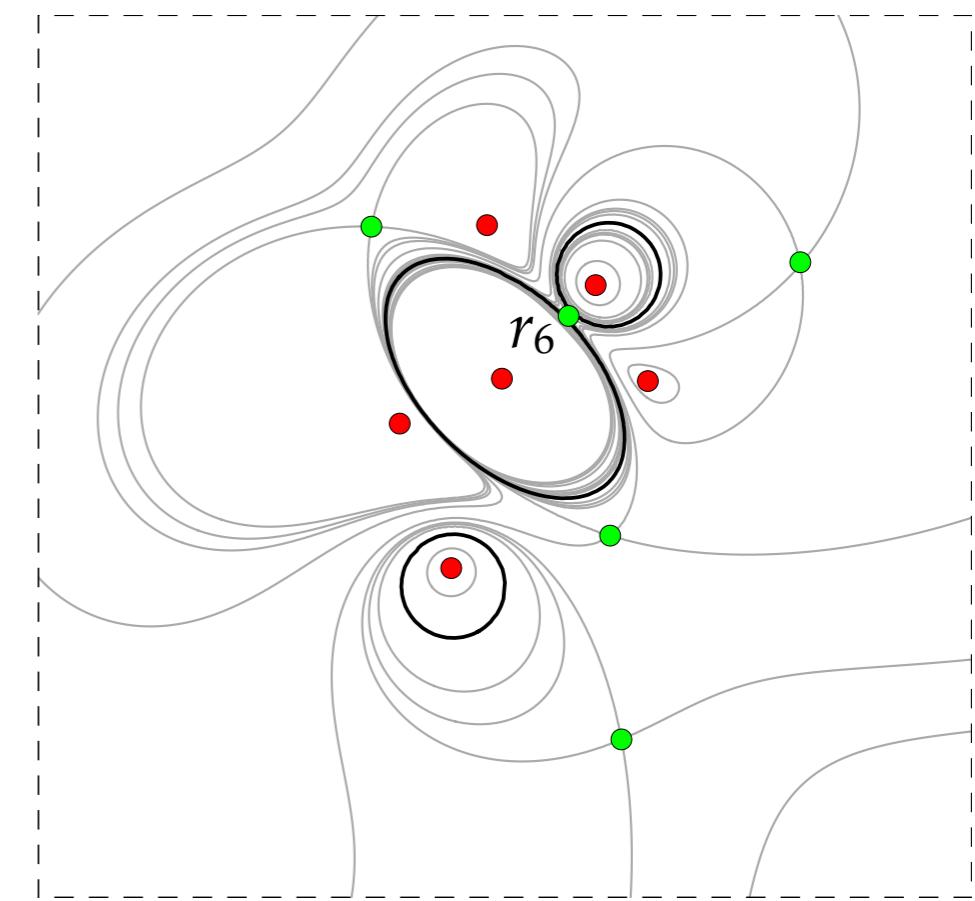
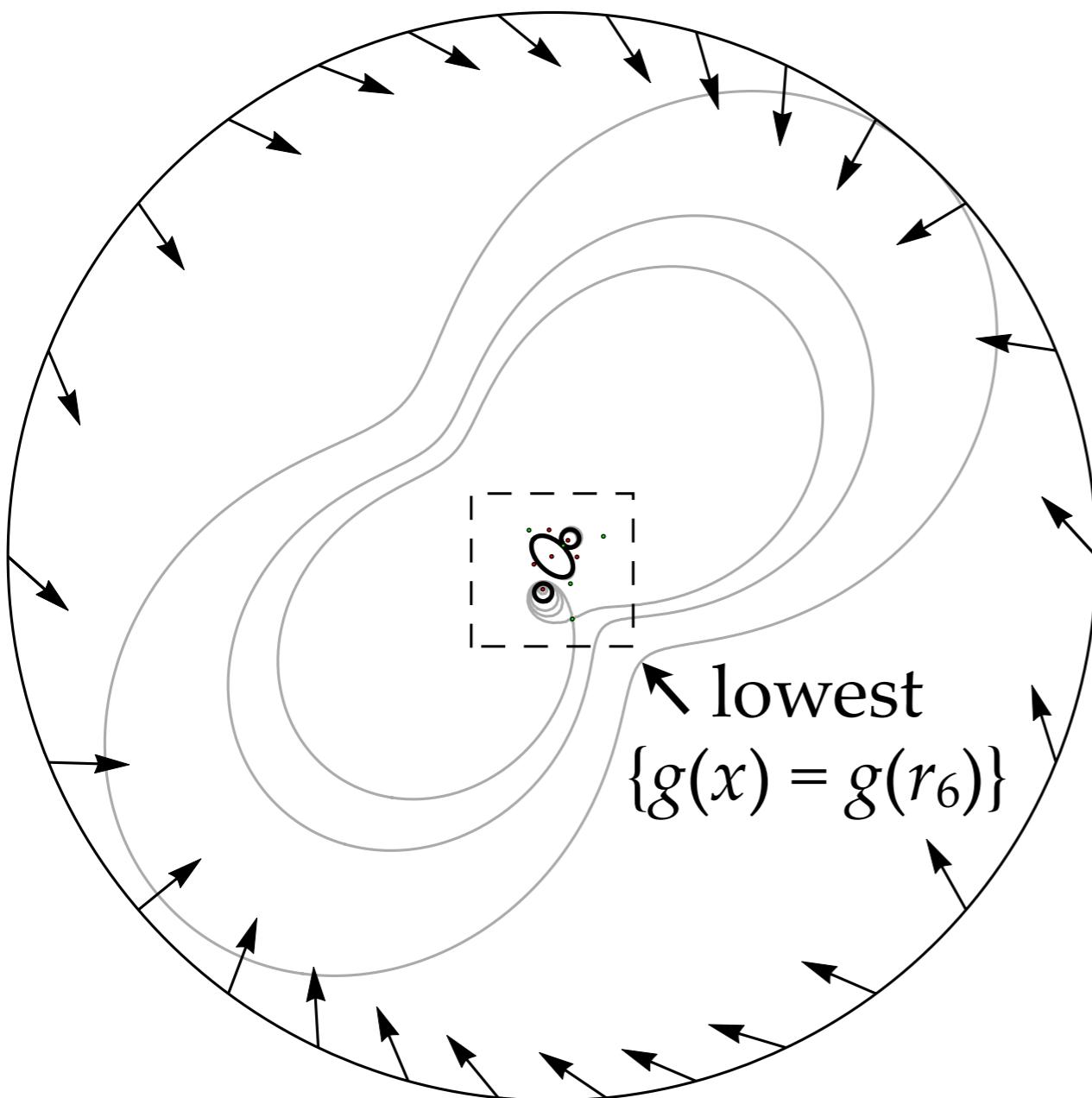
### 3. Complexity: Trajectory Bound

The length of any trajectory of  $\nabla g$  in a ball of radius  $r$  has length bounded by  $rC(n, d)$ .



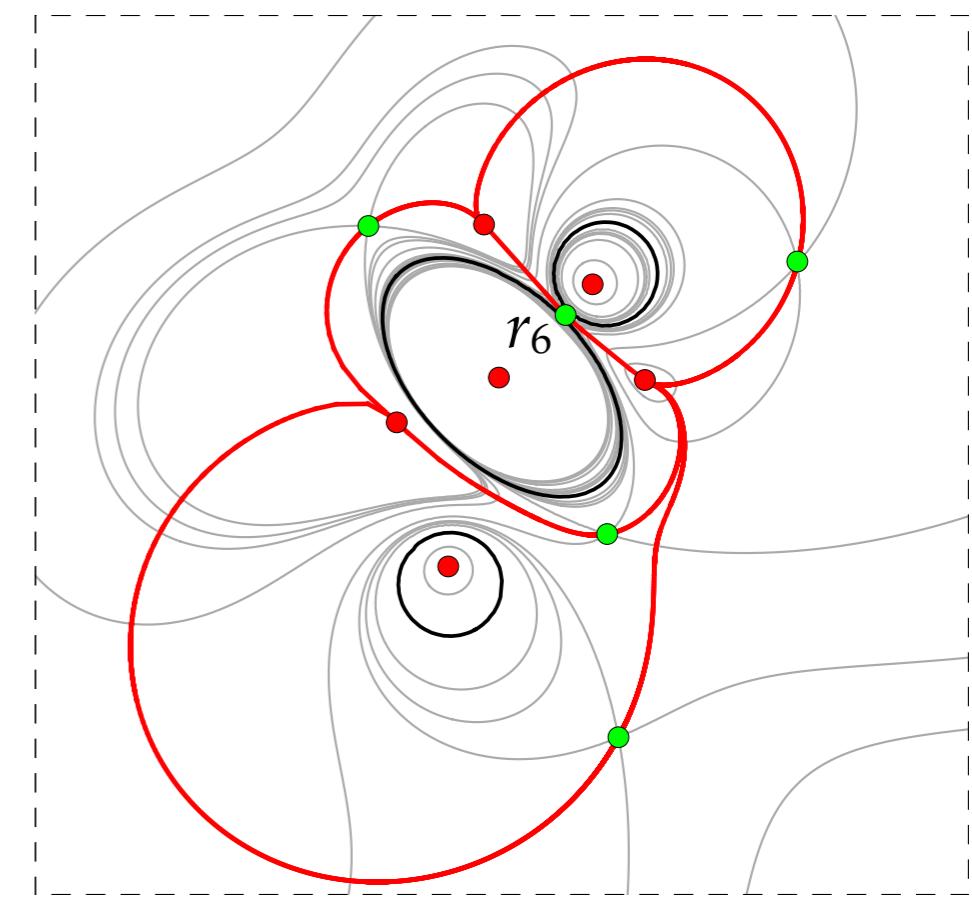
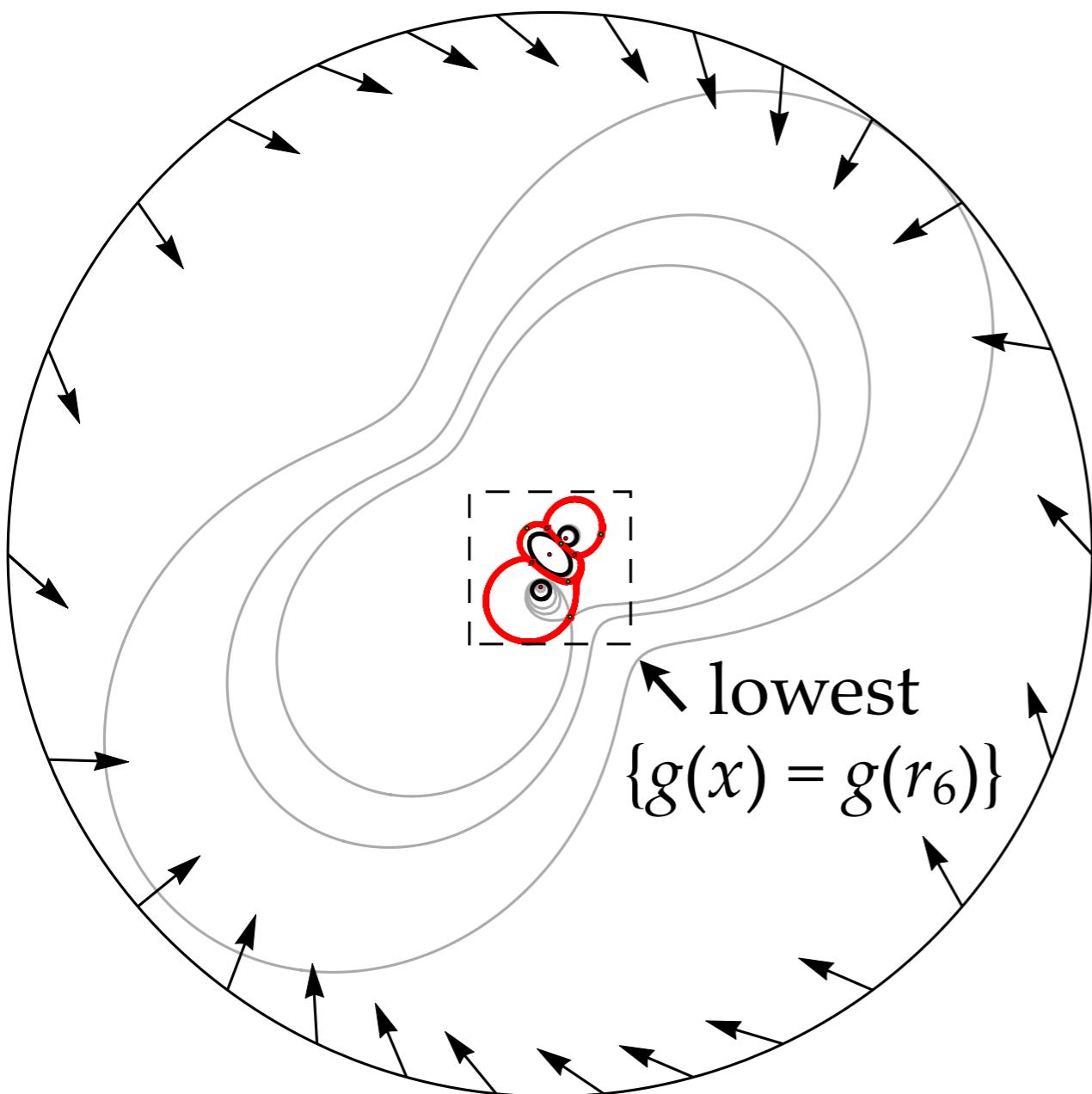
### 3. Complexity: Trajectory Bound

The length of any trajectory of  $\nabla g$  in a ball of radius  $r$  has length bounded by  $rC(n, d)$ .



### 3. Complexity: Trajectory Bound

The length of any trajectory of  $\nabla g$  in a ball of radius  $r$  has length bounded by  $rC(n, d)$ .



### 3. Complexity: Bound on $r$

### 3. Complexity: Bound on $r$

**Theorem**

### 3. Complexity: Bound on $r$

**Theorem** Assuming  $f$  has no singular points

### 3. Complexity: Bound on $r$

**Theorem** Assuming  $f$  has no singular points  
the routing point level sets of  $g$

### 3. Complexity: Bound on $r$

**Theorem** Assuming  $f$  has no singular points  
the routing point level sets of  $g$   
are contained in a ball centered at the origin of radius

### 3. Complexity: Bound on $r$

**Theorem** Assuming  $f$  has no singular points  
the routing point level sets of  $g$   
are contained in a ball centered at the origin of radius

$$r \leq \left( 2ndB \left( 1 + c_1^2 + \dots + c_n^2 \right) \right)^{(17nd)^{31n}}$$

### 3. Complexity: Bound on $r$

**Theorem** Assuming  $f$  has no singular points  
the routing point level sets of  $g$   
are contained in a ball centered at the origin of radius

$$r \leq \left( 2ndB \left( 1 + c_1^2 + \dots + c_n^2 \right) \right)^{(17nd)^{31n}}$$

**Proof Sketch**

### 3. Complexity: Bound on $r$

**Theorem** Assuming  $f$  has no singular points  
the routing point level sets of  $g$   
are contained in a ball centered at the origin of radius

$$r \leq \left( 2ndB \left( 1 + c_1^2 + \dots + c_n^2 \right) \right)^{(17nd)^{31n}}$$

**Proof Sketch**

$$\begin{aligned} \nabla g(r) = 0 \wedge g(r) \neq 0 &\iff \\ 2\nabla f(r)U(r) - \gamma f(r)\nabla U(r) = 0 &\\ \wedge f(r) \neq 0 \end{aligned}$$

### 3. Complexity: Bound on $r$

**Theorem** Assuming  $f$  has no singular points  
the routing point level sets of  $g$   
are contained in a ball centered at the origin of radius

$$r \leq \left( 2ndB \left( 1 + c_1^2 + \dots + c_n^2 \right) \right)^{(17nd)^{31n}}$$

**Proof Sketch**

$$\nabla g(r) = 0 \wedge g(r) \neq 0$$

$$\iff$$

$$2\nabla f(r)U(r) - \gamma f(r)\nabla U(r) = 0$$

$$\wedge f(r) \neq 0$$

### 3. Complexity: Bound on $r$

**Theorem** Assuming  $f$  has no singular points  
the routing point level sets of  $g$   
are contained in a ball centered at the origin of radius

$$r \leq \left( 2ndB \left( 1 + c_1^2 + \dots + c_n^2 \right) \right)^{(17nd)^{31n}}$$

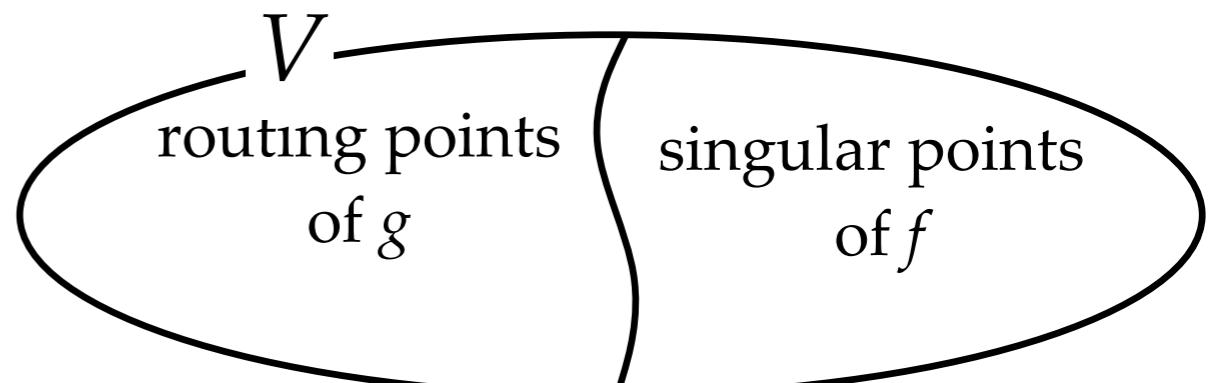
**Proof Sketch**

$$\nabla g(r) = 0 \wedge g(r) \neq 0$$

$\iff$

$$2\nabla f(r)U(r) - \gamma f(r)\nabla U(r) = 0$$

$$\wedge f(r) \neq 0$$



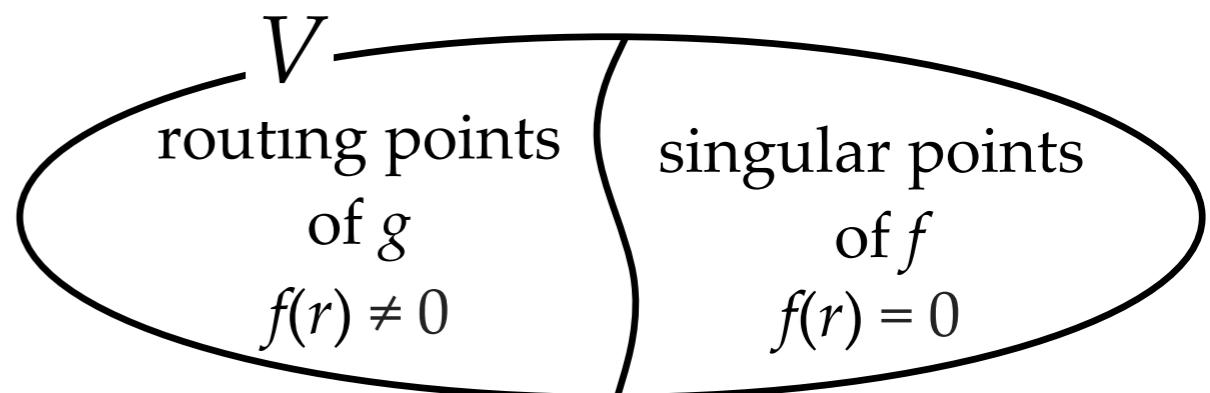
### 3. Complexity: Bound on $r$

**Theorem** Assuming  $f$  has no singular points  
the routing point level sets of  $g$   
are contained in a ball centered at the origin of radius

$$r \leq \left( 2ndB \left( 1 + c_1^2 + \dots + c_n^2 \right) \right)^{(17nd)^{31n}}$$

**Proof Sketch**

$$\begin{aligned} \nabla g(r) = 0 \wedge g(r) \neq 0 &\iff \\ 2\nabla f(r)U(r) - \gamma f(r)\nabla U(r) &= 0 \\ \wedge f(r) \neq 0 \end{aligned}$$



### 3. Complexity: Bound on $r$

**Theorem** Assuming  $f$  has no singular points  
the routing point level sets of  $g$   
are contained in a ball centered at the origin of radius

$$r \leq \left( 2ndB \left( 1 + c_1^2 + \dots + c_n^2 \right) \right)^{(17nd)^{31n}}$$

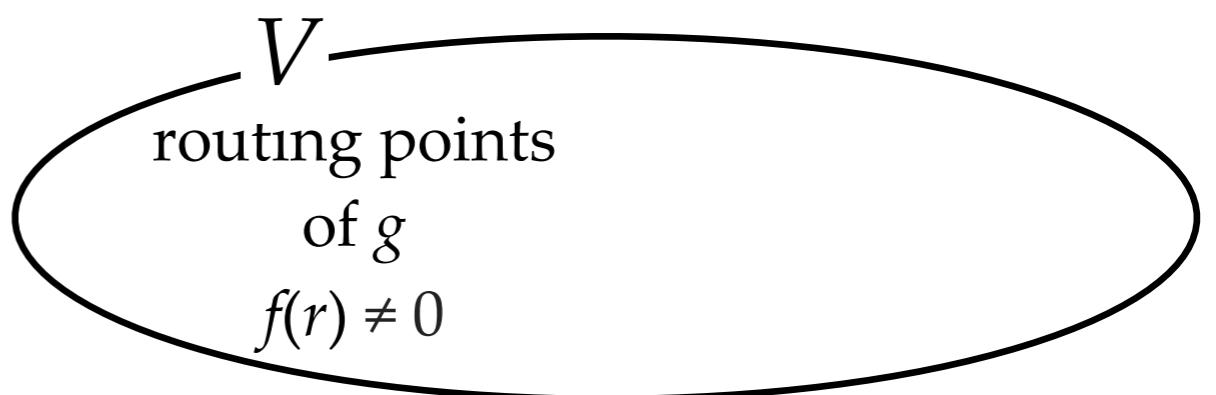
**Proof Sketch**

$$\nabla g(r) = 0 \wedge g(r) \neq 0$$

$$\iff$$

$$2\nabla f(r)U(r) - \gamma f(r)\nabla U(r) = 0$$

$$\wedge f(r) \neq 0$$



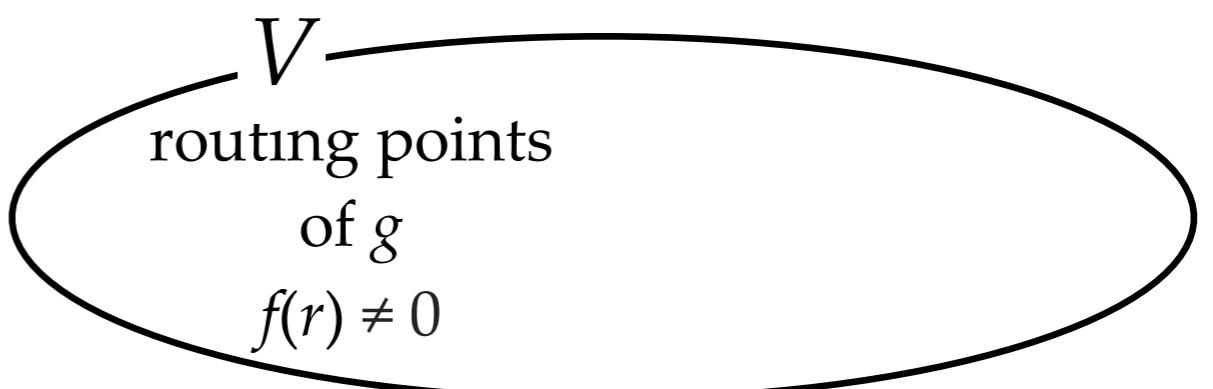
### 3. Complexity: Bound on $r$

**Theorem** Assuming  $f$  has no singular points  
the routing point level sets of  $g$   
are contained in a ball centered at the origin of radius

$$r \leq \left( 2ndB \left( 1 + c_1^2 + \dots + c_n^2 \right) \right)^{(17nd)^{31n}}$$

**Proof Sketch**

$$\begin{aligned} \nabla g(r) = 0 \wedge g(r) \neq 0 &\iff \\ 2\nabla f(r)U(r) - \gamma f(r)\nabla U(r) &= 0 \end{aligned}$$



### 3. Complexity: Bound on $r$

**Theorem** Assuming  $f$  has no singular points  
the routing point level sets of  $g$   
are contained in a ball centered at the origin of radius

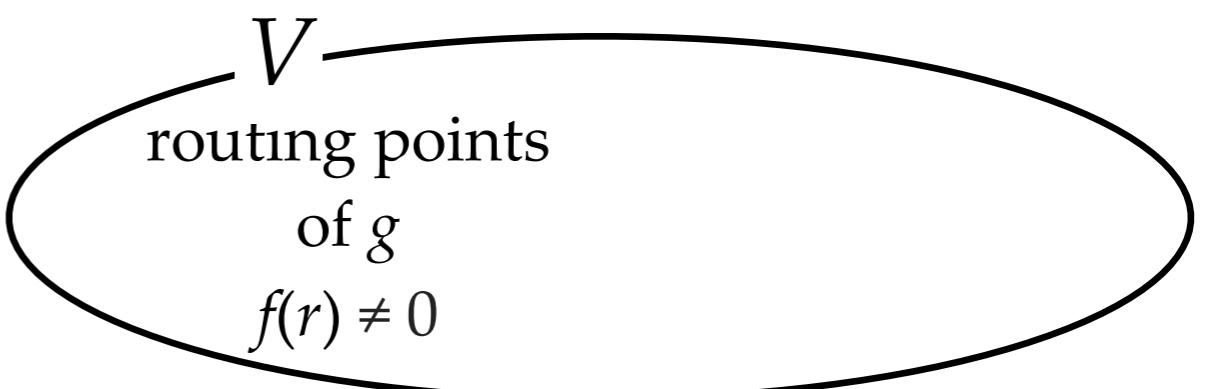
$$r \leq \left( 2ndB \left( 1 + c_1^2 + \dots + c_n^2 \right) \right)^{(17nd)^{31n}}$$

**Proof Sketch**

$$\begin{aligned} \nabla g(r) = 0 \wedge g(r) \neq 0 &\iff \\ 2\nabla f(r)U(r) - \gamma f(r)\nabla U(r) &= 0 \end{aligned}$$

---

$$2\nabla f(r)U(r) - \gamma f(r)\nabla U(r) = 0$$

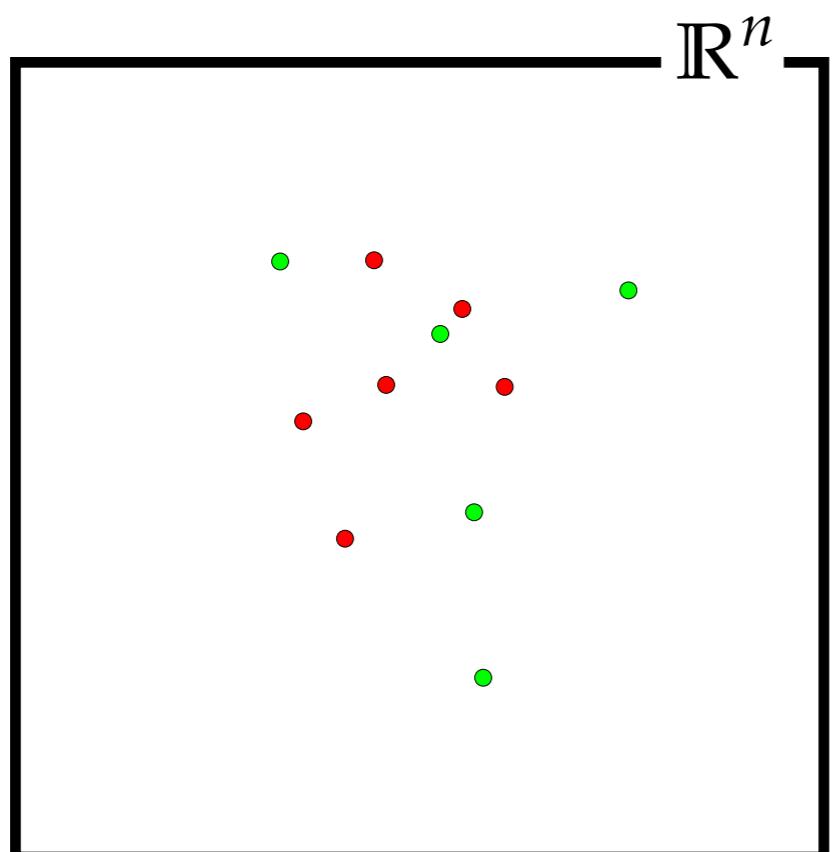


### 3. Complexity: Bound on $r$

**Theorem** Assuming  $f$  has no singular points  
the routing point level sets of  $g$   
are contained in a ball centered at the origin of radius

$$r \leq \left( 2ndB \left( 1 + c_1^2 + \dots + c_n^2 \right) \right)^{(17nd)^{31n}}$$

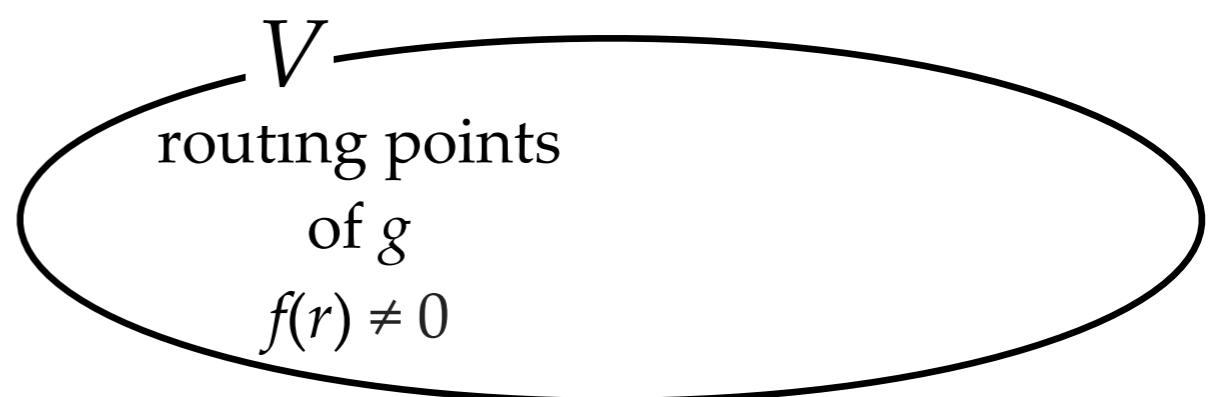
**Proof Sketch**



$$\begin{aligned} \nabla g(r) = 0 \wedge g(r) \neq 0 &\iff \\ 2\nabla f(r)U(r) - \gamma f(r)\nabla U(r) &= 0 \end{aligned}$$

---

$$2\nabla f(r)U(r) - \gamma f(r)\nabla U(r) = 0$$

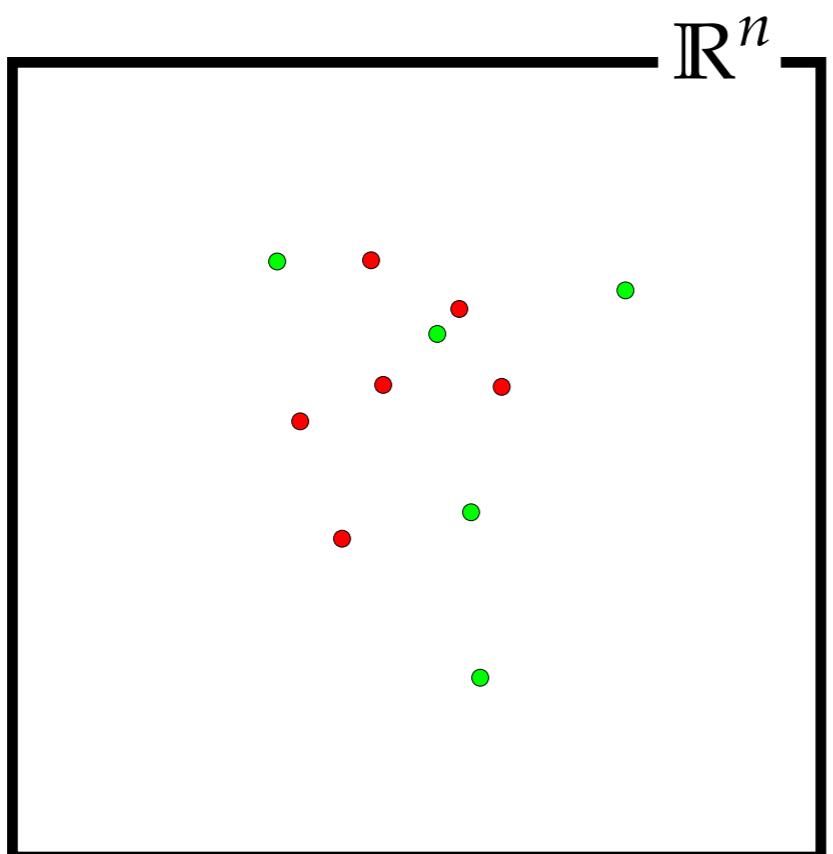


### 3. Complexity: Bound on $r$

**Theorem** Assuming  $f$  has no singular points  
the routing point level sets of  $g$   
are contained in a ball centered at the origin of radius

$$r \leq \left( 2ndB \left( 1 + c_1^2 + \dots + c_n^2 \right) \right)^{(17nd)^{31n}}$$

**Proof Sketch**

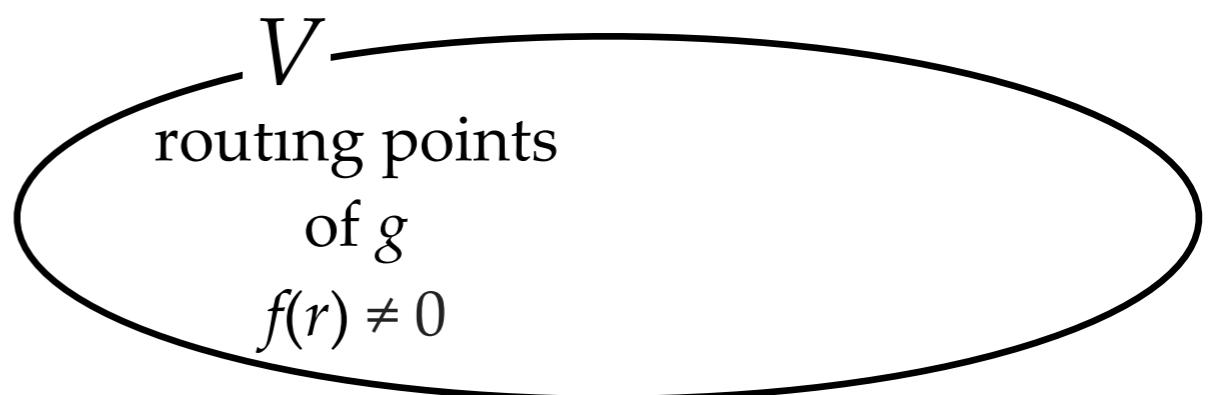


$$\begin{aligned} \nabla g(r) = 0 \wedge g(r) \neq 0 &\iff \\ 2\nabla f(r)U(r) - \gamma f(r)\nabla U(r) &= 0 \end{aligned}$$

---

$$2\nabla f(r)U(r) - \gamma f(r)\nabla U(r) = 0$$

$$g(x) = g(r)$$

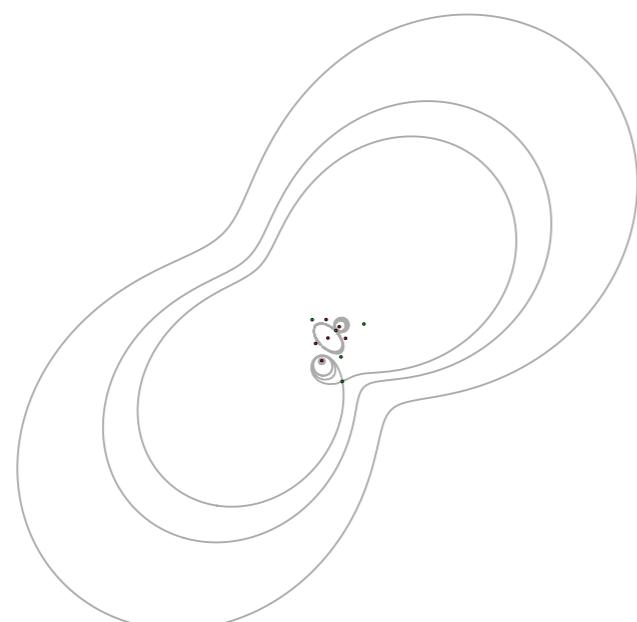


### 3. Complexity: Bound on $r$

**Theorem** Assuming  $f$  has no singular points  
the routing point level sets of  $g$   
are contained in a ball centered at the origin of radius

$$r \leq \left( 2ndB \left( 1 + c_1^2 + \dots + c_n^2 \right) \right)^{(17nd)^{31n}}$$

**Proof Sketch**



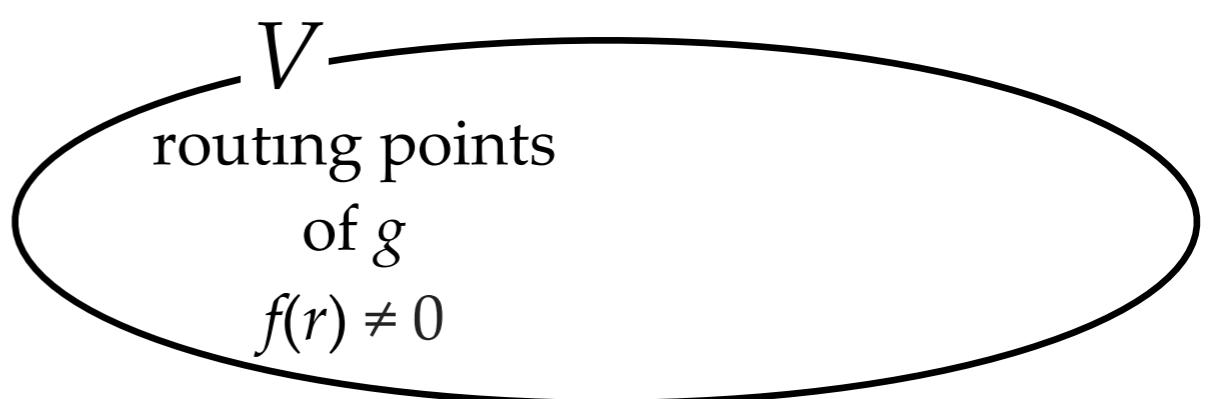
$\mathbb{R}^{2n}$

$$\begin{aligned} \nabla g(r) = 0 \wedge g(r) \neq 0 \\ \iff \\ 2\nabla f(r)U(r) - \gamma f(r)\nabla U(r) = 0 \end{aligned}$$

---

$$2\nabla f(r)U(r) - \gamma f(r)\nabla U(r) = 0$$

$$g(x) = g(r)$$

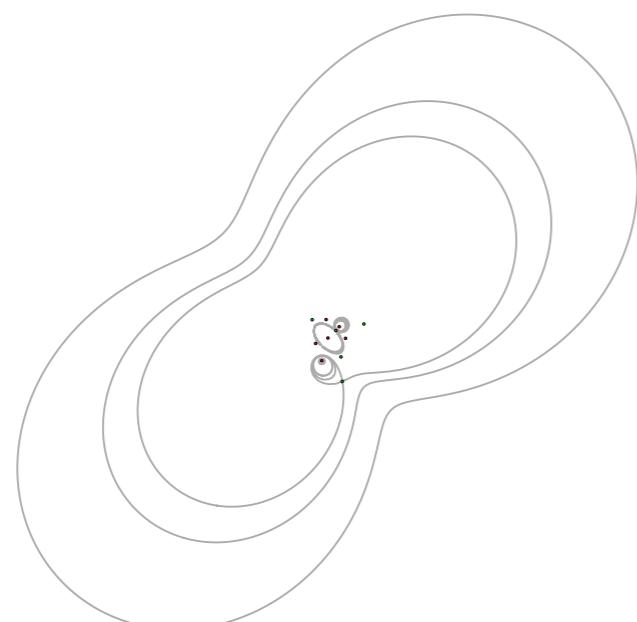


### 3. Complexity: Bound on $r$

**Theorem** Assuming  $f$  has no singular points  
the routing point level sets of  $g$   
are contained in a ball centered at the origin of radius

$$r \leq \left( 2ndB \left( 1 + c_1^2 + \dots + c_n^2 \right) \right)^{(17nd)^{31n}}$$

**Proof Sketch**



$\mathbb{R}^{2n}$

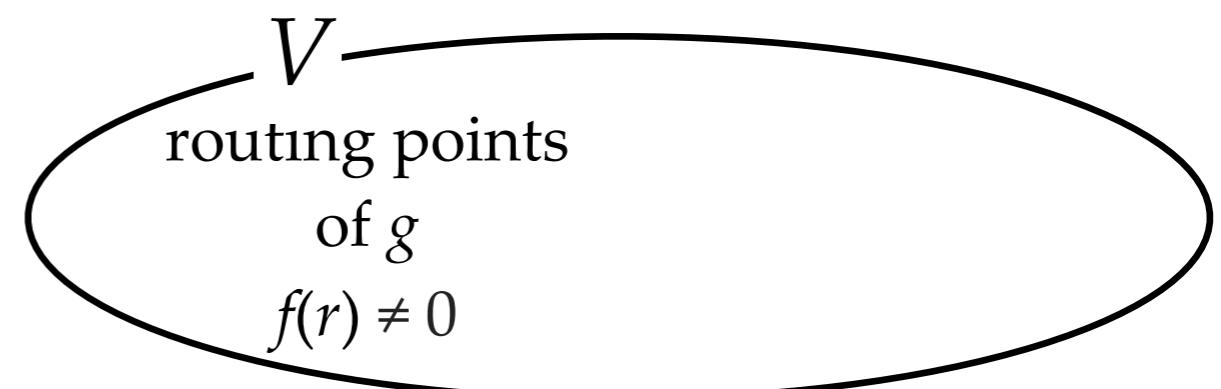
$$\nabla g(r) = 0 \wedge g(r) \neq 0$$

$$\iff 2\nabla f(r)U(r) - \gamma f(r)\nabla U(r) = 0$$

---

$$2\nabla f(r)U(r) - \gamma f(r)\nabla U(r) = 0$$

$$f(x)^2U(r)^{d+1} - f(r)^2U(x)^{d+1} = 0$$

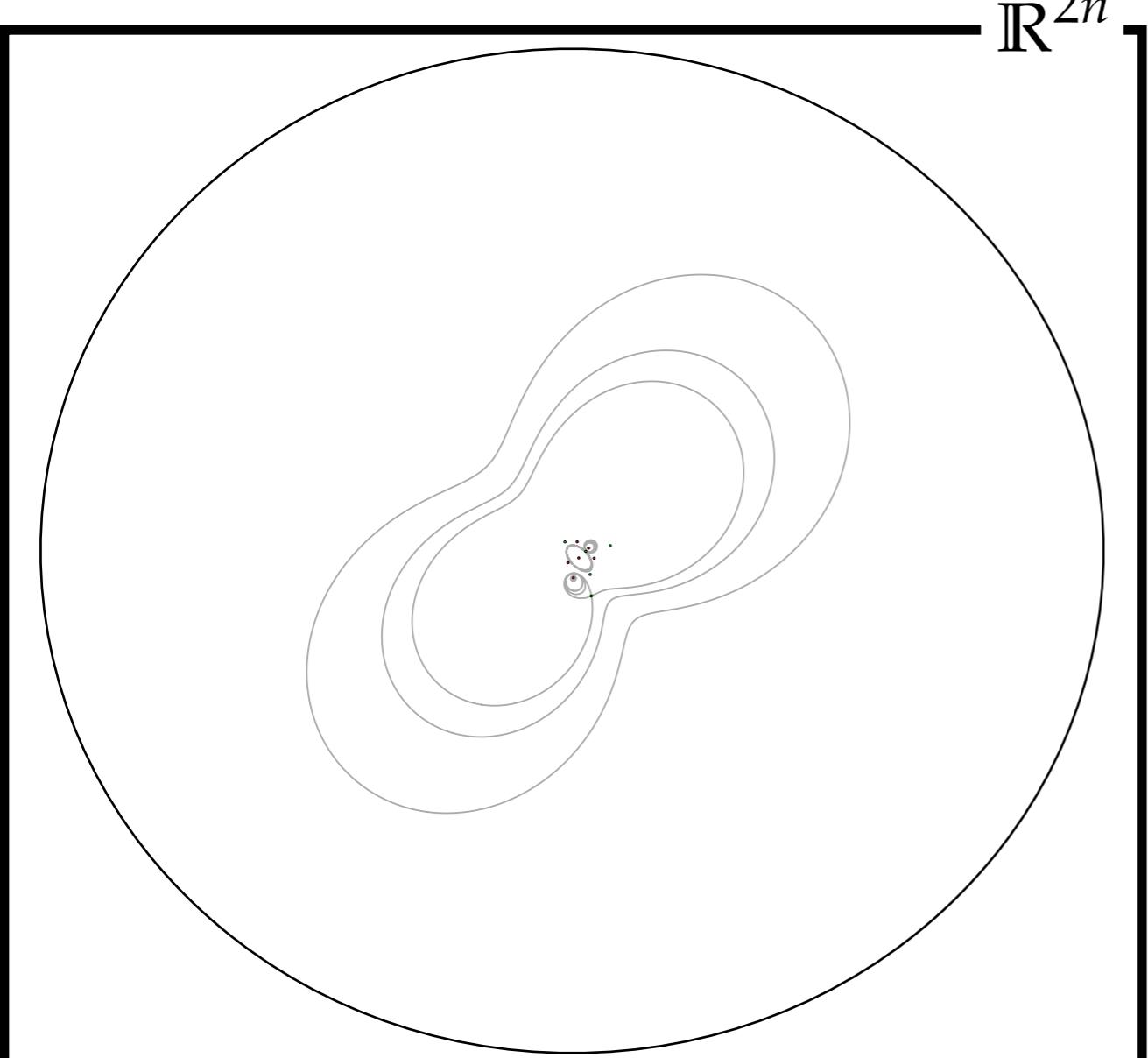


### 3. Complexity: Bound on $r$

**Theorem** Assuming  $f$  has no singular points  
the routing point level sets of  $g$   
are contained in a ball centered at the origin of radius

$$r \leq \left( 2ndB \left( 1 + c_1^2 + \dots + c_n^2 \right) \right)^{(17nd)^{31n}}$$

**Proof Sketch**



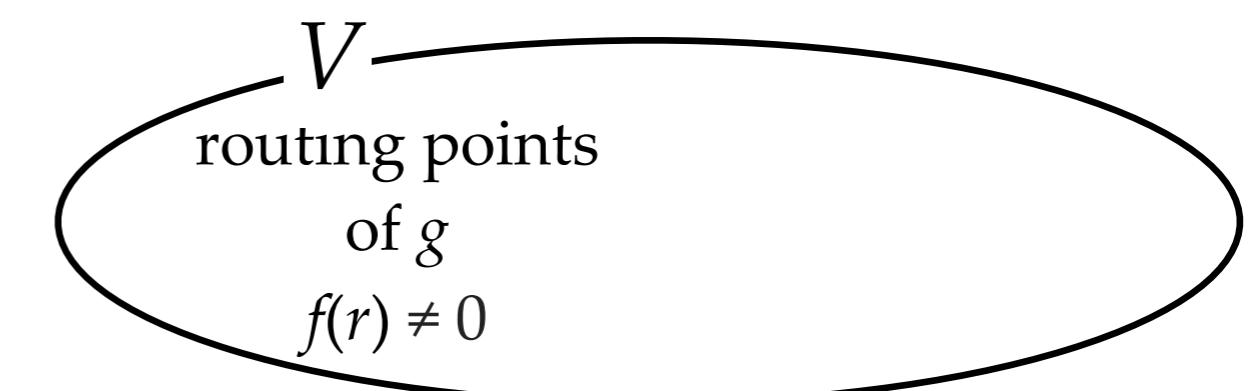
$$\nabla g(r) = 0 \wedge g(r) \neq 0$$

$$\iff 2\nabla f(r)U(r) - \gamma f(r)\nabla U(r) = 0$$

---

$$2\nabla f(r)U(r) - \gamma f(r)\nabla U(r) = 0$$

$$f(x)^2U(r)^{d+1} - f(r)^2U(x)^{d+1} = 0$$

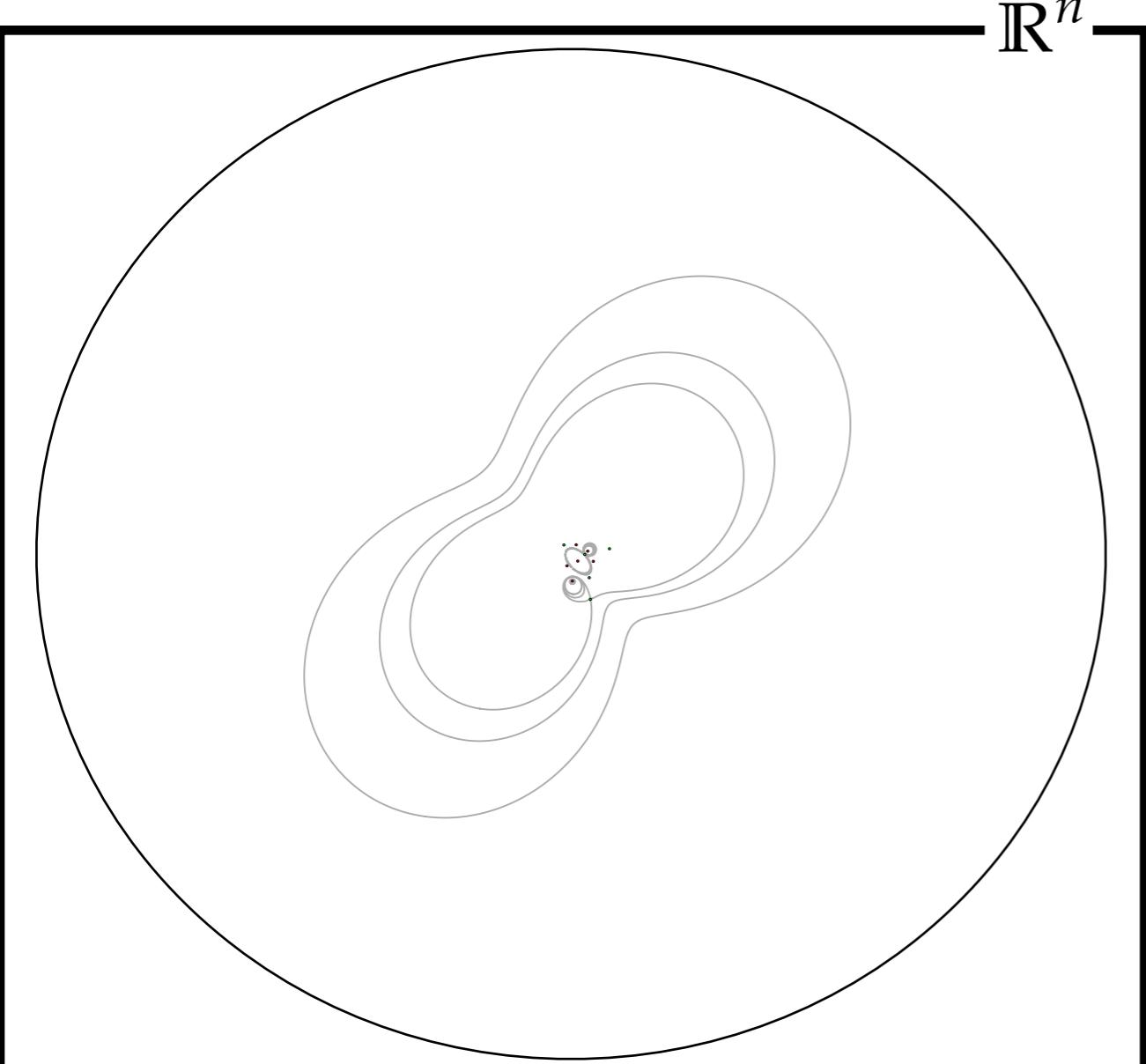


### 3. Complexity: Bound on $r$

**Theorem** Assuming  $f$  has no singular points  
the routing point level sets of  $g$   
are contained in a ball centered at the origin of radius

$$r \leq \left( 2ndB \left( 1 + c_1^2 + \dots + c_n^2 \right) \right)^{(17nd)^{31n}}$$

**Proof Sketch**



$$\begin{aligned} \nabla g(r) = 0 \wedge g(r) \neq 0 &\iff \\ 2\nabla f(r)U(r) - \gamma f(r)\nabla U(r) &= 0 \end{aligned}$$

---

$$\begin{aligned} 2\nabla f(r)U(r) - \gamma f(r)\nabla U(r) &= 0 \\ f(x)^2U(r)^{d+1} - f(r)^2U(x)^{d+1} &= 0 \end{aligned}$$

$V$   
routing points  
of  $g$   
 $f(r) \neq 0$

# Future Work

- Bounding  $C(n, d)$
- Calculating  $A$  using  $r$  and  $C(n, d)$